Direct-sum behavior of modules over one-dimensional rings

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Abstract Let R be a reduced, one-dimensional Noetherian local ring whose integral closure $\overline{R}$ is finitely generated over R. Since $\overline{R}$ is a direct product of finitely many principal ideal domains (one for each minimal prime ideal of R), the indecomposable finitely generated $\overline{R}$-modules are easily described, and every finitely generated $\overline{R}$-module is uniquely a direct sum of indecomposable modules. In this article we will see how little of this good behavior trickles down to R. Indeed, there are relatively few situations where one can describe all of the indecomposable R-modules, or even the torsion-free ones. Moreover, a given finitely generated module can have many different representations as a direct sum of indecomposable modules.

1 Finite Cohen-Macaulay type

If R is a one-dimensional reduced Noetherian local ring, the maximal Cohen-Macaulay R-modules (those with depth 1) are exactly the non-zero finitely generated torsion-free modules. One says that R has *finite Cohen-Macaulay type* provided there are, up to isomorphism, only finitely many indecomposable maximal Cohen-Macaulay modules. The following theorem classifies these rings:

Theorem 1.1. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional, reduced, local Noetherian ring. Then R has finite Cohen-Macaulay type if and only if*

(DR1) *The integral closure $\overline{R}$ of R in its total quotient ring can be generated by 3 elements as an R-module; and*
(DR2) $\mathfrak{m}(\overline{R}/R)$ *is a cyclic R-module.* □

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The two conditions above were introduced by Drozd and Roĭter in a remarkable 1967 paper [12]. They proved the theorem in the special case of a ring essentially finite over $\mathbb{Z}$ and asserted that it is true in general. In 1978 Green and Reiner [16] gave a much more detailed proof of the theorem in this special case. In 1989 R. Wiegand [43] proved necessity of the conditions (DR) in general, and sufficiency assuming only that each residue field of $\overline{R}$ is separable over $k = R/\mathfrak{m}$. Since, by (DR1), the residue field growth is of degree at most 3, this completed the proof of Theorem 1.1 except in the cases where char(k) = 2 or 3. The case of characteristic 3 was handled by indirect methods in [45], leaving only the case where k is imperfect of characteristic 2. In his 1994 Ph.D. dissertation, Nuri Cimen [6] then used explicit, and very difficult, matrix reductions to prove the remaining case of the theorem.

We will sketch some of the main ingredients of the proof, though we will not touch on the matrix reductions in [16] and [6]. The pullback representation, which we describe in more generality than needed in this section, is a common theme in most of the research leading up to the proof of the theorem. For the moment, let R be any one-dimensional Noetherian ring, not necessarily local, and let $\overline{R}$ be the integral closure of R in the total quotient ring K of R. We assume that $\overline{R}$ is finitely generated as an R-module. (This assumption is no restriction: A reduced one-dimensional ring is automatically Cohen-Macaulay. If, further, R has finite Cohen-Macaulay type, then $\overline{R}$ has to be finitely generated over R (cf. [45, Lemma 1] or Proposition 1.2 below).) The conductor $\mathfrak{f} := \{r \in R \mid r\overline{R} \subseteq R\}$ contains a non-zerodivisor of R; therefore $R/\mathfrak{f}$ and $\overline{R}/\mathfrak{f}$ are Artinian rings, and we have a pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & \overline{R} \\ \downarrow & & \downarrow \\ \frac{R}{\mathfrak{f}} & \longrightarrow & \frac{\overline{R}}{\mathfrak{f}} \end{array}$$

The bottom line of the pullback is an example of an *Artinian pair* [43], by which we mean a module-finite extension $A \hookrightarrow B$ of commutative Artinian rings. Of course this pullback has the additional property that $\overline{R}/\mathfrak{f}$ is a principal ideal ring. Given an Artinian pair $\mathbf{A} = (A \hookrightarrow B)$, one defines an $\mathbf{A}$-module to be a pair $V \hookrightarrow W$, where W is a finitely generated projective B-module and V is an A-submodule of W with the property that $BV = W$. A morphism $(V_1 \hookrightarrow W_1) \to (V_2 \hookrightarrow W_2)$ of $\mathbf{A}$-modules is, by definition, a B-homomorphism from W_1 to W_2 that carries V_1 into V_2. With submodules and direct sums defined in the obvious way, we get an additive category in which every object has finite length. We say $\mathbf{A}$ has *finite representation type* provided there are, up to isomorphism, only finitely many indecomposable $\mathbf{A}$-modules. In the local case, the bottom line tells the whole story:

Proposition 1.2 ([43, (1.9)]). *Let $(R, \mathfrak{m})$ be a one-dimensional, reduced, Noetherian local ring with finite integral closure $\overline{R}$. Then R has finite Cohen-*

Macaulay type if and only if the Artinian pair $\frac{R}{\mathfrak{f}} \hookrightarrow \frac{\overline{R}}{\mathfrak{f}}$ *has finite representation type.* □

The proof of this proposition is not very hard. The key ingredients are the following:

1. Krull-Remak-Schmidt: For an Artinian pair $\mathbf{A}$, every $\mathbf{A}$-module is uniquely (up to order and isomorphism of the factors) a direct sum of indecomposable $\mathbf{A}$-modules.
2. Dickson's Lemma [9]: $\mathbb{N}_0^n$ has no infinite antichains. (Here $\mathbb{N}_0$ is the well-ordered set of natural numbers, and $\mathbb{N}_0^n$ has the product partial order.)
3. Given a finitely generated, torsion-free R-module M, let $\overline{R}M$ be the $\overline{R}$-submodule of KM generated by M. Assume $R \neq \overline{R}$. Then $M_1 \cong M_2 \iff$ the $(R/\mathfrak{f} \hookrightarrow \overline{R}/\mathfrak{f})$-modules $(M_1/\mathfrak{f}M_1 \hookrightarrow \overline{R}M_1/\mathfrak{f}M_1)$ and $(M_2/\mathfrak{f}M_2 \hookrightarrow \overline{R}M_2/\mathfrak{f}M_2)$ are isomorphic. (The fact that R is local is crucial here.)

The proof of Theorem 1.1 then reduces to the following:

Proposition 1.3. *Let* $\mathbf{A} = (A \hookrightarrow B)$ *be an Artinian pair in which* A *is local, with maximal ideal* $\mathfrak{m}$ *and residue field* k*. Assume that* B *is a principal ideal ring. Then* $\mathbf{A}$ *has finite representation type if and only if the following conditions are satisfied:*
(dr1) $\dim_k(B/\mathfrak{m}B) \leq 3$
(dr2) $\dim_k \dfrac{\mathfrak{m}B + A}{\mathfrak{m}^2B + A} \leq 1.$ □

Green and Reiner proved Proposition 1.3 under the additional assumption that the residue fields of B are all equal to k. There is an obvious way to eliminate residue field growth, assuming one is trying to prove the more difficult implication that (dr1) and (dr2) imply finite representation type: Adjoin roots. More precisely, we observe that by (dr1) B has at most three local components, and at most one of these has a residue field properly extending k. Moreover, the degree of the extension is at most 3. Choose a primitive element θ, let $f \in A[T]$ be a monic polynomial reducing to the minimal polynomial for θ, and pass to the Artinian pair $\mathbf{A}' := (A' \hookrightarrow B')$, where $A' = A[T]/(f)$ and $B' = B \otimes_A A' = B[T]/(f)$. The problem is that if θ is inseparable then B' may not be a principal ideal ring, and all bets are off. If, however, θ is separable, all is well: The Drozd-Roĭter conditions ascend to $\mathbf{A}$, and finite representation type descends. This is not difficult, and the details are worked out in [43]. (If $k(\theta)/k$ is a non-Galois extension of degree 3, one has to repeat the construction one more time.) This proves sufficiency of the Drozd-Roĭter conditions, except when k is imperfect of characteristic 2 or 3.

We now sketch the proof of the "if" implication in Theorem 1.1 in the case of residue field growth of degree 3. By (DR1), $\overline{R}$ must be local, say with maximal ideal $\mathfrak{n}$ (necessarily equal to $\mathfrak{m}\overline{R}$) and residue field ℓ. If R is seminormal, then $\overline{R}/\mathfrak{f}$ is reduced, and therefore equal to ℓ. The ring B' described above is now a homomorphic image of $\ell[T]$ and therefore is a principal ideal ring (even if ℓ/k is not separable). The work of Green and Reiner [16] now shows that $\mathbf{A}'$ has finite representation type, and the descent argument of [43] proves that R has finite Cohen-Macaulay type.

Suppose now that R is *not* seminormal. Still assuming (DR1) and (DR1), and that $[\ell : k] = 3$, one can show [45, Lemma 4] that R is Gorenstein, with exactly one overring S (the seminormalization of R) strictly between R and $\overline{R}$. (The argument amounts to a careful computation of lengths, and both (DR1) and (DR2) are used.) Now we use an argument that goes back to Bass's "ubiquity" paper [3, (7.2)]: Given a maximal Cohen-Macaulay R-module M, suppose M has no free direct summand. Then $M^* = \mathrm{Hom}_R(M, \mathfrak{m})$, which is a module over $E := \mathrm{End_R}(\mathfrak{m})$. Clearly E contains R properly and therefore must contain S. Thus M^* is an S-module, and hence so is M^{**}, which is isomorphic to M (as R is Gorenstein and M is maximal Cohen-Macaulay). Thus every non-free indecomposable maximal Cohen-Macaulay R-module is actually an S-module. The Drozd-Roĭter conditions clearly pass to the seminormal ring S, which therefore has finite Cohen-Macaulay type. It follows that R itself has finite Cohen-Macaulay type.

The remaining case, when $\overline{R}$ has a residue field that is purely inseparable of degree two over k, was handled via difficult matrix reductions in Cimen's *tour de force* [6].

Next, we will prove *necessity* of the conditions (DR). This was proved in [43], but we will prove a stronger result here, giving a positive answer to the analog, in the present context, of the second Brauer-Thrall conjecture. Recall that a module M over a one-dimensional reduced Noetherian ring R has *constant rank n*, provided $M_P \cong R_P^{(n)}$ for each minimal prime ideal P.

Theorem 1.4. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional, reduced, local Noetherian ring with finite integral closure. Assume that either (DR1) or (DR2) fails. Let n be an arbitrary positive integer.*

(1) *There exists an indecomposable maximal Cohen-Macaulay R-module of constant rank n.*
(2) *If the residue field k is infinite, there exist $|k|$ pairwise non-isomorphic indecomposable maximal Cohen-Macaulay modules of constant rank n.*

We will prove (2) of Theorem 1.4. The additional arguments needed to prove (1) when k is finite are rather easy and are given in detail in [43]. Shifting the problem down to the bottom line of the pullback, we let $\mathbf{A} = (A \hookrightarrow B)$, where $A = R/\mathfrak{f}$ and $B = \overline{R}/\mathfrak{f}$. We keep the notation of Proposition 1.3, so that now $\mathfrak{m}$ is the maximal ideal of A. We assume that either (dr1) or (dr2) fails, and we want to build non-isomorphic indecomposable $\mathbf{A}$-modules $V \hookrightarrow W$, with $W = B^{(n)}$. Given any such $\mathbf{A}$-module, the module M defined by the pullback diagram

$$\begin{array}{ccc} M & \longrightarrow & \overline{R}^{(n)} \\ \downarrow & & \downarrow \\ V & \longrightarrow & W \end{array}$$

will be an indecomposable maximal Cohen-Macaulay R-module, and non-isomorphic $\mathbf{A}$-modules will yield non-isomorphic R-modules.

We first deal with the annoying case where (dr1) holds but (dr2) fails. (The reader might find it helpful to play along with the example $k[[t^3, t^7]]$.) Thus we assume, for the moment, that

$$\dim_k(B/\mathfrak{m}B) \leq 3 \tag{1}$$

$$\dim_k \frac{\mathfrak{m}B + A}{\mathfrak{m}^2B + A} \geq 2 \tag{2}$$

We claim that we in (2). To see this, we note that $\mathfrak{m}^2B \cap A$ is *properly* contained in $\mathfrak{m}$ (lest $\mathfrak{m}B \subseteq \mathfrak{m}^2B$). Computing lengths, we have

$$\ell_A \frac{\mathfrak{m}^2B + A}{\mathfrak{m}^2B} = \ell_A \frac{A}{\mathfrak{m}^2B \cap A} \geq 2. \tag{3}$$

Also, since $\mathfrak{m}B$ is a faithful ideal of the principal ideal ring B, we see that $\mathfrak{m}B/\mathfrak{m}^2B \cong B/\mathfrak{m}B$. Now (1) implies that

$$\ell_A(\mathfrak{m}B/\mathfrak{m}^2B) \leq 3. \tag{4}$$

Finally, we have $\ell_A \frac{A+\mathfrak{m}B}{\mathfrak{m}B} = \ell_A \frac{A}{A\cap(\mathfrak{m}B)} = 1$, and the claim now follows from (3) and (4).

Now put $C := A + \mathfrak{m}B$, and note that $C/\mathfrak{m}C \cong k[X,Y]/(X^2, XY, Y^2)$. The functor $(V,W) \mapsto (V, BV)$ from $(A \hookrightarrow C)$-mod to $\mathbf{A}$-mod is clearly faithful; and it is full, by the requirement that $CV = W$. Therefore this functor is injective on isomorphism classes, and it preserves indecomposability. Therefore we may replace B by C in this case (the only casualty being that B is now no longer a principal ideal ring).

Returning to the general case, where *either* (dr1) or (dr2) fails, we put $D := B/\mathfrak{m}B$ when (dr1) fails, and $D = C/\mathfrak{m}C$ otherwise. We now have either

$$D \text{ is a prinicpal ideal ring and } \dim_k D \geq 4, \text{ or} \tag{5}$$

$$D \cong k[X,Y]/(X^2, XY, Y^2). \tag{6}$$

We now pass to the Artinian pair $(k \hookrightarrow D)$. The functor $(V,W) \mapsto (\frac{V+\mathfrak{m}W}{\mathfrak{m}W}, \frac{W}{\mathfrak{m}W})$, from $(A \hookrightarrow B)$-mod to $(k \hookrightarrow D)$-mod is surjective on isomorphism classes and reflects indecomposables. Therefore it suffices to build our modules over the Artinian pair $(k \hookrightarrow D)$.

We now describe a general construction, a modification of constructions found in [12, 43, 7]. Let n be a fixed positive integer, and suppose we have chosen $a, b \in D$ with $\{1, a, b\}$ linearly independent over k. Let I be the $n \times n$ identity matrix, and let H the $n \times n$ nilpotent Jordan block with 1's on the superdiagonal and 0's elsewhere. For $t \in k$, we consider the $n \times 2n$ matrix $\Psi_t := \big[I \mid aI + b(tI + H)\big]$. Put $W := D^{(n)}$, viewed as columns, and let V_t be the k-subspace of W spanned by the columns of Ψ_t.

Suppose, now, that we have a morphism $(V_t, W) \to (V_u, W)$, given by an $n \times n$ matrix φ over D. The requirement that $\varphi(V) \subseteq V$ says there is a $2n \times 2n$ matrix θ *over* k such that

$$\varphi\Psi_t = \Psi_u\theta. \tag{7}$$

Write $\theta = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, where $\alpha, \beta, \gamma, \delta$ are $n \times n$ blocks. Then (7) gives the following two equations:

$$\begin{aligned} \varphi &= \alpha + a\gamma + b(uI + H)\gamma \\ a\varphi + b\varphi(tI + H) &= \beta + a\delta + b(uI + H)\delta \end{aligned} \tag{8}$$

Substituting the first equation into the second, and combining terms, we get a mess:

$$\begin{aligned} &-\beta + a(\alpha - \delta) + b(t\alpha - u\delta + \alpha H - H\delta) \\ &+(a + tb)(a + ub)\gamma + (ab + b^2)(H\gamma + \gamma H) = 0. \end{aligned} \tag{9}$$

In the "annoying" case (6), we set a and b equal to the images of X and Y, respectively. Then

$$a^2 = b^2 = ab = 0 \tag{10}$$

and from (9) and the linear independence of $\{1, a, b\}$, we get the equations

$$\beta = 0, \qquad \alpha = \delta, \qquad \alpha((t - u)I + H) = H\alpha. \tag{11}$$

If, now, φ is an isomorphism, we see from (8) that α has to be invertible. If, in addition, $t \neq u$, the third equation in (11) gives a contradiction, since the left side is invertible and the right side is not. Thus $(V_t, W) \ncong (V_u, W)$ if $t \neq u$. To see that (V_t, W) is indecomposable, we take $u = t$ and suppose φ, as above, is idempotent. Squaring the first equation in (8), and comparing "1" and "a" terms, we see that $\alpha^2 = \alpha$ and $\gamma = \alpha\gamma + \gamma\alpha$. But equation (11) says that $\alpha H = H\alpha$, and it follows that α is in $k[H]$, which is a local ring. Therefore $\alpha = 0$ or 1, and either possibility forces $\gamma = 0$. Thus $\varphi = 0$ or 1, as desired.

Having dealt with the annoying case, we assume from now on that that $\dim_k D \geq 4$ and that D is a principal ideal ring. Assume, for the moment, that there exists an element $a \in D$ such that $\{1, a, a^2\}$ is linearly independent. Choose any element $b \in D$ such that $\{1, a, a^2, b\}$ is linearly independent.

Then, for almost all $t \in k$, the set $\{1, a, b, (a + tb)^2\}$ is linearly independent. (The set of such t is open, and it is non-empty since it contains 0.) Moreover, for almost all $t \in k$, the set $\{1, a, b, (a + tb)(a + ub)\}$ is linearly independent for almost all $u \in k$. Thus, it will suffice to show that if $t \neq u$, and if $\{1, a, b, (a + tb)^2\}$ and $\{1, a, b, (a + tb)(a + ub)\}$ are linearly independent over k, then $(V_t \hookrightarrow W)$ is indecomposable and not isomorphic to $(V_u \hookrightarrow W)$.

Suppose, as before, that $\varphi : (V_t \hookrightarrow W) \to (V_u \hookrightarrow W)$ is a homomorphism. With the same notation as in (7)–(9), we claim that $\gamma = 0$. To do this, we use descending induction on i and j to show that $H^i\gamma H^j = 0$ for all $i, j = 0, \ldots, n$. If either $i = n$ or $j = n$ this is clear. Assuming $H^{i+1}\gamma H^j = 0$ and $H^i\gamma H^{j+1} = 0$, we multiply the mess (9) by H^i on the left and H^j on the right. In the resulting equation, the "$ab + b^2$" term is 0 by the inductive hypothesis. Since $\{1, a, b, (a + tb)(a + ub)\}$ is linearly independent, the "coefficient" $H^i\gamma H^j$ of $(a+tb)(a+ub)$ must be 0. This completes the induction and proves the claim. The rest of the proof that $(V_t \hookrightarrow W)$ is indecomposable and not isomorphic to $(V_u \hookrightarrow W)$ is the same as in the annoying case.

The special case, where $\{1, a, a^2\}$ is linearly dependent for every element $a \in A$, is analyzed in detail in [43]. This case reduces to the following three cases:

- Case 1: There are elements $a, b \in D$ such that $\{1, a, b\}$ is linearly independent over k and $a^2 = ab = b^2 = 0$.

- Case 2: There are elements $a, b \in D$ such that $\{1, a, b, ab\}$ is linearly independent and $a^2 = b^2 = 0$.

- Case 3: The characteristic of k is 2, and there are elements $a, b \in D$ such that $\{1, a, b, ab\}$ is linearly independent and both a^2 and b^2 are in k.

We have already dealt with Case 1. In Case 2, the mess (9) again yields equations (11), and we proceed exactly as before. In Case 3, the mess yields the equations

$$\begin{aligned} \beta = (a^2 + tub^2)\gamma + b^2(H\gamma + \gamma H), \qquad & \alpha = \delta, \\ \alpha((t - u)I + H) = H\alpha, \qquad & (t + u)\gamma + H\gamma + \gamma H = 0. \end{aligned} \tag{12}$$

Suppose $t \neq u$. Then $t+u \neq 0$ (characteristic two), and the fourth equation shows, via the same descending induction argument as before, that $\gamma = 0$. Then the third equation and a now-familiar argument show that $(V_t \hookrightarrow W) \ncong (V_u \hookrightarrow W)$.

Finally, we must show that $(V_t \hookrightarrow W)$ is indecomposable in Case 3. Suppose $t = u$ and $\varphi^2 = \varphi$. The third and fourth equations of (12) now show that α and γ are in $k[H]$. In particular, $\alpha\gamma = \gamma\alpha$. Therefore, when we square the first equation of (8) and compare "a" terms, we see that $\gamma = 2\alpha\gamma = 0$. Now $\varphi = \alpha \in k[H]$, a local ring, and it follows that $\varphi = 0$ or 1. This completes the proof of Theorem 1.4. □

One might expect that even if k is finite one could construct a countably infinite family of pairwise non-isomorphic maximal Cohen-Macaulay modules of constant rank n. In fact, this is not the case:

Proposition 1.5. *With $(R, \mathfrak{m}, k)$ as in Theorem 1.1, suppose k is a finite field. Let n be a positive integer. Then R has only finitely many isomorphism classes of maximal Cohen-Macaulay modules of constant rank n.*

Proof. Let $\mathbf{A} = (R/\mathfrak{f} \hookrightarrow \overline{R}/\mathfrak{f})$ be the Artinian pair associated with R. Recall [43, (1.7)] that two maximal Cohen-Macaulay R-modules M_1 and M_2 are isomorphic if and only if their associated $\mathbf{A}$-modules $(M_i/\mathfrak{f}M_i \hookrightarrow \overline{R}M_i/\mathfrak{f}M_i)$ are isomorphic. Therefore it is enough to show that there are only finitely many $\mathbf{A}$-modules $(V \hookrightarrow W)$ with $W = (\overline{R}/\mathfrak{f})^{(n)}$. But this is clear because $|W| < \infty$. □

1.1 Finiteness of the integral closure

Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension one, let K be the total quotient ring $\{\text{non-zerodivisors}\}^{-1}R$, and let $\overline{R}$ be the integral closure of R in K. Suppose $\overline{R}$ is not finitely generated over R. Then we can build an infinite ascending chain of finitely generated R-subalgebras of $\overline{R}$. Each algebra in the chain is a maximal Cohen-Macaulay R-module, and it is easy to see [45, Lemma 1] that no two of the algebras are isomorphic as R-modules. It follows [45, Proposition 1] that $\overline{R}$ is finitely generated as an R-module if R has finite Cohen-Macaulay type. If, now, R is Cohen-Macaulay and x is a non-zero nilpotent element, we claim that $\overline{R}$ is not finitely generated over R. To see this, choose a non-zerodivisor $t \in \mathfrak{m}$, and note that $R\frac{x}{t} \subset R\frac{x}{t^2} \subset \frac{x}{t^3} \subset \dots$ is an infinite ascending chain of R-submodules of $\overline{R}$. We have proved:

Proposition 1.6. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional, Cohen-Macaulay local ring with finite Cohen-Macaulay type. Then R is reduced, and the integral closure $\overline{R}$ is finitely generated as an R-module.* □

What if R is *not* Cohen-Macaulay? The following result, together with Theorem 1.1, gives the full classification of one-dimensional local rings of finite Cohen-Macaulay type:

Theorem 1.7 ([45, Theorem 1]). *Let $(R, \mathfrak{m})$ be a one-dimensional local ring, and let N be the nilradical of R. Then R has finite Cohen-Macaulay type if and only if*

(1) *R/N has finite Cohen-Macaulay type, and*
(2) *$\mathfrak{m}^i \cap N = (0)$ for $i >> 0$.* □

For example, $k[[X,Y]]/(X^2,XY)$ has finite Cohen-Macaulay type, since (x) is the nilradical and $(x,y)^2\cap(x)=(0)$. However $k[[X,Y]]/(X^3,X^2Y)$ has infinite Cohen-Macaulay type: For each $i\geq 1$, xy^{i-1} is a non-zero element of $(x,y)^i\cap(x)$.

Corollary 1.8 ([45, Corollary 2]). *Let $(R,\mathfrak{m})$ be a one-dimensional local ring. Then R has finite Cohen-Macaulay type if and only if the $\mathfrak{m}$-adic completion $\widehat{R}$ has finite Cohen-Macaulay type.* □

The analogous statement can fail in higher dimensions (cf. Examples 2.1 and 2.2 of [33]).

1.2 Rings containing the rational numbers

For local rings containing $\mathbb{Q}$, the rings of finite Cohen-Macaulay type have a particularly nice classification. First we recall the 1985 classification, by Greuel and Knörrer, of complete equicharacteristic-zero singularities of finite Cohen-Macaulay type. Recall that the *simple* (or "ADE") plane singularities are the following rings corresponding to certain Dynkin diagrams:

(A_n)	$k[[X,Y]]/(X^2+Y^{n+1})$	$(n\geq 1)$
(D_n)	$k[[X,Y]]/Y(X^2+Y^{n-2})$	$(n\geq 4)$
(E_6)	$k[[X,Y]]/(X^3+Y^4)$	
(E_7)	$k[[X,Y]]/X(X^2+Y^3)$	
(E_8)	$k[[X,Y]]/(X^3+Y^5)$	

Theorem 1.9 ([17]). *Let $(R,\mathfrak{m},k)$ be a one-dimensional complete local Cohen-Macaulay ring containing $\mathbb{Q}$, and assume that k is algebraically closed. Then R has finite Cohen-Macaulay type if and only if R birationally dominates a simple plane curve singularity.* □

In particular, the space curves $k[[T^3,T^4,T^5]]$ and $k[[T^3,T^5,T^7]]$ have finite Cohen-Macaulay type. To handle the case of a residue field that is not algebraically closed, we quote the following theorem (which works in all dimensions):

Theorem 1.10 ([46, Theorem 3.3]). *Let k be a field with separable closure k^s, and let f be a non-unit in the formal power series ring $k[[X_0,\dots,X_d]]$. Then $k[[X_0,\dots,X_d]]/(f)$ has finite CM type if and only if $k^s[[X_0,\dots,X_d]]/(f)$ has finite CM type.* □

As one might expect, inseparable extensions can cause difficulties:

Example 1.11. Let k be an imperfect field of characteristic 2, choose $\alpha\in k-k^2$ and put $K:=k(\alpha)$. Let $f=X^2+\alpha Y^2$, and put $R:=k[[X,Y]]/(f)$. Then

R is a one-dimensional complete local domain, and the integral closure $\overline{R}$ is generated, as an R-module, by the two elements 1 and $\frac{x}{y}$; and both x and y multiply $\frac{x}{y}$ into R. By Theorem 1.1, R has finite Cohen-Macaulay type. On the other hand, Proposition 1.6 implies that $K[[x,y]]/(f)$ does *not* have finite Cohen-Macaulay type, since it is Cohen-Macaulay and has non-zero nilpotents.

2 Bounded Cohen-Macaulay type

In this section, we consider one-dimensional local Cohen-Macaulay local rings $(R, \mathfrak{m}, k)$. We will say that R has *bounded* Cohen-Macaulay type provided there is a bound on the multiplicities of the indecomposable maximal Cohen-Macaulay R-modules. Since the notion of rank is perhaps more intuitive, we mention that if M is an R module of constant rank r, then the multiplicity $\mathrm{e}(M)$ of M satisfies

$$\mathrm{e}(M) = r \cdot \mathrm{e}(R).$$

If R is reduced, then Theorems 1.1 and 1.4 imply that finite and bounded Cohen-Macaulay types agree. In 1980 Dieterich [10] observed that the group ring $k[[X]][G]$ has bounded Cohen-Macaulay type if $|G| = 2$ and $\mathrm{char(k)} = 2$. Of course this ring is just $k[[X,Y]]/(Y^2)$. In 1987 Buchweitz, Greuel and Schreyer [5] classified the indecomposable maximal Cohen-Macaulay modules over $k[[X,Y]]/(Y^2)$ and $k[[X,Y]]/(XY^2)$, the (A_∞) and (D_∞) singularities, for *every* field k. A consequence of their classification is that these singularities have bounded Cohen-Macaulay type. Of course, by Proposition 1.6, these rings do not have finite Cohen-Macaulay type. Rather surprisingly, there is, in the complete equicharacteristic case, only one additional ring with bounded but infinite Cohen-Macaulay type:

Theorem 2.1 ([34, Theorem 2.4]). *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local Cohen-Macaulay ring, and assume that R contains a field. Then R has bounded but infinite Cohen-Macaulay type if and only if the $\mathfrak{m}$-adic completion $\widehat{R}$ is k-isomorphic to one of the following:*

(1) $A := k[[X,Y]]/(Y^2)$
(2) $B := k[[X,Y]]/(XY^2)$
(3) $C := k[[XY, YZ, Z^2]]$, *the endomorphism ring of the maximal ideal of B*

If, on the other hand, R has unbounded Cohen-Macaulay type, then R has, for each positive integer r, an indecomposable maximal Cohen-Macaulay module of constant rank r. □

The proof of the "only if" direction of this theorem involves some rather technical ideal theory. We don't know whether or not the theorem is correct without the assumption that k be infinite.

For the rings A and B of Theorem 2.1, we see from the explicit presentations in [5] that the indecomposable maximal Cohen-Macaulay modules are generated by at most two elements. This gives us a bound of six on the multiplicities of these modules. Since $C = \mathrm{End}_B(\mathfrak{m}_B)$, where $\mathfrak{m}_B$ is the maximal ideal of B, we see that C is a module-finite extension of B. Therefore every maximal Cohen-Macaulay C-module M is also maximal Cohen-Macaulay when viewed as a B-module. Moreover, since C is contained in the total quotient ring of B and M is torsion-free, we see that $\mathrm{End}_B(M) = \mathrm{End}_C(M)$. In particular, if M is indecomposable as a C-module, it is also indecomposable as a B-module. Thus the multiplicities of the indecomposable maximal Cohen-Macaulay B-modules are also bounded by six. The "if" direction of Theorem 2.1 now follows from the next theorem, on ascent to and descent from the completion.

Theorem 2.2 ([34, Theorem 2.3]). *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay local ring with completion $\widehat{R}$. Assume R contains a field and that k is infinite. Then R has bounded Cohen-Macaulay type if and only if $\widehat{R}$ has bounded Cohen-Macaulay type. Moreover, if R has unbounded Cohen-Macaulay type, then R has, for every positive integer r, an indecomposable maximal Cohen-Macaulay module of constant rank r.* □

By Lech's Theorem [32, Theorem 1] each of the rings in Theorem 2.1 is the completion of an integral domain. Suppose, for example, that $(R, \mathfrak{m}, k)$ is a one-dimensional local domain whose completion is $k[[X, Y]]/(Y^2)$. Then R has bounded but infinite Cohen-Macaulay type. Therefore the assumption, in Theorem 1.4, that $\overline{R}$ be finitely generated over R, cannot be removed.

3 Modules with Torsion

In this section we consider arbitrary finitely generated modules over local rings of dimension one. Every such ring $(R, \mathfrak{m})$ obviously has an infinite family of pairwise non-isomorphic inedecomposable modules, namely, the modules $R/\mathfrak{m}^n$. With a little more work, one can produce indecomposable modules requiring arbitrarily many generators, as long as R is not a principal ideal domain. To see this, fix $n \geq 1$, let x and y be elements of $\mathfrak{m}$ that are linearly independent modulo $\mathfrak{m}^2$, and let I and H be the $n \times n$ identity and nilpotent matrices used in the proof of Theorem 1.4. Then the cokernel of the matrix $xI + yH$ is indecomposable, and it clearly needs n generators. To prove indecomposability, one can pass to $R/\mathfrak{m}^2$ and use an argument similar to, but much easier than, the one used in the proof of Theorem 1.4. See, for example, [21, Proposition 4.1] or [39]. Similar constructions can be found in the work of Kronecker [28] and Weierstrass [40] on classifying pairs of matrices up to simultaneous equivalence. The idea is not exactly new!

It is much more difficult to build indecomposable modules of large multiplicity. Of course it is impossible to do so if R is a principal ideal ring. More generally, recall from [29, 30, 31] that a local ring $(R, \mathfrak{m}, k)$ is *Dedekind-like* provided R is reduced and one-dimensional, the integral closure $\overline{R}$ is generated by at most two elements as an R-module, and $\mathfrak{m}$ is the Jacobson radical of $\overline{R}$. In a long and difficult paper [30] Levy and Klinger classify the indecomposable finitely generated modules over most Dedekind-like rings. There is one exceptional case where the classification has not yet been worked out, namely, where $\overline{R}$ is a local domain whose residue field is purely inseparable of degree two over k. We will call these Dedekind-like rings *exceptional*. The ring of Example 1.11 is such an exception. Before stating the next result, which is a consequence of the classification in [30], we note that a Dedekind-like ring has at most two minimal prime ideals and that the localization of R at a minimal prime is a field. If R has two minimal primes P_1 and P_2, the *rank* of the R-module M is the pair (r_1, r_2), where r_i is the dimension of M_{P_i} as a vector space over R_{P_i}.

Theorem 3.1 ([30]). *Let M be an indecomposable finitely generated module over a local Dedekind-like ring R.*

(1) *If R has two minimal prime ideals P_1 and P_2, then the rank of M is $(0,0), (1,0), (0,1)$ or $(1,1)$.*
(2) *If R is a domain and R is not exceptional, then M has rank $0, 1$ or 2.* □

In a series of papers [22, 20, 21], Hassler, Klingler and the present authors proved a strong converse to this theorem:

Theorem 3.2 ([21, Theorem 1.2]). *Let R be a local ring of dimension at least one, and assume R is not a homomorphic image of a Dedekind-like ring. Let $P_1, \dots, P_s$ be an arbitrary set of pairwise incomparable non-maximal prime ideals, and let $n_1, \dots, n_s$ be non-negative integers. Then there are $|k|\aleph_0$ pairwise non-isomorphic indecomposable R-modules M_α such that $(M_\alpha)_{P_i} \cong R_{P_i}^{(n_i)}$ for each $i \le s$ and each α.* □

The proof [21] of this result is rather involved. It makes heavy use of the fact [29] that the category of finite-length R-modules has wild representation type if R is not a homomorphic image of a Dedekind-like ring.

4 Monoids of Modules

In this section we study the different ways in which a finitely generated module can be decomposed as a direct sum of indecomposable modules. Let $(R, \mathfrak{m}, k)$ be a local ring and $\mathcal{C}$ a class of modules closed under isomorphism, finite direct sums, and direct summands. We always assume that $\mathcal{C} \subseteq R{-}\mathrm{mod}$, the class of all finitely generated R-modules. There is a set $\mathrm{V}(\mathcal{C}) \subseteq \mathcal{C}$ of representatives; each element $M \in \mathcal{C}$ is isomorphic to exactly one element $[M] \in \mathrm{V}(\mathcal{C})$. We make $\mathrm{V}(\mathcal{C})$ into an additive monoid in the obvious way: $[M] + [N] = [M \oplus N]$. This monoid encodes information about direct-sum decompositions in $\mathcal{C}$. We will tacitly assume that all of our monoids are written additively, and that they are *reduced* ($x + y = 0 \implies x = y = 0$).

Suppose R is complete (in the $\mathfrak{m}$-adic topology). Then the Krull-Remak-Schmidt theorem holds for finitely generated modules, that is, each $M \in R$-mod is uniquely a direct sum of indecomposable modules (up to isomorphism and ordering of the summands). In the language of monoids, $\mathrm{V}(R\text{−mod}) \cong \mathbb{N}_0^{(\mathrm{I})}$, the free monoid with basis $\{b_i \mid i \in I\}$, where the b_i range over a set of representatives for the indecomposable finitely generated R-modules.

For a general local ring R, we can exploit the monoid homomorphism

$$j : \mathrm{V}(R\text{−mod}) \to \mathrm{V}(\widehat{\mathrm{R}}\text{−mod})$$

taking $[M]$ to $[\widehat{R} \otimes_R M]$. This homomorphism is injective [11, (2.5.8)], and it follows that the monoid R−mod is *cancellative*: $x + z = y + z \implies x = y$. (Cf. [14], [38].) Since, in this section, we will deal only with local rings, all of our monoids are tacitly assumed to be cancellative.

The homomorphism j actually satisfies a much stronger condition. If $x, y \in \mathrm{V}(R\text{−mod})$ and $j(x) \mid j(y)$, then $x \mid y$. (For elements x and y in a monoid Λ we say x *divides* y, written "$x \mid y$" provided there is an element $\lambda \in \Lambda$ such that $x + \lambda = y$.) Here is a proof, given by Reiner and Roggenkamp [36] in a slightly different context: Suppose M' and M are finitely generated modules over a local ring R, and that $\widehat{R} \otimes_R M' \mid \widehat{R} \otimes_R M$. We identify $\widehat{R} \otimes_R M'$ and $\widehat{R} \otimes_R M$ with the completions $\widehat{M'}$ and $\widehat{M}$ of M' and M. Choose $\widehat{R}$-homomorphisms $\varphi : \widehat{M'} \to \widehat{M}$ and $\psi : \widehat{M} \to \widehat{M'}$ such that $\psi\varphi = 1_{\widehat{M'}}$. Since $H := \mathrm{Hom}_R(M', M)$ is a finitely generated R-module, it follows that $\widehat{H} = \widehat{R} \otimes_R H = \mathrm{Hom}_{\widehat{R}}(\widehat{M}, \widehat{N})$. Therefore φ can be approximated to any order by an element of H. In fact, order 1 suffices: Choose $f \in \mathrm{Hom}_R(M, N)$ such that $\widehat{f} - \varphi \in \widehat{\mathfrak{m}}\widehat{H}$. Similarly, we can choose $g \in \mathrm{Hom}_R(N, M)$ with $\widehat{g} - \psi \in \widehat{\mathfrak{m}}\mathrm{Hom}_{\widehat{R}}(\widehat{N}, \widehat{M})$. Then the image of $\widehat{g}\widehat{f} - 1_{\widehat{M}}$ is in $\widehat{\mathfrak{m}}\widehat{M}$, and now Nakayama's lemma implies that $\widehat{g}\widehat{f}$ is surjective, and therefore an isomorphism. It follows that $\widehat{g}$ is a split surjection (with splitting map $\widehat{f}(\widehat{g}\widehat{f})^{-1}$). By faithful flatness g is a split surjection.

A *divisor homomorphism* $j : \Lambda_1 \to \Lambda_2$ (between reduced, cancellative monoids) is a homomorphism such that, for all $x, y \in \Lambda_1$, $j(x) \mid j(y) \implies x \mid y$. The result we just proved is a special case of the following theorem:

Theorem 4.1 ([24, Theorem 1.3]). *Let $R \to S$ be a flat local homomorphism of Noetherian local rings. Then the map $\mathrm{V}(R\text{−mod}) \to \mathrm{V}(\mathrm{S}\text{−mod})$ taking $[M]$ to $[S \otimes_R M]$ is a divisor homomorphism.* □

Definition 4.2. A *Krull monoid* is a monoid that admits a divisor homomorphism into a free monoid.

Every finitely generated Krull monoid admits a divisor homomorphism into $\mathbb{N}_0^{(t)}$ for some positive integer t. Conversely, it follows easily from Dickson's Lemma (Item 2 following Proposition 1.2) that a monoid admitting a divisor homomorphism to $\mathbb{N}_0^{(t)}$ must be finitely generated.

Finitely generated Krull monoids are called *positive normal affine semigroups* in [4]. From [4, Exercise 6.1.10, p. 252], we obtain the following characterization of these monoids:

Proposition 4.3. *The following conditions on a monoid Λ are equivalent:*

(1) *Λ is a finitely generated Krull monoid.*
(2) *$\Lambda \cong G \cap \mathbb{N}^{(t)}$ for some positive integer t and some subgroup G of $\mathbb{Z}^{(t)}$.*
(3) *$\Lambda \cong W \cap \mathbb{N}^{(u)}$ for some positive integer u and some $\mathbb{Q}$-subspace W of $\mathbb{Q}^{(n)}$.*
(4) *There exist positive integers m, n and an $m \times n$ matrix α over $\mathbb{Z}$ such that $\Lambda \cong \mathbb{N}^{(n)} \cap \ker(\alpha)$.* □

Item (4) says that a finitely generated Krull monoid can be regarded as the collection of non-negative integer solutions of a homogeneous system of linear equations. For this reason these monoids are sometimes called *Diophantine* monoids.

In order to study uniqueness of direct-sum decompositions, it's really enough to examine a small piece of the class R-mod of all finitely generated modules. Given a finitely generated module M, we let add(M) be the class of modules that are isomorphic to direct summands of direct sums of finitely many copies of M. We note that $+(M) := \mathrm{V}(\mathrm{add(M)})$ is a finitely generated Krull monoid, since the divisor homomorphism $j : \mathrm{V}(R\mathrm{-mod}) \to \mathrm{V}(\widehat{\mathrm{R}}\mathrm{-mod})$ carries $+(M)$ into the free monoid generated by the isomorphism classes of the indecomposable direct summands of $\widehat{M}$.

The key to understanding the monoids $\mathrm{V}(R\mathrm{-mod})$ and $+(M)$ is knowing which modules over the completion actually come from R-modules. More generally, if $R \to S$ is a ring homomorphism, we say that the S-module N is *extended* (from R) provided there is an R-module M such that $S \otimes_R M \cong N$. In dimension one, a beautiful result due to Levy and Odenthal [35] tells us exactly which $\widehat{R}$-modules are extended. First, we define, for any one-dimensional local ring $(R, \mathfrak{m}, k)$ the *Artinian localization* $\mathfrak{a}(R)$ as follows:

$$\mathfrak{a}(R) = (R - (P_1 \cup \cdots \cup P_s))^{-1} R,$$

where $P_1, \ldots, P_s$ are the minimal prime ideals of R (the prime ideals distinct from $\mathfrak{m}$). If R is Cohen-Macaulay, this is just the classical quotient ring. If R is *not* Cohen-Macaulay, the natural map $R \to \mathfrak{a}(R)$ is not one-to-one.

Theorem 4.4 ([35]). *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local ring, and let N be a finitely generated $\widehat{R}$-module. Then N is extended from R if and only if $\mathfrak{a}(\widehat{R}) \otimes_{\widehat{R}} N$ is extended from $\mathfrak{a}(R)$.* □

We refer the reader to [24, Theorem 4.1] for the proof of a somewhat more general result.

We return now to the situation of Section 1, where $(R, \mathfrak{m}, k)$ is a local ring whose completion $\widehat{R}$ is reduced. The localizations at the minimal primes are then fields. If $A := L_1 \times \cdots \times L_t$ is a K-algebra, where K and the L_j are fields, a finitely generated A-module N is extended from K if and only if $\dim_{L_i}(L_i N) = \dim_{L_j}(L_j N)$ for all i, j. Therefore Theorem 4.4 has the following consequence:

Corollary 4.5. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local ring whose completion $\widehat{R}$ is reduced, and let N be a finitely generated $\widehat{R}$-module. Then N is extended from R if and only if $\dim_{R_P}(N_P) = \dim_{R_Q}(N_Q)$ whenever P and Q are prime ideals of $\widehat{R}$ lying over the same prime ideal of R. In particular, if R is a domain, then N is extended if and only if N has constant rank.* □

This gives us a strategy for producing strange direct-sum behavior:

(1) Find a one-dimensional domain R whose completion has lots of minimal primes.
(2) Build indecomposable $\hat{R}$-modules with highly non-constant ranks.
(3) Put them together in different ways to get constant-rank modules.

Suppose, for example, that R is a domain whose completion $\widehat{R}$ has two minimal primes P and Q. Suppose we can build indecomposable $\hat{R}$-modules U, V, W and X, with ranks $(2,0), (0,2), (2,1)$ and $(1,2)$, respectively. Then $U \oplus V$ is extended, say, $U \oplus V \cong \widehat{M}$. Similarly, there are R-modules N, F and G such that $V \oplus W \oplus W \cong \widehat{N}, W \oplus X \cong \widehat{F}$ and $U \oplus X \oplus X \cong \widehat{G}$. Using the Krull-Remak-Schmidt theorem over $\widehat{R}$, we see easily that no nonzero proper direct summand of any of the modules $\widehat{M}, \widehat{N}, \widehat{F}, \widehat{G}$ has constant rank. It follows from Corollary 4.5 that M, N, F and G are indecomposable, and of course no two of them are isomorphic since (again by Krull-Remak-Schmidt) their completions are pairwise non-isomorphic. Finally, we see that $M \oplus F \oplus F \cong N \oplus G$, since the two modules have isomorphic completions. Thus we easily obtain a mild violation of Krull-Remak-Schmidt uniqueness over R.

It's easy to accomplish (1), getting a one-dimensional domain with a lot of splitting. In order to facilitate (2), however, we want to ensure that each analytic branch has infinite Cohen-Macaulay type. The following example from [47] does the job nicely:

Example 4.6 ([47, (2.3)]). Fix a positive integer s, and let k be any field with $|k| \geq s$. Choose distinct elements $t_1, \ldots, t_s \in k$. Let Σ be the complement of the union of the maximal ideals $(X - t_i)k[X],\ i = 1, \ldots, s$. We define $R = R_s$ by the pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & \Sigma^{-1}k[X] \\ \downarrow & & \downarrow \pi \\ k & \longrightarrow & \frac{\Sigma^{-1}k[X]}{(X-t_1)^4\cdot\dots\cdot(X-t_s)^4} \end{array}$$

where π is the natural map. Then R is a one-dimensional local domain, and $\widehat{R}$ is reduced with exactly s minimal prime ideals.

Let $P_1, \dots, P_s$ be the minimal prime ideals of $\widehat{R}$. By the *rank* of a finitely generated $\widehat{R}$-module N, we mean the s-tuple $(r_1, \dots, r_s)$, where r_i is the dimension of N_{P_i} as a vector space over R_{P_i}. A jazzed-up version of the argument used to prove Theorem 1.4 yields the following:

Theorem 4.7 ([47, (2.4)]). *Fix a positive integer s, and let $(r_1, \dots, r_s)$ be any non-trivial sequence of non-negative integers. Then $\widehat{R_s}$ has an indecomposable maximal Cohen-Macaulay module N with* $\operatorname{rank}(N) = (r_1, \dots, r_s)$. □

Thus even the case $s = 2$ of Example 4.6 yields the pathology discussed after Corollary 4.5

Recalling (4) of Proposition 4.3, we say that the finitely generated Krull monoid Λ *can be defined by m equations* provided $\Lambda = \mathbb{N}_0^{(n)} \cap \ker(\alpha)$ for some n and some $m \times n$ integer matrix α. Given such an embedding of Λ in $\mathbb{N}_0^{(n)}$, we say a column vector $\lambda \in \Lambda$ is *strictly positive* provided each of its entries is a positive integer. By decreasing n (and removing some columns from α) if necessary, we can harmlessly assume (without changing m) that Λ contains a strictly positive element (cf. [49, Remark 3.1]).

Corollary 4.8 ([47, Theorem 2.1]). *Fix a non-negative integer m, and let R be the ring R_{m+1} of Example 4.6. Let Λ be a finitely generated Krull monoid defined by m equations and containing a strictly positive element λ. Then there exist a maximal Cohen-Macaulay R-module M and a commutative diagram*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\subseteq} & \mathbb{N}_0^{(n)} \\ \varphi \downarrow \cong & & \psi \downarrow \cong \\ \mathrm{V}(+(M)) & \xrightarrow{i} & \mathrm{V}(+(\widehat{M})) \end{array} ,$$

in which

(1) *i is the natural map taking $[F]$ to $[\widehat{F}]$,*
(2) *φ and ψ are monoid isomorphisms, and*
(3) *$\varphi([M]) = \lambda$.*

Proof. We have $\Lambda = \mathbb{N}_0^{(n)} \cap \ker(\alpha)$, where $\alpha = [a_{ij}]$ is an $m \times n$ matrix over $\mathbb{Z}$. Choose a positive integer h such that $a_{ij} \geq 0$ for all i, j. For $j = 1, \dots, n$, choose, using Theorem 4.7, a maximal Cohen-Macaulay $\widehat{R}$-module L_j such that $\operatorname{rank}(L_j) = (a_{1j} + h, \dots, a_{mj} + h, h)$.

Given any column vector $\beta = [b_1\ b_2\ \dots b_n]^{\mathrm{tr}} \in \mathbb{N}_0^{(n)}$, put $N_\beta = L_1^{(b_1)} \oplus \cdots \oplus L_n^{(b_n)}$. The rank of N_β is

$$\left(\sum_{j=1}^{n}(a_{1j}+h)b_j, \dots, \sum_{j=1}^{n}(a_{mj}+h)b_j, (\sum_{j=1}^{n} b_j)h\right).$$

Since R is a domain, Corollary 4.5 implies that N_β is in the image of $j : \mathrm{V}(R\text{−mod}) \to \mathrm{V}(\widehat{\mathrm{R}}\text{−mod})$ if and only if $\sum_{j=1}^{n}(a_{ij}+h)b_j = (\sum_{j=1}^{n} b_j)h$ for each i, that is, if and only if $\beta \in \mathbb{N}_0^{(n)} \cap \ker(\alpha) = \Lambda$. To complete the proof, we let M be the R-module (unique up to isomorphism) such that $\widehat{M} \cong N_\lambda$. □

This corollary makes it very easy to demonstrate spectacular failure of Krull-Remak-Schmidt uniqueness:

Example 4.9. Let $\Lambda = \{\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in \mathbb{N}_0^{(3)} \mid 72x + y = 73z\}$. This has three atoms (minimal non-zero elements), namely

$$\alpha := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \beta := \begin{bmatrix} 0 \\ 73 \\ 1 \end{bmatrix}, \qquad \gamma := \begin{bmatrix} 73 \\ 0 \\ 72 \end{bmatrix}.$$

Note that $73\alpha = \beta + \gamma$, Taking $s = 2$ in Example 4.6, we get a local ring R and indecomposable R-modules M, F, G such that $M^{(t)}$ has only the obvious direct-sum decompositions for $t \leq 72$, but $M^{(73)} \cong F \oplus G$.

We define the *splitting number* $\mathrm{spl}(R)$ of a one-dimensional local ring R by

$$\mathrm{spl}(R) = |\mathrm{Spec}(\widehat{R})| - |\mathrm{Spec}(R)|.$$

The splitting number of the ring R_s in Example 4.6 is $s - 1$. Corollary 4.8 says that every finitely generated Krull monoid defined by m equations can be realized as $+(M)$ for some finitely generated module over a one-dimensional local ring (in fact, a domain essentially of finite type over $\mathbb{Q}$) with splitting number m. This is the best possible:

Theorem 4.10. *Let M be a finitely generated module over a one-dimensional local ring R with splitting number m. Then the Krull monoid $+(M)$ is defined by m equations.*

Proof. Write $\widehat{M} = V_1^{(e_1)} \oplus \cdots \oplus V_n^{(e_n)}$, where the V_j are pairwise non-isomorphic indecomposable $\widehat{R}$-modules and the e_i are all positive. We have an embedding $+(M) \hookrightarrow \mathbb{N}_0^{(n)}$ taking $[F]$ to $[b_1\ \dots b_n]^{\mathrm{tr}}$, where $\widehat{F} \cong V_1^{(b_1)} \oplus \cdots \oplus V_n^{(b_n)}$, and we identify $+(M)$ with its image Λ in $\mathbb{N}_0^{(n)}$. Given a prime $P \in \mathrm{Spec}(R)$ with, say, t primes $Q_1, \dots, Q_t$ lying over it, there are $t - 1$ homogeneous linear equations on the b_j that say that $\widehat{N}$ has constant rank

on the fiber over P (cf. Corollary 4.5). Letting P vary over $\mathrm{Spec}(R)$, we obtain exactly $m = \mathrm{spl}(R)$ equations that must be satified by elements of Λ. Conversely, if the b_j satisfy these equations, then $N :\cong V_1^{(b_1)} \oplus \cdots \oplus V_n^{(b_n)}$ has constant rank on each fiber of $\mathrm{Spec}(\widehat{R}) \to \mathrm{Spec}(R)$. By Corollary 4.5, N is extended from an R-module, say $N \cong \widehat{F}$. Clearly $\widehat{F} \mid \widehat{M^{(u)}}$ if u is large enough, and it follows from Theorem 4.1 that $F \in +(M)$, whence $[b_1 \ \ldots b_n]^{\mathrm{tr}} \in \Lambda$.
□

In [27] Karl Kattchee showed that, for each m, there is a finitely generated Krull monoid Λ that cannot be defined by m equations. Thus no single one-dimensional local ring can realize *every* finitely generated Krull monoid in the form $+(M)$ for a finitely generated module M.

We have seen that the monoids $+(M)$ have a very rich structure. In contrast, the monoids $\mathrm{V}(R\mathrm{-mod})$, for R a one-dimensional reduced local ring, are pretty boring. For certain Dedekind-like rings we will encounter the submonoid Γ of the free monoid $\mathbb{N}_0^{(\aleph_0)}$ consisting of (finitely non-zero) sequences $[a_i]$ satisfying $\sum_i (-1)^i a_i = 0$. For rings that are not Dedekind-like, we fix a positive integer q and let $v_1, v_2, v_3, \ldots$ be an enumeration of the elements of $\mathbb{Z}^{(q)}$. Let F be the free monoid with countably infinite basis $\{b_1, b_2, b_3, \ldots\}$, and define $f : F \to \mathbb{Z}^{(q)}$ by $b_i \mapsto v_i$. Now let τ be an infinite cardinal, and define $g : F^{(\tau)} \to \mathbb{Z}^{(q)}$ by taking the map f on each component. We let $\Lambda(q, \tau) = \ker(g)$. Finally, we let $\Lambda(0, \tau) = \mathbb{N}_0^{(\tau)}$, the free monoid with basis of cardinality τ.

The following theorem, from [15] and [21], is an easy consequence of Theorem 2.2 and Theorem 4.4:

Theorem 4.11. *Let $(R, \mathfrak{m}, k)$ be a reduced one-dimensional local ring, with splitting number $q = \mathrm{spl}(R)$. Put $\tau = |k|\aleph_0$.*

(1) *If R is not Dedekind-like, then* $\mathrm{V}(R\mathrm{-mod}) \cong \Lambda(q, \tau)$.
(2) *If R is a discrete valuation ring, then* $\mathrm{V}(R\mathrm{-mod}) \cong \Lambda(0, \aleph_0) = \mathbb{N}_0^{(\aleph_0)}$.
(3) *If R is Dedekind-like, R is not a discrete valuation ring, and $q = 0$, then* $\mathrm{V}(R\mathrm{-mod}) \cong \mathbb{N}_0^{(\aleph_0)}$.
(4) *If R is Dedekind-like and $q > 0$, then $q = 1$ and* $\mathrm{V}(R\mathrm{-mod}) \cong \Gamma \oplus \mathbb{N}_0^{(\tau)}$.

In every case, the divisor class group of $\mathrm{V}(R\mathrm{-mod})$ *is* $\mathbb{Z}^{(q)}$. □

The theorem raises two questions. First, what if R has non-zero nilpotents? The problem is that we do not have, in this case, a useful criterion for an $\widehat{R}$-module to be extended. Theorem 4.4 reduces the problem to the case of Artinian rings, but that does not eliminate the difficulty. The interested reader is referred to [24, Section 6] for a discussion of this problem.

Secondly, is there a similar classification of the monoid $\mathcal{C}(R)$ of isomorphism classes of maximal Cohen-Macaulay modules (say, when the completion is reduced)? If R has finite Cohen-Macaulay type, such a classification has been worked out by Nicholas Baeth and Melissa Luckas in [1] and [2]. At the other extreme, when *each* analytic branch has infinite Cohen-Macaulay type, Andrew Crabbe and Silvia Saccon [8] have a result similar to Theorem 4.7 above, from which one can decode the structure of $\mathcal{C}(R)$. The intermediate case, e.g., $R = k[[X, Y]]/X(X^3 - Y^4)$, where R has infinite Cohen-Macaulay type but at least one branch has finite Cohen-Macaulay type, is discussed in [8], but much less is known about the possible ranks of the indecomposables in this case.

5 Direct-sum cancellation

Let R be a commutative Noetherian ring. In very general terms, the direct-sum cancellation question is this: If M, N, and L are R-modules in some fixed subcategory $\mathcal{C} \subseteq R{-}\mathrm{mod}$, where $R{-}\mathrm{mod}$ is the category of all finitely generated R-modules, when does $M \oplus L \cong N \oplus L$ always imply $M \cong N$? When this is the case, we say that cancellation holds for R (with respect to the chosen category). Otherwise, we say cancellation fails for R.

Evans [14] and Vasconcelos [38] showed that cancellation of arbitrary finitely generated modules always holds over semilocal rings. Since the cancellation question is interesting only if the ring is not semilocal, we focus largely on *non*-semilocal rings in this section. However, the localizations of a ring R do play a role in answering some kinds of cancellation questions over R itself.

The cancellation question gained prominence in 1955 through its connection with the celebrated conjecture of Serre [37]: If R is the polynomial ring in a finite number of variables over a field, is every finitely generated projective R-module free? Serre reduced his question to a cancellation question involving projective modules: If P and Q are finitely generated projective R-modules such that $P \oplus R \cong Q \oplus R$, are P and Q necessarily isomorphic?

Well before the proof of Serre's Conjecture by Quillen and Suslin in 1976, the cancellation question had taken on a life of its own. The emphasis shifted to other categories of modules and other rings. In 1962, Chase [6] studied cancellation of finitely generated torsion-free modules over two-dimensional rings. He proved, for example, that torsion-free cancellation holds for the ring $R = k[X, Y]$ when k is an algebraically closed field of characteristic zero. He also produced non-isomorphic torsion-free modules A and B over $R = \mathbb{R}[X, Y]$ such that $A \oplus R \cong B \oplus R$.

The first known failure of cancellation for finitely generated modules is perhaps due to Kaplansky, who used the non-triviality of the tangent bundle on the two-sphere to produce a module T over $R = \mathbb{R}[X, Y]/(X^2+Y^2+Z^2-1)$ such that $T \oplus R \cong R^3$ and yet $T \ncong R^2$. For quite a while, every known failure of cancellation for finitely generated modules over commutative rings involved rings of dimension greater than one. Even as late as 1973, Eisenbud and Evans

[13] raised the following question: Does cancellation hold for arbitrary finitely generated modules over one-dimensional Noetherian rings?

In the 1980's, effective techniques, such as those in [48], were developed for studying the cancellation of finitely generated torsion-free modules over one-dimensional rings. We will sketch some of the main ideas. We assume, from now on, that all modules are finitely generated.

Borrowing from the notation we used previously for local rings, we let R be a one-dimensional domain such that the integral closure $\overline{R}$ of R in its quotient field is a finitely generated R-module. As before, the conductor of $\overline{R}$ in R is denoted by $\mathfrak{f}$. (The reader may find it helpful to refer to the pullback that precedes Proposition 1.2.) The main technique in [48] is to examine the relationship between $M/\mathfrak{f}M$ and $\overline{R}M/\mathfrak{f}M$ for torsion-free R-modules M.

Given a torsion-free R-module M, one defines the so-called "delta group" of M, denoted Δ_M. This is the subgroup of $(\overline{R}/\mathfrak{f})^\times$ consisting of determinants of automorphisms of $\overline{R}M/\mathfrak{f}M$ that carry $M/\mathfrak{f}M$ into itself. (See [48] for the basic properties of Δ_M.) There are two important facts we need:

(1) $\Delta_{M\oplus N} = \Delta_M \cdot \Delta_N$.
(2) If $M_\mathfrak{m} \cong N_\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$, then $\Delta_M = \Delta_N$.

The first fact allows one to restrict attention to *indecomposable* torsion-free R-modules. The second fact says that the delta group is an invariant of the local isomorphism class of M. Now let $\Lambda_\mathfrak{f}$ be the image of $(\overline{R})^\times$ in $(\overline{R}/\mathfrak{f})^\times$. We call this the group of *liftable units* with respect to $\mathfrak{f}$. The next theorem follows directly from Lemma 1.6 and Proposition 1.9 in [48].

Theorem 5.1. *Let R, $\overline{R}$, and $\mathfrak{f}$ be as above. Then R has torsion-free cancellation if and only if $(\overline{R}/\mathfrak{f})^\times \subseteq \Delta_M \cdot \Lambda_\mathfrak{f}$ for all torsion-free R-modules M.* □

Next, consider the cancellation question for arbitrary finitely generated modules. We shall call this the *mixed* cancellation question. Is there a result similar to the preceeding theorem that pertains to the mixed cancellation question? Such a result appears in [23]. Let S denote the complement of the union of the maximal ideals of R that contain $\mathfrak{f}$. Then $S^{-1}R$ is a semilocal domain of dimension one. One defines a delta group for $S^{-1}M$, denoted $\Delta_{S^{-1}M}$. From Corollary 4.4 of [23] one gets the following result, where now Λ_S denotes the group of units of $S^{-1}\overline{R}$ that lift to units of $\overline{R}$.

Theorem 5.2. *Let R, $\overline{R}$, and S be as above. Then R has mixed cancellation if and only if $(S^{-1}\overline{R})^\times \subseteq \Delta_{S^{-1}M} \cdot \Lambda_S$ for every finitely generated R-module M.*

An important question one can raise at this point is whether torsion-free cancellation implies mixed cancellation. It was shown in [23] that the two kinds of cancellation are not equivalent in general. We will give an example from that paper in Subsection 5.2 below.

Suppose, now, that R is an order in an algebraic number field K. That is, suppose $\mathcal{O}_K$ is the ring of algebraic integers of K and R is a subring of $\mathcal{O}_K$ such that $\mathbb{Q}R = K$. (Then $\overline{R} = \mathcal{O}_K$.) If R is a *quadratic* order then R has finite Cohen-Macaulay type. In [41], definitive results were obtained for torsion-free cancellation over quadratic orders. In [26], one can find decisive answers to the torsion-free cancellation question for a large family of cubic orders having finite Cohen-Macaulay type. In these two papers, each of the present authors used methods based on the calculation of delta groups. We will revisit these results in more detail below.

In [25] and [26], a connection between cancellation and finite Cohen-Macaulay type is exploited. The work is based on the idea that the failure of finite Cohen-Macaulay type often implies the failure of cancellation. In these two papers, negative answers to the torsion-free cancellation question are given for many quartic and higher-degree orders.

In the remainder of this section, we will focus on the cancellation question for one-dimensional Noetherian domains R, although many of the results given below are known to hold for other classes of rings as well, especially for reduced rings. Throughout, R will be a one-dimensional domain with quotient field K. Also, $\overline{R}$ will always be the integral closure of R in K. We insist that $\overline{R}$ be finitely generated as an R-module.

5.1 Torsion-free cancellation over one-dimensional domains

Let $D(R)$ denote the kernel of the natural map on Picard groups $\operatorname{Pic} R \to \operatorname{Pic} \overline{R}$. If $D(R) \neq 0$ then one can show that R has an invertible ideal $I \ncong R$ such that $I \oplus \overline{R} \cong R \oplus \overline{R}$ (cf. [41, Corollary 2.4]). This is one of the easiest ways in which torsion-free cancellation can fail. For certain kinds of rings, $D(R)$ is exactly the obstruction to torsion-free cancellation. For example, the following is from Theorem 0.1 of [44]:

Theorem 5.3. *Let R be as above. Assume further that R is finitely generated as a k-algebra for some infinite perfect field k. Then R has torsion-free cancellation if and only if $D(R) = 0$.* □

For examples of affine k-domains where $D(R) = 0$, we have Dedekind domains and the rings $F + XK[X]$, where $k \subseteq F \subseteq K$ are field extensions of finite degree. In fact [44, (1.7)], up to analytic isomorphism, these are the only examples! In particular [41, Corollary 3.3], an affine domain over an algebraically closed field has torsion-free cancellation if and only if it is a Dedekind domain.

Another case where $D(R)$ controls torsion-free cancellation is provided by Theorem 2.7 of [41]:

Theorem 5.4. *Let R be as above. Assume that every ideal of R is two-generated. Then R has torsion-free cancellation if and only if $D(R) = 0$.* □

In [41], the theorem above is applied to orders in quadratic number fields. We state the following classification result for imaginary quadratic orders (Theorem 4.5 of [41]):

Theorem 5.5. *Let d be a squarefree negative integer, and let R be an order in $\mathbb{Q}(\sqrt{d})$. Then R has torsion-free cancellation if and only if either $R = \overline{R}$ or else R satisfies one of the following:*

(1) $R = \mathbb{Z}[\sqrt{d}]$ *where* $d \equiv 1 \bmod 8$
(2) $R = \mathbb{Z}[2\sqrt{-1}]$
(3) $R = \mathbb{Z}[\sqrt{-3}]$
(4) $R = \mathbb{Z}[\frac{3}{2}(1 + \sqrt{-3})]$ □

For real quadratic orders R, the situation is more complicated. The condition $D(R) = 0$ depends on subtle arithmetical properties of the fundamental unit of $\overline{R}$, and it is extremely difficult to give a version of Theorem 5.5 that classifies those real quadratic orders having torsion-free cancellation. But given any *specific* real quadratic order R, a finite calculation involving the fundamental unit of $\overline{R}$ will determine whether or not torsion-free cancellation holds.

The cancellation question can be answered decisively if one knows all the delta groups that come from indecomposable torsion-free R-modules. In cases where R has finite Cohen-Macaulay type, one has some hope of calculating these delta groups. This is indeed the case for quadratic orders. The following result is equivalent to Corollary 4.2 of [41] but is stated in terms of data intrinsic to the ring. Recall that $\mathfrak{f}$ is the conductor of $\overline{R}$ in R.

Theorem 5.6. *Suppose $R = \mathbb{Z} + f\mathcal{O}_K$ is an order in a quadratic number field K, where $f \in \mathbb{Z}$ is a nonzero nonunit. Then torsion-free cancellation holds for R if and only if $(\overline{R}/\mathfrak{f})^\times \subseteq (R/\mathfrak{f})^\times \cdot \Lambda_{\mathfrak{f}}$.* □

Let's compare this with Theorem 5.1, where the statement of the condition for cancellation to hold depends on the entire family of isomorphism classes of indecomposable torsion-free R-modules. For a quadratic order R, it is known [3] that every indecomposable torsion-free R-module has rank one. Furthermore, there are only finitely many isomorphism classes of such modules. This makes it possible to replace the condition in Theorem 5.1 with a condition that depends only on subgroups of $(\overline{R}/\mathfrak{f})^\times$.

We now state some results for cubic orders. It is well known that every quadratic order R in K has the form $R = \mathbb{Z} + f\mathcal{O}_K$ for some nonzero rational integer f. While this is not necessarily true for cubic orders, one can consider cubic orders of that same form. Now, a cubic order R having finite Cohen-Macaulay type may have indecomposable torsion-free modules of rank greater than 1. The following result is a special case of Theorem 31 in [26] and depends crucially on the existence of indecomposable torsion-free R-modules of rank two:

Theorem 5.7. *Suppose $R = \mathbb{Z} + p\mathcal{O}_K$ is an order in a cubic number field K, where $p \in \mathbb{Z}$ is nonzero. Further, suppose $p\overline{R}$ is a prime ideal. Then torsion-free cancellation holds for R if and only if*

(1) $(\overline{R}/\mathfrak{f})^\times \subseteq (R/pR)^\times \cdot \Lambda_\mathfrak{f}$, *and*
(2) $(\overline{R}/\mathfrak{f})^\times \subseteq ((\overline{R}/\mathfrak{f})^\times)^2 \cdot \Lambda_\mathfrak{f}$ □

This is similar to Theorem 5.6. Once again, the torsion-free cancellation question for R is answered in terms of subgroups of $(\overline{R}/\mathfrak{f})^\times$. Using this result, one can find examples of cubic orders R for which $D(R) = 0$ and yet torsion-free cancellation fails for R.

There also exist many cubic orders that do not have finite Cohen-Macaulay type. Moreover, most orders in number fields of degree four and higher do not have finite Cohen-Macaulay type. Using the Drozd-Roĭter conditions [12] (cf. Theorem 1.1 in Section 1), we easily get the following (see Proposition 19 of [26]).

Lemma 5.8. *Let K be a number field of degree d and suppose $R = \mathbb{Z} + f\mathcal{O}_K$ is an order, where $f \in \mathbb{Z}$ is a nonzero nonunit. Then R has finite Cohen-Macaulay type if and only if either (i) $d = 2$ or (ii) $d = 3$ and f is square-free.* □

Failure of finite Cohen-Macaulay type often leads to failure of torsion-free cancellation. Many such examples can be given using the following result, which is a specialized version of Theorem 26 in [26].

Theorem 5.9. *Let K be a number field of degree at least four. Suppose $R = \mathbb{Z} + f\mathcal{O}_K$ is an order, where $f \in \mathbb{Z}$ is a nonzero nonunit. Then torsion-free cancellation holds for R if and only if $(\overline{R}/\mathfrak{f})^\times \subseteq \Lambda_\mathfrak{f}$.* □

The condition appearing in the result above is quite satisfying, given that the category of torsion-free R-modules for these orders is generally intractable. It turns out that the condition $(\overline{R}/\mathfrak{f})^\times \subseteq \Lambda_\mathfrak{f}$ is rarely satisfied. For example, the next result follows directly from Corollary 7.1 in [25].

Corollary 5.10. *Let K be a number field of degree four or higher. Then there are only finitely many primes $p \in \mathbb{Z}$ for which the order $R = \mathbb{Z} + p\mathcal{O}_K$ has torsion-free cancellation.* □

5.2 Mixed cancellation for one-dimensional domains

In [23], Hassler and Wiegand found a way to extend the techniques in [48] to handle arbitrary finitely generated modules. The original motivation for the work in [23] was the following question: When does torsion-free cancellation imply mixed cancellation? The following theorem gives a class of rings for which the answer is affirmative (see Theorem 6.1 of [23]):

Theorem 5.11. *Let R be a one-dimensional Noetherian domain. Further, suppose R is finitely generated as k-algebra, where k is an infinite field of characteristic zero. The following are equivalent:*

(1) $D(R) = 0$
(2) *R has torsion-free cancellation*
(3) *R has mixed cancellation* □

In the same paper [23], the class of Dedekind-like rings is considered. A one-dimensional, reduced, Noetherian ring R is defined to be Dedekind-like if $R_{\mathfrak{m}}$ is Dedekind-like for all maximal ideals $\mathfrak{m}$ of R. (See Section 3 for the definition of local Dedekind-like rings.) The following is from Corollary 6.11 in [23] and depends heavily on Levy and Klingler's classification [30] of indecomposable modules over local Dedekind-like rings:

Theorem 5.12. *Suppose R is a Dedekind-like order in a number field. Then torsion-free cancellation implies mixed cancellation.* □

Likewise, Hassler [18] has proved the following theorem. (We note that orders in quadratic number fields need not be Dedekind-like. For example, $\mathbb{Z}[2\sqrt{-1}]$ is not Dedekind-like.)

Theorem 5.13. *Suppose R is an order in an imaginary quadratic field. Then torsion-free cancellation implies mixed cancellation.* □

Now, suppose R is an order in a *real* quadratic field such that R is not Dedekind-like. Does torsion-free cancellation still imply mixed cancellation over R? The authors in [23] show that the order $R = \mathbb{Z}[17\frac{1+\sqrt{17}}{2}]$ has torsion-free cancellation but does not have mixed cancellation!

Finally, we remark that when R is an order in a real quadratic field, Hassler has shown in [19] that the mixed cancellation question for R can often be answered by a computation that involves the fundamental unit of R. The computation is a more complicated version of the one mentioned in the paragraph that follows Theorem 5.5 above.

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