THE TOR GAME

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0 INTRODUCTION

Let M and N be non-zero finitely generated modules over a local (Noetherian) ring $(R,\mathfrak{m})$. Assume that $\operatorname{Tor}_i^R(M,N)=0$ for $i>>0$, and let $q=q_R(M,N)$ be the largest integer i for which $\operatorname{Tor}_i^R(M,N)\neq 0$. For the case $q=0$, Auslander [2, Theorem 1.2] found a useful formula relating the depths of M, N and R. In view of the Auslander-Buchsbaum formula [4, Theorem 1.3.3] Auslander's theorem goes as follows:

$$\begin{gathered}\operatorname{depth}(M)+\operatorname{depth}(N)-\operatorname{depth}(R)=\operatorname{depth}(M\otimes_R N)\\ \text{provided } M \text{ has finite projective dimension and } q(M,N)=0.\end{gathered}\tag{1}$$

For larger values of q, Auslander proved the following result (still assuming that M has finite projective dimension):

$$\operatorname{depth}(M)+\operatorname{depth}(N)-\operatorname{depth}(R)=\operatorname{depth}(\operatorname{Tor}_q(M,N))-q,\tag{2}$$

provided $\operatorname{depth}(\operatorname{Tor}_q^R(M,N))\leq 1$.

The past decade has seen a renewed interest in these formulas. The authors showed in [7] that (1) holds for complete intersections, even without the assumption that M have finite projective dimension. Indeed, there seem to be many situations where homological properties of a module hold if *either* (a) the module has finite projective dimension *or* (b) the ring is a complete interesection. This phenomenon led Avramov, Gasharov and Peeva [3] to investigate a condition (see §1 for the

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definition) called "finite complete intersection dimension" (finite CI dimension for short), a condition that generalizes both (a) and (b). In [1, Theorem 2.5] Araya and Yoshino obtained formulas (1) and (2) in this general context. They also showed by example that (2) can fail if $\operatorname{Tor}_q^R(M,N)$ has depth 2. Jorgensen [9, Theorem 2.2] showed, assuming M has finite CI dimension, that $\min\{\operatorname{depth}(M_p)+\operatorname{depth}(N_p)-\operatorname{depth}(R_p) \mid p\in\operatorname{Spec}(R)\} = -q$; and Theorem 3 of [5] shows that the minimum value $-q$ is achieved precisely at those primes p for which $\operatorname{depth}(\operatorname{Tor}_q^R(M,N)_p)=0$. In particular, one has

$$\operatorname{depth}(M)+\operatorname{depth}(N)-\operatorname{depth}(R)\geq -q, \tag{3}$$

if M has finite CI dimension. Choi and Iyengar [5] raised the interesting question of whether there is always *some* value j between 0 and q such that

$$\operatorname{depth}(M)+\operatorname{depth}(N)-\operatorname{depth}(R)=\operatorname{depth}(\operatorname{Tor}_j^R(M,N))-j. \tag{4}$$

They gave examples showing that this too can fail.

In this paper we describe a "game" whose goal is to prove (4) and to identify the "winning" subscript j. This approach indicates rather clearly the obstruction to winning the game, that is, the reason formula (4) can fail.

To state our main theorem we adopt some notation. Throughout this paper R is a local ring with maximal ideal $\mathfrak{m}$. Our convention is that local rings are always Noetherian. Let M and N be finitely generated non-zero R-modules, and put $q_R(M,N)=\max\{i \mid \operatorname{Tor}_i^R(M,N)\neq 0\}$. (This invariant is denoted by $\operatorname{fd}(M,N)$ in [5].) For each $i\in\mathbb{Z}$ we put $d_i(M,N)=\operatorname{depth}(\operatorname{Tor}_i^R(M,N))$. (We define the depth of the zero module to be ∞. Thus $d_i=\infty$ if $i>q$ or $i<0$.) Finally, we let

$$D_R(M,N)=\operatorname{depth}(M)+\operatorname{depth}(N)-\operatorname{depth}(R).$$

Our main result is Theorem 2.4, reproduced here for convenience:

MAIN THEOREM 0.1. *Let M and N be non-zero finitely generated modules over a local ring R. Assume M has finite CI dimension and $q_R(M,N)<\infty$. Put $m=m(M,N)=\min\{d_i-i \mid i\in\mathbb{Z}\}$ and $j=j(M,N)=\max\{i \mid d_i-i=m\}$. Assume*

$$d_i-i\geq m+2 \text{ for } i>j. \tag{†}$$

Then $D(M,N)=m$.

The definition of j forces

$$d_i-i\geq m+1 \text{ for } i>j.$$

The assumption of the theorem is simply that the next possible value for d_i-i, namely $d_j-j+1=m+1$, is not attained. Theorem 2.4 recovers the earlier results mentioned above. For example, suppose that the depth of the last non-zero Tor is 0 or 1. Then d_q-q is either $-q$ or $1-q$, while all other d_i-i are $\geq 1-q$. Theorem 2.4 immediately gives that $D(M,N)=d_q-q$ in this case (which recovers formula (2) above).

1 PRELIMINARIES

We recall the following definition from [3]:

DEFINITION 1.1. A finitely generated module M over a local ring R has *finite CI dimension* provided there exist a local ring $(S,\mathfrak{n})$, a regular sequence $(x_1,\ldots,x_c)$ in $\mathfrak{n}$, and a flat local homomorphism $R \to R' := S/(x_1,\ldots,x_c)$ such that $\operatorname{pd}_S(M \otimes_R R') < \infty$.

We will need the following result, proved by Auslander [2, Proposition 1.1] in the case of finite projective dimension:

PROPOSITION 1.2. *Let M and N be non-zero finitely generated modules over a local ring $(R,\mathfrak{m})$. Assume M has finite CI dimension and that $q = q_R(M,N) < \infty$. If* $\operatorname{depth} N = 0$ *then $d_q(M,N) = 0$.*

Proof. With the notation of (1.1), we may assume that $R = R'$. For $0 \le j \le c$, put $S_j = S/(x_1,\ldots,x_{c-j})$ (so that $S_0 = R$ and $S_c = S$), and put $T_i^{S_j} = \operatorname{Tor}_i^{S_j}(M,N)$. For $i \ge 1$ and $0 \le j \le c-1$, there is an exact sequence

$$\mathrm{T}_{i+1}^{S_j} \to \mathrm{T}_{i-1}^{S_j} \to \mathrm{T}_i^{S_{j+1}} \to \mathrm{T}_i^{S_j}. \tag{1.2.1}$$

(See, for example, [7, (2.1)].) We see that

$$0 \neq \mathrm{T}_q^{S_0} \cong \mathrm{T}_{q+1}^{S_1} \cong \ldots \cong \mathrm{T}_{q+c}^{S_c} \tag{1.2.2}$$

and that $T_i^{S_j} = 0$ for all $i > q+j$. Since M has finite projective dimension over S, Auslander's result [2, Proposition 1.1] $\mathrm{T}_{q+c}^{S_c}$ has depth 0. By (1.2.2), then, $\mathrm{T}_q^{S_0}$ also has depth 0, as desired. □

DEFINITION 1.3. Let M be a finitely generated module over a local ring $(R,\mathfrak{m})$. An element $x \in \mathfrak{m}$ is *general with respect to M*, provided x is a non-zerodivisor on $M/\mathrm{H}_{\mathfrak{m}}^0(M)$.

LEMMA 1.4. *Let M be a finitely generated module over a local ring $(R,\mathfrak{m})$, and let $x \in \mathfrak{m}$ be general with respect to M.*

(1) $\operatorname{Ann}_M(x)$ *has finite length.*

(2) If $\operatorname{depth}(M) = 0$, *then* $\operatorname{depth}(M/xM) = 0$.

Proof. If $z \in \operatorname{Ann}_M(x)$, then $xz = 0$, and since x is general this forces $z \in \mathrm{H}_{\mathfrak{m}}^0(M)$. Thus $\operatorname{Ann}_M(x) \subseteq \mathrm{H}_{\mathfrak{m}}^0(M)$, and (1) follows. Suppose now that M has depth 0. Then $\operatorname{Ann}_M(x) \neq 0$, and $\operatorname{Ann}_M(x)$ has finite length by part (1). By [8, Lemma 6.2], $\operatorname{depth}(M/xM) = 0$. □

2 THE TOR GAME

The idea of the Tor Game is to choose a general element x, replace N by $\bar{N} := N/xN$, and compute the depths of the new modules $\mathrm{Tor}_i^R(M,\bar{N})$. Repeating this procedure, we eventually decrease d_q to 0 or 1, at which point we can invoke formula (2) in the introduction. Then, with a little luck, we can backtrack and identify the winning index j for which $D(M,N) = d_j(M,N) - j$. The next theorem gives the recipe for computing the new depths, except for one annoying siutation—when $d_i = 2$ and $d_{i-1} = 0$. In a Tor Game where this situation never occurs at any stage, we win.

RULES OF THE TOR GAME 2.1. Let M and N be non-zero finitely generated modules over a local ring R. Assume that $q(M,N) < \infty$, and define $d_i = d_i(M,N)$ as above. Assume that N has positive depth, and let $x \in \mathfrak{m}$ be general with respect to N and with respect to each $\mathrm{Tor}_i^R(M,N), 0 \le i \le q(M,N)$. Let $\bar{N} = N/xN$ and put $\bar{d_i} = \mathrm{depth}(\mathrm{Tor}_i^R(M,\bar{N}))$. In the following rules, we declare the zero module to have depth ∞ and apply the usual rules of arithmetic involving ∞, e.g., $\infty - 1 = \infty$. Also, we take $\mathrm{Tor}_i^R(M,N) = 0$ if $i < 0$.

(R1) If $d_i > 0$ and $d_{i-1} > 0$, then $\bar{d_i} = d_i - 1$.

(R2) If $d_i = 0$ then $\bar{d_i} = 0$.

(R3) If $d_i \neq 2$ and $d_{i-1} = 0$, then $\bar{d_i} = 0$.

(R4) If $d_i < \infty$ then $\bar{d_i} < \infty$.

Proof. We put $\bar{N} = N/xN$, and tensor the short exact sequence $0 \to N \xrightarrow{x} N \to \bar{N} \to 0$ with M. The resulting long exact sequence provides, for each $i \in \mathbb{Z}$, a short exact sequence

$$0 \to \frac{\mathrm{Tor}_i^R(M,N)}{x\mathrm{Tor}_i^R(M,N)} \to \mathrm{Tor}_i^R(M,\bar{N}) \to \mathrm{Ann}_{\mathrm{Tor}_{i-1}^R(M,N)}(x) \to 0.$$

For ease of notation, we rewrite this exact sequence as

$$0 \to U \to V \to W \to 0.$$

If $d_{i-1} > 0$, then $W = 0$. Therefore $U \cong V$, and $\bar{d_i} = \mathrm{depth}(U)$. If also $d_i > 0$, then $\mathrm{depth}(U) = d_i - 1$, since x is a non-zerodivisor on $\mathrm{Tor}_i^R(M,N)$ (even if $d_i = \infty$). This proves (R1).

To prove (R2), suppose $d_i = 0$. Then $\mathrm{depth}(U) = 0$ by Lemma 1.4. Since V contains U, $\bar{d_i} = \mathrm{depth}(V) = 0$.

For (R3) we assume that $d_{i-1} = 0$ and $\bar{d_i} > 0$. Since x is general, W has finite length by (1) of Lemma 1.4. Also, $W \neq 0$ because $d_{i-1} = 0$. Therefore $\mathrm{depth}(W) = 0$. Since $\mathrm{depth}(V) = \bar{d_i} > 0$, the “Depth Lemma” [6, Lemma 1.1] implies that $\mathrm{depth}(U) = 1$. Now $d_i > 0$ by (R2), and since x is general x is a non-zero divisor on $\mathrm{Tor}_i^R(M,N)$. Therefore $d_i = 1 + \mathrm{depth}(U) = 2$, as desired.

Finally, we note that if $d_i < \infty$ then $U \neq 0$ by Nakayama’s Lemma. Therefore $V \neq 0$ and we have (R4). □

Before stating the main theorem, we show how to use the Tor Game to prove an inequality obtained by Choi and Iyengar.

PROPOSITION 2.2. *[5, Remark 7] Let M and N be non-zero finitely generated modules over a local ring R. Assume that M has finite CI dimension and that $q_R(M,N) < \infty$. Then $D(M,N) \geq m := \min\{d_i - i \mid i \in \mathbb{Z}\}$.*

Proof. If $\operatorname{depth}(N) = 0$, then $d_q = 0$ by Proposition 1.2. Therefore $m = -q$, which equals $D(M,N)$ by formula (2) (proved by Araya and Yoshino [1] in the present context). We assume now that $\operatorname{depth}(N) > 0$ and proceed by induction on $\operatorname{depth}(N)$. We play the Tor Game, noting that $D(M,\bar{N}) = D(M,N) - 1$. It will suffice to show that $\bar{d_i} - i \geq m - 1$ for each i, for then we will have by induction that $D(M,\bar{N}) \geq m - 1$.

If $d_{i-1} > 0$, then $\bar{d_i} \geq d_i - 1$ (by (R1) and (R2)), and $\bar{d_i} - i \geq d_i - i - 1 \geq m - 1$. If, on the other hand, $d_{i-1} = 0$, then $\bar{d_i} - i \geq -i = d_{i-1} - (i-1) - 1 \geq m - 1$. □

Note that inequality (3) in the introduction follows easily from (2.2). In fact, we get the following fact, proved by Choi and Iyengar:

PROPOSITION 2.3. *[5, Theorem 3] Let M and N be non-zero finitely generated modules over a local ring R. Assume M has finite CI dimension and $q := q_R(M,N) < \infty$. Then $D(M,N) \geq -q$, with equality if and only if $d_q = 0$.*

Proof. If $d_q = 0$ we have equality by formula (2), as observed in the proof of (2.2). Assume now that $d_q > 0$, and let $d_j - j = m := \min\{d_i - i \mid i \in \mathbb{Z}\}$. Then $D(M,N) \geq d_j - j$ by (2.2). If $j = q$, then $D(M,N) \geq d_q - q > -q$; and if $j < q$ then $D(M,N) \geq -j > -q$. □

Now we come to our main theorem, which gives a sufficient condition for winning the Tor Game.

MAIN THEOREM 2.4. *Let M and N be non-zero finitely generated modules over a local ring R. Assume M has finite CI dimension and $q_R(M,N) < \infty$. Put $m = m(M,N) = \min\{d_i - i \mid i \in \mathbb{Z}\}$ and $j = j(M,N) = \max\{i \mid d_i - i = m\}$. Assume*

$$d_i - i \geq m + 2 \text{ for } i > j. \tag{†}$$

Then $D(M,N) = m$.

The proof will be deferred to Section 5 of the paper. As we shall see in Example 3.2, (†) is not a necessary condition for winning the Tor Game. It is, however, both necessary and sufficient if we play strictly by the rules (see §5).

An immediate corollary is worth stating separately.

COROLLARY 2.5. *Let M and N be non-zero finitely generated modules over a local ring R. Assume M has finite CI dimension and $q_R(M,N) < \infty$. Put $m = m(M,N) = \min\{d_i - i \mid i \in \mathbb{Z}\}$ Suppose that $m = d_q - q$. Then $D(M,N) = m$.*

Proof. This follows immediately from Theorem 2.4 since the main condition needed in that theorem is vacuously satisfied. $\square$

3 EXAMPLES

Here we give some examples to indicate some of the possibilities that can occur in the Tor Game.

EXAMPLE 3.1. **A losing Tor Game.** Let $R = k[[X,Y,U,V]]$, where k is a field, and put $f = XY - UV$. Let $M = (X,U)/(f)$ and $N = (Y,V)/(f)$. Both M and N have depth 3 (cf. [7, Example 4.1]), whence $D(M,N) = 2$. As modules over $R/(f)$, both M and N are free on the punctured spectrum of $R/(f)$ (cf. [8, Example 1.8]). If d_0 were positive, $M \otimes_R N$ would satisfy Serre's condition (S_1) as an $R/(f)$-module and therefore be torsion-free as an $R/(f)$-module. But then $M \otimes_R N$, which requires 4 generators, would be isomorphic to the ideal product $(x,u)(y,v)$ in $R/(f)$. But $(x,u)(y,v) = (xy,xv,uy,uv) = (xy,xv,uy)$, contradiction. Thus $d_0 = 0$.

To compute d_1, we choose an exact sequence $0 \to F \to G \to M \to 0$, with F and G free R-modules. Tensoring this with N, we get an exact sequence

$$0 \to \mathrm{Tor}_1^R(M,N) \to F \otimes_R N \to G \otimes_R N \to M \otimes_R N \to 0.$$

Since the two middle terms have depth 3 and $M \otimes_R N$ has depth 0, it follows from the Depth Lemma that $d_1 = 2$.

To summarize, we have $q = 1, D(M,N) = 2, d_0 = 0$ and $d_1 = 2$. Since neither d_0 nor $d_1 - 1$ is equal to $D(M,N)$, we lose the Tor Game.

EXAMPLE 3.2. **A more general losing Tor Game.** Let R be a Cohen-Macaulay ring of dimension d and let p be a height-one prime ideal of R. Assume that R/p is not Cohen-Macaulay, and set its depth equal to s. Choose a non-zerodivisor $a \in p$ and write $p = (a : b)$, where b is also a non-zerodivisor. Set $M = R/Ra$ and $N = R/Rb$. Observe that

$$D(M,N) = (d-1) + (d-1) - d = d-2,$$

and both M and N have projective dimension 1. We compute the depths d_0 and d_1. Since $\mathrm{Tor}_1^R(M,N) \cong p/(a)$, the short exact sequence

$$0 \to p/(a) \to R/(a) \to R/p \to 0,$$

together with the fact that $\mathrm{depth}(R/p) = s < d-1 = \mathrm{depth}(R/(a))$, shows that $d_1 = s+1$. Likewise, since $\mathrm{Tor}_0^R(M,N) \cong R/(a,b)$, the short exact sequence

$$0 \to R/p \to R/(a) \to R/(a,b) \to 0$$

proves that $d_0 = s-1$. Hence the minimum value of $d_1 - 1$ and $d_0 - 0$ is $s-1$ which is not equal to $d-2 = D(M,N)$. This example shows very clearly that in general there is no chance of proving a relationship between $D(M,N)$ and the values $d_i - i$. The gap of 2 between d_1 and d_0 is fatal. Our main result in some sense shows that this is only way the Tor Game fails.

EXAMPLE 3.3. **Cheating in the Tor Game.** Let $R = k[[X,Y,Z]]$ and put $f = XY + Z^2$. Let $M = N = (X,Z)/(f)$. As in the first example, it is helpful to view M as an ideal of the hypersurface $R/(f)$. Certainly $M \otimes_R N$ has non-zero torsion as an $R/(f)$-module (since it needs four generators and the ideal $(x,z)^2$ of $R/(f)$ needs only 3). As shown in [7, Example 4.2], M is a reflexive $R/(f)$-module and therefore has depth 2. Therefore $D_R(M,N) = 1$ and $q(M,N) \leq 1$. Since $R/(f)$ is an isolated singularity M is free on the punctured spectrum of $R/(f)$, and it follows, essentially as in Example 4.1, that $d_0 = 0$ and $d_1 = 2$. We see that $m(M,N) = 0$, $j(M,N) = 0$, but (†) fails because $d_1 - 1 = m + 1$. In a sense we have won the Tor Game, since $D(M,N) = d_1 - 1$, but we have not followed the rules, and the answer is not the one "$D(M,N) = m(M,N)$" given by the theorem. In the next section we will describe a slightly modified game that keeps the player from having to think and prevents "accidental" wins like this.

EXAMPLE 3.4. **Fluctuating depths.** One might hope that in general the d_i are monotone, as a kind of strong rigidity of Tor. However, this is not the case as the following example (pointed out to us by Bernd Ulrich) shows. Let R be a regular local ring of dimension d and I a perfect height two ideal. Assume that I is not generated by a regular sequence. Set $M = N = R/I$. Then $\operatorname{Tor}_0^R(M,N) = R/I$ has depth $d-2$. The module $\operatorname{Tor}_1^R(M,N) \cong I/I^2$, and this module has depth $d-3$ (see [10]). A resolution of R/I is given by

$$0 \to R^{n-1} \to R^n \to R \to R/I \to 0.$$

Tensoring with R/I shows that the following is exact:

$$0 \to \operatorname{Tor}_2^R(R/I,R/I) \to (R/I)^{n-1} \to (R/I)^n \to I/I^2 \to 0.$$

This proves that $\operatorname{Tor}_2^R(R/I,R/I)$ is Cohen-Macaulay having depth $d-2$, strictly greater than $\operatorname{depth}(\operatorname{Tor}_1^R(R/I,R/I))$, which in turn is strictly less than the depth of $\operatorname{Tor}_0^R(R/I,R/I)$. Hence the depths fluctuate. In this case, however, the conditions of Theorem 2.4 are met and we win the Tor game: $D(R/I,R/I) = (d-2) + (d-2) - d = d-4$. Then $(d_0 - 0, d_1 - 1, d_2 - 2) = (d-2, d-4, d-4)$. Since the minimum value occurs at the last non-zero Tor, we automatically win according to Corollary 2.5.

EXAMPLE 3.5. **Another losing game.** We refer to [5, Prop. 11, Example 12] which constructs a losing Tor game over a regular local ring. Specifically, they construct two cyclic modules M and N over a regular local ring of dimension 5 such that $q_R(M,N) = 1, D(M,N) = 2, d_0 = 0$ and $d_1 = 2$. Hence $\{d_0 - 0, d_1 - 1\} = \{0, 1\}$ and $D(M,N)$ is not in this set.

4 THE VIRTUAL TOR GAME

In order to expand the market for our game, we have developed a version that is much easier to play. It requires very little skill, and there is no strategy whatsoever. You just follow the rules and hope for the best.

In the Virtual Tor Game, Player I chooses a sequence $d_0,\dots,d_q$, where each d_i is either a non-negative integer, the symbol ∞, or the symbol $\star$ (indicating an "unknown" depth). It is required that $d_0 \in \{0,1,2,\dots\}$ and $d_q \neq \infty$. Player I understands that $d_i = \infty$ if $i < 0$. Values of d_i for $i > q$ are irrelevant. The symbol $\star$ is incomparable with integers and with ∞. Thus, if we hypothesize, for example, that $d_i > 0$, it is assumed that $d_i \neq \star$.

If $d_q \in \{0,\star\}$, the game is over (see below). Otherwise Player II computes the sequence $\bar{d}_0,\dots,\bar{d}_q$ according to the rules (which we shall describe shortly), and, if the game is not over, repeats the process on the new sequence (computing $\bar{\bar{d}}_0,\dots,\bar{\bar{d}}_q$) and so on. A consequence of the Rules is that at each stage one has $d_{-1} = \infty$, and Player I can simply be assured of this at the outset.

The rules of the Virtual Tor Game are the same as in the actual Tor Game, except that we have introduced the symbol $\star$ to deal with the undecided case $d_i = 2$, $d_{i-1} = 0$. Also, we allow some of the original data to include unknown entries.

The rules of the Virtual Tor Game are the following:

(VR1) If $d_i > 0$ and $d_{i-1} > 0$, then $\bar{d}_i = d_i - 1$.

(VR2) If $d_i \leq 1$ then $\bar{d}_i = 0$.

(VR3) If $d_i > 2$ and $d_{i-1} = 0$, then $\bar{d}_i = 0$.

(VR4) If $d_i = \star$ then $\bar{d}_i = \star$.

(VR5) If $d_i = 2$ and $d_{i-1} = 0$, then $\bar{d}_i = \star$.

(VR6) If $d_i \geq 2$ and $d_{i-1} = \star$, then $\bar{d}_i = \star$.

Player II wins if, after a finite sequence of iterations, d_q becomes 0. Player II loses if he eventually gets $d_q = \star$. It is easy to see that Player II either wins or loses. (Of course, Player I is expected to make an interesting choice of the orginal d_i, so that neither a win nor a loss is obvious from the start!)

We will give some examples of winning and losing Virtual Tor Games. In deference to Macaulay 2, we will list the d_i in the order $d_0\ d_1\ \dots\ d_q$. This list appears in the second line of each display, the first line (in bold) being just the indices $0\ 1\ \dots\ q$. The third line will be $\bar{d}_0\ \bar{d}_1\ \dots\ \bar{d}_q$, the result of one move of Virtual Tor Game, and so on.

EXAMPLE 4.1. **An inauspicious beginning, but we win in the end.**

i	:	**0**	**1**	**2**	**3**	**4**	**5**	**6**
d_i	:	0	2	3	5	4	7	9
$\bar{d}_i$	:	0	$\star$	2	4	3	6	8
$\bar{\bar{d}}_i$	:	0	$\star$	$\star$	3	2	5	7
·		0	$\star$	$\star$	$\star$	1	4	6
·		0	$\star$	$\star$	$\star$	0	3	5
·		0	$\star$	$\star$	$\star$	0	0	4
·		0	$\star$	$\star$	$\star$	0	0	0

Here we get stars right from the start, and they move ominously to the right, but eventually they are blocked by 0's. In this example we have $m = 0$ and $j = 4$. Note that condition $(\dagger)$ is satisfied.

EXAMPLE 4.2. **A promising beginning with a devastating finale.**

i	:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
d_i	:	2	4	4	7	8	10	9	11
$\bar{d}_i$	:	1	3	3	6	7	9	8	10
$\bar{\bar{d}}_i$	:	0	2	2	5	6	8	7	9
·		0	$\star$	1	4	5	7	6	8
·		0	$\star$	0	3	4	6	5	7
·		0	$\star$	0	0	3	5	4	6
·		0	$\star$	0	0	0	4	3	5
·		0	$\star$	0	0	0	0	2	4
·		0	$\star$	0	0	0	0	$\star$	3
·		0	$\star$	0	0	0	0	$\star$	$\star$

Here stars crop up early but are quickly blocked. Trouble lies ahead, however, and the culprit is d_6: Note that $m = 2$ and $j = 2$; condition $(\dagger)$ fails, since $d_6 - 6 = m + 1$.

The configuration $d_{i+1} = d_i + 2$ will often, but not always, lead to a losing Virtual Tor Game. In fact, one can lose the virtual game even when this configuration is not present at the outset.

EXAMPLE 4.3. **Another losing Tor Game.**

i	:	**0**	**1**	**2**	**3**	**4**	**5**
d_i	:	1	1	4	4	5	6
$\bar{d}_i$	:	0	0	3	3	4	5
$\bar{\bar{d}}_i$	:	0	0	0	2	3	4
·		0	0	0	$\star$	2	3
·		0	0	0	$\star$	$\star$	2
·		0	0	0	$\star$	$\star$	$\star$

5 PROOF OF THE MAIN THEOREM

In this section we will show that condition (†) of (1.4), slightly modified to allow some unknown depths in the original data, is necessary and sufficient for winning the Virtual Tor Game. The Main Theorem will follow as a special case.

NOTATION AND ASSUMPTIONS 5.1. Suppose that we are given a list $d_0, \dots, d_q$, with each $d_i \in \{0, 1, 2, \dots, \infty, \star\}$. We assume always that d_0 is an integer, that $d_q \neq \infty$, and that $d_i = \infty$ if $i < 0$ or $i > q$. We put $b = \sup\{i \mid d_i = \star\}$. (If $\star$ does not occur among the d_i we put $b = -\infty$.) Let $m = \inf\{d_i - i \mid i > b\}$ and $j = \sup\{i > b \mid d_i - i = m\}$. (Note that $m = j = \infty$ if $b = q$; otherwise $m, j < \infty$.) Assuming that $d_q \notin \{0, \star\}$, we compute $\bar{d}_0, \dots, \bar{d}_q$ according to the rules of the Virtual Tor Game in Section 4. Note that we still have $\bar{d}_0 \in \{0, 1, 2, \dots\}$, $\bar{d}_q \neq \infty$, and $\bar{d}_i = \infty$ if $i < 0$ or $i > q$. (Informally, $\bar{q} = q$.) We define $\bar{b}$, $\bar{m}$ and $\bar{j}$ in the obvious way. We consider the following condition, which amounts to (†) in the case where there are no stars ($b = -\infty$):

$$\begin{aligned} &\text{(a) } d_i - i \geq m + 2 \text{ for each } i > j,\\ &\text{(b) } d_q \neq \star, \text{ and}\\ &\text{(c) } j \geq b + d_j. \end{aligned} \qquad (\ddagger)$$

Part (b) guarantees that j and d_j are finite, and (c) ensures that j is far enough to the right of the last star (if there are any).

THEOREM 5.2. *Assume* $d_q \notin \{0, \star\}$.

(1) Condition (‡) holds for the d_i *if and only if (‡) holds for the* $\bar{d}_i$.

(2) If (‡) holds and $d_j > 0$, *then* $\bar{j} = j$, $\bar{d}_j = d_j - 1$ *and* $\bar{m} = m - 1$.

(3) If (‡) holds and $d_j = 0$, *then* $\bar{j} = j + 1$, $\bar{d}_{j+1} = 0$ *and* $\bar{m} = m - 1$.

Proof. Suppose first that $d_j > 0$. We claim that if (c) of (‡) holds for $d_0, \dots, d_q$, then $\bar{j} = j$, $\bar{m} = m - 1$, and both (b) and (c) hold for $\bar{d}_0, \dots, \bar{d}_q$. To see this, let $i \geq j - d_j + 1$. Since $j - d_j \geq b$, we have $d_i - i \geq d_j - j$, whence $d_i \geq d_j - j + i \geq d_j - j + (j - d_j + 1) = 1$. We have shown that $d_i \geq 1$ for all $i \geq j - d_j + 1$, and it follows that

$$\bar{d}_i = d_i - 1 \text{ for all } i \geq j - d_j + 2. \tag{5.2.1}$$

From (5.2.1) and (VR2) we see that $\bar{d}_j = j - 1$. To complete the proof of the claim, we just need to verify that $\bar{d}_i - i \geq \bar{d}_j - j$ for each $i > \bar{b}$. Let $i > \bar{b}$. Then $i > b$ (by VR6)), and hence $d_i - i \geq d_j - j$. If $\bar{d}_i - i < \bar{d}_j - j$ $(= d_j - j - 1)$ then $\bar{d}_i \leq d_i - 2$, which forces $d_{i-1} = 0$. Therefore $i - 1 > b$, and we have $\bar{d}_i - i \geq -i = d_{i-1} - (i-1) - 1 \geq d_j - j - 1 = \bar{d}_j - j$. This proves the claim. Assertion (2) and the “only if” part of (1) in the case $d_j > 0$ follow easily.

Still assuming $d_j > 0$, suppose now that (‡) fails for the d_i. If (c) holds for the d_i, then (a) must fail. By the claim in the paragraph above, (a) fails also for $\bar{d}_0, \dots, \bar{d}_q$. Assume now that (c) fails, that is, $j - d_j < b$. Then, for each $i > b$ we have $d_i - i \geq d_j - j > -b$, whence $d_i > i - b \geq 1$. In particular, $d_{b+1} \geq 2$, and now

(VR6) implies that $\bar{d}_{b+1} = \star$. Moreover, $\bar{d}_i = d_i - 1$ for each $i > b+1$, and we see that $\bar{b} = b+1$ and $\bar{j} = j$. Now $j - \bar{d}_j = j - d_j + 1 < b+1 = \bar{b}$, and (‡) fails for $\bar{d}_0, \dots, \bar{d}_q$.

This completes the proof in the case $d_j > 0$. Assume now that $d_j = 0$. Then (c) is automatically satisfied for $d_0, \dots, d_q$ (and (b) is part of the hypotheses of the theorem). If $i > j$ we have $d_i - i > d_j - j = -j$. Therefore

$$d_i \geq 2 \text{ for each } i > j. \tag{5.2.2}$$

If $d_{j+1} \geq 3$, then, by (5.2.2), $\bar{d}_{j+1} = 0$ and $\bar{d}_i = d_i - 1$ for each $i > j+1$. Assertions (1) and (3) now follow easily. Therefore we assume that $d_{j+1} = 2$, in which case (‡) fails for $\bar{d}_0, \dots, \bar{d}_q$. To complete the proof, we need only verify that (‡) fails for $d_0, \dots, d_q$. We may assume that $\bar{d}_q \neq \star$. Since $\bar{d}_{j+1} = \star$ by (VR5), we have $j+1 \leq \bar{b} < \bar{j} \leq q$. Noting that $\bar{d}_{\bar{j}} = d_{\bar{j}} - 1$ by (5.2.2), we have $\bar{b} + \bar{d}_{\bar{j}} \geq j + 1 + d_{\bar{j}} - 1 = j + \bar{j} + d_{\bar{j}} - \bar{j} > j + \bar{j} + d_j - j = \bar{j}$. This shows that (c) fails for $\bar{d}_0, \dots, \bar{d}_q$, and the proof is complete. □

COROLLARY 5.3. *Given the initial data* $d_0, \dots, d_q \in \{0, 1, 2, \dots, \infty, \star\}$, *with* $d_0 \in \mathbb{Z}$ *and* $d_q \neq \infty$, *we win the Virtual Tor Game if and only if (‡) is true.*

Proof. The rules of the Virtual Tor Game dictate that if $d_i \neq \star$, then either $\bar{d}_i < d_i$ or $\bar{d}_i = \star$. Thus, after a finite number of iterations of the Tor Game, d_q becomes either 0 or $\star$. By (1) of (5.1), this happens precisely when (‡) is, respectively, is not true. □

Proof of the Main Theorem (2.4) Put $b = -\infty$, so that (†) and (‡) say the same thing. We play the Tor Game until d_q drops to 0. By (1.2), N has positive depth at the beginning of each play; therefore $D(M,N)$ drops by 1 at each stage. But also m drops by 1 at each stage, by (2) and (3) of (5.2). Since the Theorem is true when $d_q = 0$ (by formula (2) of the introduction—proved in the context of finite CI dimension by Araya and Yoshino [1]), it must be true in general. □

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