Direct-sum decompositions of modules with semilocal endomorphism rings

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Let R be a ring and $\mathcal{C}$ a class of right R-modules closed under finite direct sums. If we suppose that $\mathcal{C}$ has a set of representatives, that is, a set $\mathrm{V}(\mathcal{C}) \subseteq \mathcal{C}$ such that every $M \in \mathcal{C}$ is isomorphic to a unique element $[M] \in \mathrm{V}(\mathcal{C})$, then we can view $\mathrm{V}(\mathcal{C})$ as a monoid, with the monoid operation $[M_1] + [M_2] = [M_1 \oplus M_2]$. Recent developments in the theory of commutative monoids (e.g., [4], [15]) suggest that one might obtain useful insights on decompositions of modules by considering the monoids $\mathrm{V}(\mathcal{C})$.

Suppose, for example, that R is a semilocal ring and $\mathcal{C} = \text{proj-}R$ (see §1 for definitions and notation). Then $\mathrm{V}(\mathcal{C})$ is a *positive normal monoid*, equivalently [4, Exercise 6.4.16], it is isomorphic to the submonoid of $\mathbf{N}^t$ $(t \in \mathbf{N})$ consisting of solutions to a finite system of homogeneous linear equations with integer coefficients. (These monoids have also been called *Diophantine monoids* in the literature.) The first author and Herbera [13] showed, conversely, that given a positive normal monoid M, there is semilocal ring R such

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that $M \cong \mathrm{V}(\text{proj-}R)$. Thus the direct-sum behavior of finitely generated projective modules over a semilocal ring is precisely delineated by the structure of positive normal monoids.

For another example, let R be a commutative Noetherian local ring and A a finitely generated R-module. Let $\mathrm{add}(A)$ denote the class of R-modules that are isomorphic to direct summands of direct sums of finitely many copies of A. Then $E := \mathrm{End}_R(A)$ is a semilocal ring, and $\mathrm{V}(\mathrm{add}(A))$ is naturally isomorphic to $\mathrm{V}(\text{proj-}E)$. (See, for example, [9].) Thus $\mathrm{V}(\mathrm{add}(A))$ is a positive normal monoid. Moreover, the second author showed in [17] that every positive normal monoid arises as $\mathrm{V}(\mathrm{add}(A))$ for some finitely generated module A over a suitable commutative Noetherian local ring. A consequence of this realization theorem is the fact [17, (4.2)] that every positive normal monoid arises also in the form $\mathrm{V}(\mathrm{add}(A))$ for some Artinian module A over a suitable ring. Since unique factorization can fail in positive normal monoids, this shows how much the Krull-Schmidt theorem can fail for direct-sum decompositions of finitely generated modules over Noetherian local rings and for direct-sum decompositions of Artinian modules. Further examples of modules with semilocal endomorphism ring are finite rank torsion-free modules over commutative semilocal principal ideal domains or valuation domains, linearly compact modules, and modules of finite Goldie dimension and finite dual Goldie dimension; cf. [11, §4.3].

In this paper we prove a version of the realization theorems described above not restricted by finite generation assumptions. Suppose $\mathcal{C}$ is a class of right R-modules with the following three properties:

- $\mathcal{C}$ has a *set of representatives* $\mathrm{V}(\mathcal{C})$ (that is, $\mathrm{V}(\mathcal{C})$ is a set—not a proper class—contained in $\mathcal{C}$, and every $A \in \mathcal{C}$ is isomorphic to a unique $[A] \in \mathrm{V}(\mathcal{C})$);
- $\mathcal{C}$ is closed under finite direct sums, under direct summands and under isomorphisms; and
- $\mathrm{End}_R(A)$ is semilocal for each $A \in \mathcal{C}$.

The first author showed in [12] that in this case $\mathrm{V}(\mathcal{C})$ is a reduced Krull monoid (see definitions at the beginning of the next section). Moreover, if M_R is the direct sum of the modules in $\mathrm{V}(\mathcal{C})$, E is the endomorphism ring $\mathrm{End}(M_R)$, $\mathcal{S}_E$ denotes the full subcategory of Mod-E consisting of finitely generated projective right E-modules with semilocal endomorphism ring, and $\mathcal{C}$ is viewed as a full subcategory of Mod-R, then the categories $\mathcal{C}$ and $\mathcal{S}_E$ turn out to be equivalent via the functors $\mathrm{Hom}_R(M_R, -)\colon \mathcal{C} \to \mathcal{S}_E$ and $- \otimes_E M\colon \mathcal{S}_E \to \mathcal{C}$. In particular, the monoids $\mathrm{V}(\mathcal{C})$ and $\mathrm{V}(\mathcal{S}_E)$ are isomorphic. The main theorem of this paper states that every reduced Krull monoid arises in this fashion, as $\mathrm{V}(\mathcal{S}_R)$ for a suitable ring R. Thus reduced Krull monoids coincide both with the monoids that can be realized as $\mathrm{V}(\mathcal{C})$ for some class $\mathcal{C}$ of modules satisfying the three properties above, and with the monoids that can be realized as $\mathrm{V}(\mathcal{S}_R)$ for some ring R. To compare our result with the earlier realization theorems, we note that the classes $\mathcal{C}$ in [13] and [17] that represent a given positive normal monoid do indeed satisfy the three properties above, and that the positive normal monoids are exactly the *finitely generated* reduced Krull monoids (see Proposition 1.4).

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1 Definitions, notation and conventions

Monoids are always assumed to be commutative and will be written additively, with 0 as the identity element. Recall that the monoid M is said to be *cancellative* provided $a+b=a+c \Longrightarrow b=c$ for all $a,b,c \in M$, and *reduced* provided $a+b=0 \Longrightarrow a=b=0$ for all $a,b \in M$. A monoid homomorphism $f\colon M_1 \to M_2$ is a *divisor homomorphism* provided whenever $f(x)+z=f(y)$ (with $x,y \in M_1$ and $z \in M_2$) there is an element $w \in M_1$ such that $x+w=y$. (In multiplicative language: If $f(x)$ divides $f(y)$ then x divides y.) A submonoid M' of a cancellative monoid M is said to be a *full* submonoid provided the inclusion $M' \to M$ is a divisor homomorphism, equivalently, $M' = G \cap M$, where G is a subgroup of the quotient group $\mathcal{Q}(M) := \{\, x-y \mid x,y \in M \,\}$ of M.

If G is an abelian group or, more generally, a commutative additive monoid, and Ω is a set, we shall denote by $G^{(\Omega)}$ the direct sum of $|\Omega|$ copies of G, and by G^{Ω} the direct product. We denote by $\mathbf{N}$ the monoid of non-negative integers and by $\mathbf{Z}$ the additive group of integers. The free monoid on a set Ω is then the direct sum $\mathbf{N}^{(\Omega)}$.

Definition 1.1 A *Krull monoid* is a commutative cancellative monoid admitting a divisor homomorphism into some free monoid $\mathbf{N}^{(\Omega)}$.

Remark 1.2 It is easy to see that if M is a reduced monoid any divisor homomorphism from M into a cancellative monoid must be injective, and hence that M is cancellative as well. Up to isomorphism, a reduced Krull monoid is one of the form $M := G \cap \mathbf{N}^{(\Omega)}$, where Ω is a set and G is a subgroup of the free abelian group $\mathbf{Z}^{(\Omega)}$. ∎

A (possibly non-commutative) ring R is said to be *semilocal* provided $R/J(R)$ is Artinian, where $J(R)$ is the Jacobson radical of R. For any module A_R over a ring R we shall denote by $[A_R]$ the isomorphism class of A_R. We denote by

Mod-R the category of all right R-modules and by mod-R the full subcategory of Mod-R consisting of all finitely generated right R-modules. The full subcategory of mod-R consisting of finitely generated projective right R-modules will be denoted by proj-R.

Example 1.3 Let $(R, \mathbf{m})$ be a commutative Noetherian local ring with $\mathbf{m}$-adic completion $\hat{R}$. The map $\mathrm{V}(\text{mod-}R) \to \mathrm{V}(\text{mod-}\hat{R})$ taking $[A]$ to $[A \otimes_R \hat{R}]$ is a divisor homomorphism [17, (1.3)]. Moreover, by the Krull-Schmidt theorem for direct-sum decompositions over complete local rings, $\mathrm{V}(\text{mod-}\hat{R})$ is a free monoid (on the set $\{ [C] \mid C$ is an indecomposable element of mod-$\hat{R} \}$). Thus $\mathrm{V}(R)$ is a reduced Krull monoid. ∎

More generally, let R be an arbitrary ring and $\mathcal{C}$ a class of right R-modules closed under isomorphism, finite direct sums, and direct summands, and such that $\mathrm{End}_R(M)$ is semilocal for every $M \in \mathcal{C}$. Assume that $\mathcal{C}$ has a set of representatives $\mathrm{V}(\mathcal{C})$. In [12] the first author defined a set $\mathrm{Spec}(\mathcal{C})$ and a divisor homomorphism $\phi\colon \mathrm{V}(\mathcal{C}) \to \mathbf{N}^{(\mathrm{Spec}(\mathcal{C}))}$, thereby showing that $\mathrm{V}(\mathcal{C})$ is a reduced Krull monoid.

The main theorem in this paper is a strong converse to the theorem in [12]. We show not only that every reduced Krull monoid (with a divisor homomorphism ϕ into a free monoid) arises in this fashion (for a suitable ring R) but also that the class $\mathcal{C}$ that realizes a given reduced Krull monoid can be taken to be the class $\mathcal{S}_R$ of all finitely generated projective R-modules with semilocal endomorphism ring:

$$\mathcal{S}_R := \{ P \in \text{proj-}R \mid \mathrm{End}_R(P) \text{ is semilocal} \}$$

Moreover, the divisor homomorphism ϕ is represented very simply, as the natural map $\tau\colon \mathrm{V}(\mathcal{S}_R) \to \mathrm{V}(\mathcal{S}_{R/J(R)})$.

Recall that a *positive normal monoid* is a monoid isomorphic to one of the form $G \cap \mathbf{N}^t$ for some $t \in \mathbf{N}$ and some subgroup G of $\mathbf{Z}^t$. The following well-known result relates the results of this paper to the earlier results discussed in the introduction:

Proposition 1.4 *Let M be a monoid. Then M is a positive normal monoid if and only if M is a finitely generated reduced Krull monoid.*

Proof. If M is a finitely generated reduced Krull monoid, one can clearly replace the index set Ω in Remark 1.2 by a finite set, so that M is positive normal. Conversely, any positive normal monoid is finitely generated (see, e.g., [13, Lemma 3.2]). ∎

Modules with semilocal endomorphism rings form a natural class of modules for our study, since direct-sum cancellation is assured for these modules. This cancellation theorem is due essentially to Evans [10] (see [11, Theorem 4.5]).

2 The Main Theorem

Here is the precise statement of our main result:

Theorem 2.1 (Main Theorem) *Let k be a field, M a reduced Krull monoid, Ω a set and $T\colon M \to \mathbf{N}^{(\Omega)}$ a divisor homomorphism. Then there exist a k-algebra R and two monoid isomorphisms $M \to \mathrm{V}(\mathcal{S}_R)$ and $h\colon \mathbf{N}^{(\Omega)} \to \mathrm{V}(\mathcal{S}_{R/J(R)})$ such that if $\tau\colon \mathrm{V}(\mathcal{S}_R) \to \mathrm{V}(\mathcal{S}_{R/J(R)})$ is the homomorphism induced by the natural surjection $\pi\colon R \to R/J(R)$, then the diagram of monoids and monoid homomorphisms*

$$\begin{array}{ccc} M & \xrightarrow{T} & \mathbf{N}^{(\Omega)} \\ \downarrow\cong & & {}_h\downarrow\cong \\ \mathrm{V}(\mathcal{S}_R) & \xrightarrow{\tau} & \mathrm{V}(\mathcal{S}_{R/J(R)}) \end{array} \tag{1}$$

commutes.

Remark 2.2 The map h in (1) actually comes from a bijection between Ω (the set of atoms of $\mathbf{N}^{(\Omega)}$) and the set of atoms of $\mathrm{V}(\mathcal{S}_{R/J(R)})$. Indeed, this is automatic: Any isomorphism of free monoids has to carry the atoms of one to the atoms of the other, and any bijection on the atoms of two free monoids induces a unique monoid isomorphism. ∎

The monoid homomorphism $\tau\colon \mathrm{V}(\mathcal{S}_R) \to \mathrm{V}(\mathcal{S}_{R/J(R)})$ is the obvious map $[P_R] \mapsto [P/PJ(R)]$. To see that this makes sense, we need to know that $P/PJ(R)$ has semilocal endomorphism ring whenever $P \in \mathcal{S}_R$. In fact, the endomorphism ring of $P/PJ(R)$ is semisimple Artinian:

Lemma 2.3 [1, Corollary 17.12] *Let R be any ring and P a finitely generated projective right R-module with endomorphism ring E. The natural homomorphism $E \to \mathrm{End}_{R/J(R)}(P/PJ(R))$ is surjective with kernel $J(E)$. In particular $P \in \mathcal{S}_R$ if and only if $\mathrm{End}_{R/J(R)}(P/PJ(R))$ is semisimple Artinian.* ∎

Notice, however, that base change does not in general make $R \mapsto \mathrm{V}(\mathcal{S}_R)$ into a functor from the category of rings to the category of monoids. For instance, if I is an infinite index set, the diagonal embedding $\varepsilon\colon k \to k^I$ does not induce a homomorphism $\mathrm{V}(\mathcal{S}_k) \to \mathrm{V}(\mathcal{S}_{k^I})$, because the k-module k is in $\mathcal{S}_k$ but k^I is not in $\mathcal{S}_{k^I}$. For a Noetherian example, one could take $k \to k[x]$, where x is an indeterminate.

Of course the map τ in (1) is a divisor homomorphism. In fact, more is true:

Lemma 2.4 [14, (2.10)] *For any ring R the homomorphism*

$$\mathrm{V}(\text{proj-}R) \to \mathrm{V}(\text{proj-}R/J(R))$$

is a divisor homomorphism. ■

Moreover, the target $\mathrm{V}(\mathcal{S}_{R/J(R)})$ of the homomorphism τ is always a free monoid:

Proposition 2.5 *Let R be a ring with $J(R) = 0$. Then $\mathcal{S}_R$ is the class of all finitely generated projective semisimple right R-modules. In particular, the Krull-Schmidt Theorem holds for $\mathcal{S}_R$, that is, $\mathrm{V}(\mathcal{S}_R)$ is a free commutative monoid generated by the simple projectives.*

Proof. Let $P \in \mathcal{S}_R$. By Lemma 2.3 $E := \mathrm{End}_R(P)$ is semisimple Artinian. Therefore there is a complete set $\{e_1, \dots, e_n\}$ of idempotents of E such that e_iEe_i is a division ring for every i. Thus $P_R = e_1P_R \oplus \cdots \oplus e_nP_R$. By [1, Proposition 17.19], each $e_iP_R = e_iP/e_iPJ(R)$ is simple, whence P_R is semisimple. Conversely, if $P \in \text{proj-}R$ is semisimple, clearly $P \in \mathcal{S}_R$. ■

3 Preliminary results

To prove the main theorem, we must build a k-algebra R with a prescribed monoid $\mathrm{V}(\mathcal{S}_R)$. We will build it in finite stages and then take the direct limit. The basic building block is provided by Lemma 3.4, which is a synthesis of two remarkable results, Theorems 3.1 and 3.2, due to Bergman [2]. We recall some terminology from [2] that we will need here and later. Given a field k and a k-algebra R, an *R-ring$_k$* is a k-algebra S together with a k-algebra homomorphism $R \to S$. A morphism $S_1 \to S_2$ of R-rings$_k$ is a k-algebra homomorphism $f\colon S_1 \to S_2$ making the following diagram commute:

$$\begin{array}{ccc} S_1 & \xrightarrow{f} & S_2 \\ & \nwarrow \quad \nearrow & \\ & R & \end{array} \tag{2}$$

Theorem 3.1 (Bergman [2]) *Let k be a field and let P be a nonzero finitely generated projective right module over a k-algebra R. Then there exist an R-ring$_k$ S and an idempotent S-module endomorphism φ of $P \otimes_R S$ such that given any R-ring$_k$ T and any idempotent T-module endomorphism ψ of $P \otimes_R T$, there exists a unique homomorphism $S \to T$ of R-rings$_k$ such that $\psi = \varphi \otimes_S T$.*

Moreover, the commutative monoid $\mathrm{V}(\text{proj-}S)$ *may be obtained from* $\mathrm{V}(\text{proj-}R)$

by adjoining two new generators $[P']$ *and* $[Q']$ *and one relation* $[P']+[Q']=[P]$*, where* P' *and* Q' *are the image and the kernel of* φ *respectively.*

Finally, if R *is hereditary, then so is* S*.* ■

Theorem 3.2 (Bergman [2]) *Let* k *be a field and let* P, Q *be two nonzero finitely generated projective right modules over a* k*-algebra* R*. Then there exist an* R*-ring*$_k$ S *and an* S*-module isomorphism* $\varphi\colon P\otimes_R S\to Q\otimes_R S$ *such that given any* R*-ring*$_k$ T *and any* T*-module isomorphism* $\psi\colon P\otimes_R T\to Q\otimes_R T$*, there exists a unique homomorphism* $S\to T$ *of* R*-rings*$_k$ *such that* $\psi=\varphi\otimes_S T$*.*

Moreover, the commutative monoid $\mathrm{V}(\text{proj-}S)$ *is obtained from* $\mathrm{V}(\text{proj-}R)$ *by imposing the relation* $[P]=[Q]$*.*

Finally, if R *is hereditary, then so is* S*.* ■

Notation 3.3 Let W be a finite index set, and let $\mathrm{Free}(2W+1)$ denote the free monoid on the $2|W|+1$ symbols I, p_w and q_w ($w\in W$). Next, we fix positive integers n_w, $w\in W$. Suppose S is a ring and φ_w^S is an idempotent endomorphism of the right S-module S^{n_w} for each $w\in W$. Denote by P_w and Q_w the image and the kernel of φ_w^S. Given an element $x=aI+\sum_{w\in W}b_wp_w+\sum_{w\in W}c_wq_w\in\mathrm{Free}(2W+1)$ (with $a,b_w,c_w\in\mathbf{N}$), we denote by $x(\varphi_w^S\mid w\in W)$ the finitely generated projective S-module $S^a\oplus\left(\bigoplus_{w\in W}P_w^{b_w}\right)\oplus\left(\bigoplus_{w\in W}Q_w^{c_w}\right)$. Fix a finite set $A\subseteq(\mathrm{Free}(2W+1)-\{0\})\times(\mathrm{Free}(2W+1)-\{0\})$.

Lemma 3.4 *Let* k *be a field and* R *a* k*-algebra, and keep the notation established in* (3.3)*. There exist an* R*-ring*$_k$ S*,* $|W|$ *idempotent* S*-module endomorphisms* φ_w^S *of* S^{n_w} $(w\in W)$*, and* $|A|$ S*-module isomorphisms* $\psi_{(x,y)}^S\colon x(\varphi_w^S\mid w\in W)\to y(\varphi_w^S\mid\beta\in W)$ $((x,y)\in A)$*, with the following universal property: Given any* R*-ring*$_k$ T*,* $|W|$ *idempotent* T*-module endomorphisms* φ_w^T *of* T^{n_w} $(w\in W)$*, and* $|A|$ T*-module isomorphisms* $\psi_{(x,y)}^T\colon x(\varphi_w^T\mid w\in W)\to y(\varphi_w^T\mid w\in W)$ $((x,y)\in A)$*, there exists a unique homomorphism* $S\to T$ *of* R*-rings*$_k$ *such that* $\varphi_w^T=\varphi_w^S\otimes_S T$ *for every* $w\in W$ *and* $\psi_{(x,y)}^T=\psi_{(x,y)}^S\otimes_S T$ *for every* $(x,y)\in A$*.*

Moreover, the commutative monoid $\mathrm{V}(\text{proj-}S)$ *may be obtained from* $\mathrm{V}(\text{proj-}R)$ *by adjoining* $2|W|$ *new generators* $[P_w]$ *and* $[Q_w]$ $(w\in W)$ *and* $|W|+|A|$ *relations* $[P_w]+[Q_w]=n_w[S]$ $(w\in W)$ *and* $[x(\varphi_w^S\mid\beta\in W)]=[y(\varphi_w^S\mid\beta\in W)]$ $((x,y)\in A)$*, where* P_w *and* Q_w *are the image and the kernel of* φ_w^S *respectively, for each* $w\in W$*.*

Finally, if R *is hereditary, then so is* S*.*

Proof. Proceed by induction on $|W|+|A|$, taking $S=R$ when $|W|=|A|=0$. One applies Theorem 3.1 when $|W|$ increases and Theorem 3.2 when $|A|$ increases. The details are left to the reader. ■

§3.5 Localization

A crucial step in our construction will be a version of non-commutative localization due to Cohn. Suppose we are given a k-algebra R and a collection Σ of matrices over R. Then there is a universal Σ-inverting R-ring$_k$ $\lambda\colon R \to R_\Sigma$ [8, Theorem 7.2.1]. That is, the image of each matrix in Σ is invertible over R_Σ and if S is any R-ring$_k$ over which each matrix in Σ becomes invertible, then there is a unique homomorphism of R-rings$_k$ from R_Σ to S.

Lemma 3.6 (Cohn [6]) *Let R be a k-algebra and $\varphi\colon R \to D$ be a surjective k-algebra homomorphism, with D commutative and von Neumann regular. Let Σ be the set of square matrices over R that become invertible over D, and let $\bar{\varphi}\colon R_\Sigma \to D$ be the map coming from the universal property of R_Σ. Then the kernel of $\overline{\varphi}$ is the Jacobson radical of R_Σ. Moreover,* $\mathrm{V}(\text{proj-}R_\Sigma)$ *is a cancellative monoid.*

Proof. Since $R_\Sigma/\ker(\overline{\varphi}) \cong D$ is von Neumann regular, we have that $J(R_\Sigma) \subseteq \ker(\overline{\varphi})$. For the converse, we appeal to [6, Theorem 3.1], which says that if C is a square matrix over R_Σ such that $\overline{\varphi}(C)$ is invertible over D, then C is already invertible over R_Σ. Therefore, if $x \in \ker(\overline{\varphi})$, then $1 + x$ is invertible in R_Σ. As $\ker(\overline{\varphi})$ is a two-sided ideal, it follows that $x \in J(R_\Sigma)$. Therefore $\ker\overline{\varphi} = J(R_\Sigma)$.

For the second statement, we recall that every finitely generated projective D-module is isomorphic to a direct sum of principal ideals generated by idempotents. It follows easily that $\mathrm{V}(\text{proj-}D)$ is cancellative. Since $D \cong R_\Sigma/J(R_\Sigma)$, $\mathrm{V}(\text{proj-}R_\Sigma) \to \mathrm{V}(\text{proj-}D)$ is a divisor homomorphism (Lemma 2.4) and hence $\mathrm{V}(\text{proj-}R_\Sigma)$ is cancellative as well. ∎

We also need the following proposition, whose standard proof is left to the reader:

Proposition 3.7 [3, p. 320] *Let Δ be a directed set and $\{\, R_\delta \mid \delta \in \Delta \,\}$ be a direct system of rings with direct limit R. The canonical monoid homomorphism* $\varinjlim_{\delta} \mathrm{V}(\text{proj-}R_\delta) \to \mathrm{V}(\text{proj-}R)$ *is an isomorphism.* ∎

4 Two special cases of the main theorem

In this section we prove two special cases of the main theorem. We begin by fixing some notation that will be used throughout the paper.

Notation 4.1 Since in the statement of the main theorem $T\colon M \to \mathbf{N}^{(\Omega)}$ is a divisor homomorphism, we can suppose that there exists a subgroup G of $\mathbf{Z}^{(\Omega)}$

such that $M = G \cap \mathbf{N}^{(\Omega)}$, $T\colon M = G \cap \mathbf{N}^{(\Omega)} \to \mathbf{N}^{(\Omega)}$ is the inclusion and G is the subgroup of $\mathbf{Z}^{(\Omega)}$ generated by M. As $M \subseteq \mathbf{N}^{(\Omega)}$, every element of M is a sum of indecomposable elements. The set of all indecomposable elements of M is the unique minimum generating set of the monoid M and consists exactly of the minimal non-zero elements of M (relative to the partial ordering induced by the product ordering on $\mathbf{N}^{(\Omega)}$). (Since M is a *full* (§1) submonoid of $\mathbf{N}^{(\Omega)}$, this partial ordering agrees with the *intrinsic* partial ordering ("divides" in multiplicative terminology) on M.)

For every $w \in \mathbf{Z}^{(\Omega)}$, let $\operatorname{supp}(w)$ be the set of all $i \in \Omega$ with $w(i) \neq 0$; thus $\operatorname{supp}(w)$ is a finite subset of Ω. Set $\operatorname{supp}(M) = \bigcup_{w \in M} \operatorname{supp}(w)$.

§4.2 The finite case

Suppose $\operatorname{supp}(M) = \Omega$ and Ω is a finite set. In this case, M has only finitely many indecomposable elements (because they form an antichain in $\mathbf{N}^{\Omega}$). Therefore the sum u of all indecomposable elements of M is an order-unit of M (that is, all of the coordinates of u for the embedding $M \subseteq \mathbf{N}^{\Omega}$ are positive). In this case, by [13, Theorem 6.1] there exist a semilocal hereditary k-algebra R and a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & \mathbf{N}^{(\Omega)} \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{V}(\text{proj-}R) & \longrightarrow & \mathrm{V}(\text{proj-}R/J(R)). \end{array}$$

Since R is semilocal, proj-$R = \mathcal{S}_R$ and proj-$R/J(R) = \mathcal{S}_{R/J(R)}$.

§4.3 The case $M = \{0\}$

When $M = 0$ (equivalently, $\operatorname{supp}(M) = \emptyset$), there is still something to prove, since we have to replicate the given divisor homomorphism $M \to \mathbf{N}^{(\Omega)}$. To do this, we must build a k-algebra R such that $\mathcal{S}_R = \{(0)\}$ but $\mathrm{V}(\mathcal{S}_{R/J(R)}) \cong \mathbf{N}^{(\Omega)}$. We put $D = \frac{k^{\mathbf{N}}}{k^{(\mathbf{N})}} \times k^{\Omega}$. (The factor $\frac{k^{\mathbf{N}}}{k^{(\mathbf{N})}}$ is necessary only in the case that Ω is finite.) We will choose R so that $R/J(R) \cong D$, and we will compute $\mathrm{V}(\mathcal{S}_D)$ using the following lemma:

Lemma 4.4 *Let A be a commutative von Neumann regular ring. Then* $\mathrm{V}(\mathcal{S}_A)$ *is the free monoid generated by* $\{\, [A/\mathbf{m}] \mid \mathbf{m}$ *is an isolated point of* $\operatorname{Spec}(A)\,\}$.

PROOF. By Proposition 2.5, $\mathrm{V}(\mathcal{S}_A)$ is the free monoid generated by the simple projective A-modules. The simple projectives are exactly the modules $A/\mathbf{m}$, where $\mathbf{m}$ is a principal maximal ideal of A, that is, an isolated point of $\operatorname{Spec}(A)$. ∎

Since $\mathrm{Spec}(\frac{k^{\mathbf{N}}}{k^{(\mathbf{N})}})$ has no isolated points and since the isolated points of $\mathrm{Spec}(k^{\Omega})$ are the maximal ideals $\mathbf{m}_i := \{\, f \in k^{\Omega} \mid f(i) = 0 \,\}$, for $i \in \Omega$, we see that $\mathrm{V}(\mathcal{S}_D) \cong \mathbf{N}^{(\Omega)}$, as desired.

Now let X be a set of noncommuting indeterminates over k with $|X| = |D|$, and let $\varphi\colon k\langle X\rangle \to D$ be a surjective k-algebra homomorphism from the free k-algebra $k\langle X\rangle$ onto D. It is well known that $k\langle X\rangle$ is a two-sided fir [8, Corollary to Proposition 2.4.2]. Let Σ be the set of square matrices over $k\langle X\rangle$ whose images in D are invertible, and let $\lambda\colon k\langle X\rangle \to R := k\langle X\rangle_{\Sigma}$ be the universal Σ-inverting $k\langle X\rangle$-ring$_k$ (§3.5). By Lemma 3.6 we have $D \cong R/J(R)$, and we need to show that $\mathcal{S}_R = \{(0)\}$.

Suppose A, B are (not necessarily square) matrices with entries in $k\langle X\rangle$ such that $AB \in \Sigma$. If A is $r \times s$ then $r \le s$, since $\varphi(A)\varphi(B)$ is invertible over the commutative ring D. We claim that one can adjoin rows to $\lambda(A)$ and columns to $\lambda(B)$ to get invertible $s \times s$ matrices over R. This will show that Σ is *factor-complete* [7]. Since $\lambda(A)\lambda(B)$ becomes invertible when we pass to D, [6, Theorem 3.1] implies that $\lambda(A)\lambda(B)$ is already invertible over R. Therefore the kernel of $\lambda(A)$ and the cokernel of $\lambda(B)$ are stably free R-modules and hence free, by Lemma 3.6. The claim follows easily.

Since Σ is factor-complete, [7, Theorem 1] implies that the localization $R = k\langle X\rangle_{\Sigma}$ is a semifir, so that every finitely generated projective module is free of unique rank. Since R is not semilocal (i.e., $R/J(R)$ is not Artinian), it follows that $\mathcal{S}_R = \{(0)\}$, as desired.

5 Reduction to the case where $\mathrm{supp}(M) = \Omega$ and Ω is infinite

Returning to the general case of the main theorem, let us write Ω as the disjoint union of $\Omega_1 := \mathrm{supp}(M)$ and its complement $\Omega_2 := \Omega - \mathrm{supp}(M)$. Suppose we can prove the theorem in the case where Ω is infinite and $\Omega_1 = \Omega$. By §4.2 we will then know that the theorem is true whenever $\Omega_1 = \Omega$.

In the general case, the image of the divisor homomorphism $T\colon M \to \mathbf{N}^{(\Omega)}$ is contained in $\mathbf{N}^{(\Omega_1)}$. Moreover, the induced homomorphism $T_1\colon M \to \mathbf{N}^{(\Omega_1)}$ is a divisor homomorphism. By the case $\Omega_1 = \Omega$, we obtain a k-algebra R_1 and a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{T_1} & \mathbf{N}^{(\Omega_1)} \\ \downarrow \cong & & {\scriptstyle h_1} \downarrow \cong \\ \mathrm{V}(\mathcal{S}_{R_1}) & \xrightarrow{\tau_1} & \mathrm{V}(\mathcal{S}_{R_1/J(R_1)}). \end{array} \tag{3}$$

Also, by the case treated in §4.3, we obtain a k-algebra R_2 and a commutative diagram

$$\begin{array}{ccc} \{0\} & \longrightarrow & \mathbf{N}^{(\Omega_2)} \\ \downarrow \cong & & {}_{h_2}\downarrow \cong \\ \mathrm{V}(\mathcal{S}_{R_2}) & \xrightarrow{\tau_2} & \mathrm{V}(\mathcal{S}_{R_2/J(R_2)}). \end{array} \tag{4}$$

Taking $R = R_1 \times R_2$, we obtain the direct sum of the two diagrams (3) and (4), which yields the desired diagram in Theorem 2.1.

• Thus we may assume from now on that $\mathrm{supp}(M) = \Omega$ and Ω is infinite. The proof in this case will occupy the rest of the paper.

6 Proof of the Main Theorem

We now begin the proof of the main theorem in the remaining case, when $\mathrm{supp}(M) = \Omega$ and Ω is infinite. We preserve the notation established in Notation 4.1. The proof proceeds in several steps.

§6.1 The monoids $M_F \subseteq C_F$

The first step is to write M as a convenient direct limit of finitely generated monoids. We shall actually enlarge M a bit in order to keep track of its relations. After taking the direct limit, we will be able to identify and discard the unwanted stuff (see §6.9).

Let $\mathbf{1}$ be the element of the direct product $\mathbf{Z}^\Omega$ with $\mathbf{1}(i) = 1$ for every $i \in \Omega$. Let $\mathcal{F}(\Omega)$ be the directed set of all finite subsets of Ω. For each $F \in \mathcal{F}(\Omega)$, let C_F be the submonoid of $\mathbf{N}^\Omega$ consisting of elements that are constant on $\Omega - F$. Thus $C_F = \mathbf{N}^F \oplus \chi_{\Omega - F}\mathbf{N}$, where $\chi_{\Omega-F}$ is the characteristic function, taking the value 0 on F and 1 on $\Omega - F$. In particular, C_F is a free monoid of rank $|F| + 1$. Put $M_F := ((\mathbf{Z}^F \cap G) \oplus \mathbf{1Z}) \cap \mathbf{N}^\Omega$. Note that M_F is a full submonoid of C_F.

If $F, F' \in \mathcal{F}(\Omega)$ and $F' \subseteq F$, we have a commutative diagram of monoids:

$$\begin{array}{ccc} M_{F'} & \subseteq & C_{F'} \\ \downarrow \subseteq & & \downarrow \subseteq \\ M_F & \subseteq & C_F \end{array} \tag{5}$$

The direct limit (union) of the embeddings $M_F \subseteq C_F$ is

$$(G \oplus \mathbf{1Z}) \cap \mathbf{N}^\Omega \subseteq C := \mathbf{N}^{(\Omega)} \oplus \mathbf{1N}. \tag{6}$$

§6.2 The generating sets W_F

We shall fix, for each $F \in \mathcal{F}(\Omega)$, a finite set $W_F \subseteq M_F - \{\mathbf{1}\}$ such that M_F is generated as a monoid by $W_F \cup \{\mathbf{1}\}$. Moreover, we shall choose these sets in such a way that $W_{F'} \subseteq W_F$ whenever $F' \subseteq F$. We do this by induction on $|F|$, starting with $W_\emptyset = \emptyset$. For $F \neq \emptyset$, we note that M_F is finitely generated, being a full submonoid of C_F. Now we choose any W_F containing the union of the finitely many finite sets $W_{F'}$, F' ranging over the proper subsets of F, such that $\mathbf{1} \notin W_F$ and $W_F \cup \{\mathbf{1}\}$ generates M_F.

If $w \in (G \oplus \mathbf{1Z}) \cap \mathbf{N}^\Omega = \bigcup_{F \in \mathcal{F}(\Omega)} M_F$, we write $w = g + z\mathbf{1}$ with $g \in G$ and $z \in \mathbf{Z}$. Since $G \subseteq \mathbf{Z}^{(\Omega)}$ and $w \in \mathbf{N}^\Omega$, we must have $z \geq 0$. Choose a positive integer z' such that $-g + z'\mathbf{1} \in \mathbf{N}^\Omega$. We put $u_w := -g + z'\mathbf{1}$ and $n_w = z + z'$. Thus we have defined functions $\bigcup_{F \in \mathcal{F}(\Omega)} M_F \to \bigcup_{F \in \mathcal{F}(\Omega)} M_F$, $w \mapsto u_w$, and $\bigcup_{F \in \mathcal{F}(\Omega)} M_F \to \mathbf{N} - \{0\}$, $w \mapsto n_w$, with the property that $w + u_w = n_w\mathbf{1}$. Notice that if $w \in M_F$ then $u_w \in M_F$ too (since the elements g and $-g$ are then in $\mathbf{Z}^F \cap G$). We emphasize, however, that the element u_w and the positive integer n_w depend only on w and not on the particular monoid M_F containing w.

§6.3 The presentation $T_F\colon \mathrm{Free}_F \to M_F$

Recalling Notation 3.3 and the generating sets W_F from §6.2, we put $\mathrm{Free}_F := \mathrm{Free}(2W_F + 1)$ for each $F \in \mathcal{F}(\Omega)$. We have a surjective homomorphism of monoids $T_F\colon \mathrm{Free}_F \to M_F$ taking the images of I, p_w and q_w to $\mathbf{1}, w$ and u_w respectively. These maps are compatible with the inclusions $F' \subseteq F$, and we can enlarge (5) to the following commutative diagram, in which the horizontal arrows are surjections:

$$\begin{array}{ccccc} \mathrm{Free}_{F'} & \xrightarrow{T_{F'}} & M_{F'} & \subseteq & C_{F'} \\ \downarrow \subseteq & & \downarrow \subseteq & & \downarrow \subseteq \\ \mathrm{Free}_F & \xrightarrow{T_F} & M_F & \subseteq & C_F. \end{array} \tag{7}$$

Let $\sim_F$ be the kernel of the homomorphism $T_F\colon \mathrm{Free}_F \to C_F$ (that is, the congruence $\{(x, y) \in \mathrm{Free}_F \times \mathrm{Free}_F \mid T_F(x) = T_F(y)\}$). Since Free_F is finitely generated, $\sim_F$ is finitely generated as well [5, Theorem 9.28]. We note that the $|W_F|$ elements $(p_w + q_w, n_wI)$ are in $\sim_F$. We shall fix a finite set $A_F \subseteq \mathrm{Free}_F \times$

Free_F that generates $\sim_F$. Moreover, we choose the sets A_F, by induction on $|F|$ (as we did for the sets W_F in §6.2), so that $A_{F'} \subseteq A_F$ whenever $F' \subseteq F$. (This is possible since $(x, y) \in A_{F'}$ implies $x \sim_F y$, by the commutativity of diagram (7).) Notice that the only element of Free_F that is mapped to zero via T_F is the zero element of Free_F, because the generators I, p_w and q_w of Free_F are mapped to the positive elements $\mathbf{1}$, w and u_w of $\mathbf{N}^\Omega$. Thus the only element in $\sim_F$ of the form $(x, 0)$ or $(0, y)$ is $(0, 0)$, and we may suppose that all the elements of A_F are pairs of non-zero elements of Free_F. We record the result of this discussion:

Lemma 6.4 *For each $F \in \mathcal{F}(\Omega)$, we have $A_F \subseteq (\mathrm{Free}_F - \{0\}) \times (\mathrm{Free}_F - \{0\})$. The monoid M_F is obtained from the free monoid $I\mathbf{N} \cong \mathbf{N}$ by adjoining $2|W_F|$ new generators p_w, q_w ($w \in W_F$) and imposing $|A_F|$ relations $x = y$ ($(x, y) \in A_F$).* ∎

§6.5 The isomorphisms $h_F \colon C_F \to \mathrm{V}(\text{proj-}D_F)$

In this section and the next we establish a context to which we can apply Lemma 3.4. We will define a direct system of rings and certain projective modules over these rings, as well as certain isomorphisms among these projective modules. In order to do this, we need to specify exactly where these projective modules "live", and creating this habitat results in some technical complications that we have been unable to avoid.

Let $D = k^{(\Omega)} \oplus k \cdot 1_{k^\Omega}$, the k-subalgebra of k^Ω generated by the ideal $k^{(\Omega)}$. For every $F \in \mathcal{F}(\Omega)$ (the set of finite subsets of Ω), put $D_F := k^F \oplus k \cdot 1_{k^\Omega} \subseteq D$, a semisimple Artinian subring of D. Letting e_i^F be the idempotent of D_F with support $\{i\}$ and e_∞^F the idempotent with support $\Omega - F$, we see that a k-basis of D_F is given by the set $\{ e_i^F \mid i \in F \cup \{\infty\} \}$. (We have put the unnecessary superscripts on the "finite" e_i in order to allow simultaneous treatment of the idempotents e_i^F and e_∞^F.)

Let $\{ v_{(j,\ell,t)} \mid j, \ell, t \in \mathbf{N} \}$ be the standard basis of the free D-module of countable rank $D^{(\mathbf{N}\times\mathbf{N}\times\mathbf{N})}$. (Thus the value of $v_{(j,\ell,t)}$ is 1 at (j, ℓ, t) and 0 elsewhere.) We linearly order the standard basis by means of the lexicographic order on $\mathbf{N}\times\mathbf{N}\times\mathbf{N}$. All of our projective modules will be embedded as explicit submodules of $D^{(\mathbf{N}\times\mathbf{N}\times\mathbf{N})}$. This means that change of rings (as in Lemma 3.4) can be made very explicit as well: If $F' \subseteq F \in \mathcal{F}(\Omega)$ and P is a direct summand of $D_{F'}^{(\mathbf{N}\times\mathbf{N}\times\mathbf{N})}$, then $P \otimes_{D_{F'}} D_F$ can be identified with PD_F, the D_F-submodule of $D^{(\mathbf{N}\times\mathbf{N}\times\mathbf{N})}$ generated by P.

For $F \in \mathcal{F}(\Omega)$ and $j, \ell \in \mathbf{N}$, we will now define a natural function $g_{(F,j,\ell)}$ from the free monoid C_F of §6.1 to the class proj-D_F of finitely generated projective D_F-modules. The image $g_{(F,j,\ell)}(c)$ of each element $c \in C_F$ will be a direct summand of the free D_F-module $\bigoplus_{t\in\mathbf{N}} v_{(j,\ell,t)} D_F \subseteq D^{(\mathbf{N}\times\mathbf{N}\times\mathbf{N})}$. Given an element $c \in C_F$, we let $c_i \in \mathbf{N}$ be the i^{th} coordinate of c for each $i \in F$ and c_∞ the constant value of c on $\Omega - F$. We put

$$g_{(F,j,\ell)}(c) := \bigoplus_{i \in F \cup \{\infty\}} \bigoplus_{t=1}^{c_i} v_{(j,\ell,t)} e_i^F D_F.$$

Thus $g_{(F,j,\ell)}(c)$ has a canonical k-basis consisting of the elements $v_{(j,\ell,t)} e_i^F$ ($i \in F \cup \{\infty\}$, $t = 1, \ldots, c_i$). The modules $g_{(F,j,\ell)}(c)$ are all isomorphic (as j and ℓ vary), and their sum inside $D_F^{(\mathbf{N}\times\mathbf{N}\times\mathbf{N})}$ is a direct sum. The maps $g_{(F,j,\ell)}$ are compatible with inclusions in $\mathcal{F}(\Omega)$. In detail, suppose $F' \subseteq F \in \mathcal{F}(\Omega)$. We have the equations

$$e_\infty^{F'} = e_\infty^F + \sum_{i \in F - F'} e_i^F \quad \text{and} \quad (\text{for} \quad i \in F \cup \{\infty\} - F') \quad e_i^F = e_i^F e_\infty^{F'}. \tag{8}$$

If, now, $c \in C_{F'}$, it follows from (8) that

$$g_{(F',j,\ell)}(c) \subseteq g_{(F,j,\ell)}(c) \quad \text{and} \quad g_{(F,j,\ell)}(c) = \Big(g_{(F',j,\ell)}(c)\Big) D_F \tag{9}$$

in $D^{(\mathbf{N}\times\mathbf{N}\times\mathbf{N})}$.

All of the maps $g_{(F,j,\ell)}$ induce the same monoid isomorphism $h_F \colon C_F \to \mathrm{V}(\text{proj-}D_F)$, and this monoid isomorphism is compatible with inclusions. That is, if $F, F' \in \mathcal{F}(\Omega)$ with $F' \subseteq F$, we have a commutative diagram

$$\begin{array}{ccc} C_{F'} & \xrightarrow{h_{F'}} & \mathrm{V}(\text{proj-}D_{F'}) \\ \downarrow \subseteq & & \downarrow \\ C_F & \xrightarrow{h_F} & \mathrm{V}(\text{proj-}D_F) \end{array} \tag{10}$$

in which the horizontal arrows are isomorphisms.

§6.6 The k-algebra surjections $\xi_F \colon R_F \to D_F$

We begin this section by constructing certain idempotent endomorphisms $\varphi_w^{D_F}$ of the free D_F-modules $D_F^{n_w}$, where the n_w are the positive integers defined in §6.2. Here F is an arbitrary finite subset of Ω, and $w \in M_F$. In keeping with our plan to embed everything explicitly in $D^{(\mathbf{N}\times\mathbf{N}\times\mathbf{N})}$, we identify the free module D_F^a with $g_{(F,0,0)}(a\mathbf{1})$ for each $a \in \mathbf{N}$. We will define the idempotent endomorphism $\varphi_w^{D_F}$ by specifying its image P_w and kernel Q_w (in the notation of §3.3). Since we will need to handle direct sums of many copies of P_w and Q_w, we will create these copies via a direct-sum decomposition of *each* of the free modules $g_{(F,j,\ell)}(n_w\mathbf{1})$, where $j, \ell \in \mathbf{N}$.

For $F \in \mathcal{F}(\Omega)$ and $w \in M_F$, we recall the setup in §6.2. We have the relation $w + u_w = n_w\mathbf{1}$, whence $w_i + (u_w)_i = n_w$ for each $i \in F \cup \{\infty\}$. The "(j, ℓ)

copy" of P_w will be the direct summand $g_{(F,j,\ell)}(w)$ of $g_{(F,j,\ell)}(n_w\mathbf{1})$. For the complementary summand (corresponding to Q_w), we let

$$K_{(F,j,\ell)}(w) := \bigoplus_{i\in F\cup\{\infty\}} \bigoplus_{t=w_i+1}^{n_w} v_{(j,\ell,t)} e_i^F D_F.$$

Then $K_{(F,j,\ell)}(w)$ has a canonical k-basis consisting of the elements $v_{(j,\ell,t)}e_i^F$ with $i \in F \cup \{\infty\}$ and $t = w_i + 1, \dots, n_w$. The modules $K_{(F,j,\ell)}(w)$ are D_F-submodules of $D^{(\mathbf{N}\times\mathbf{N}\times\mathbf{N})}$ and are isomorphic to $g_{(F,j,\ell)}(u_w)$ (and of course their isomorphism class does not vary with (j,ℓ)). If $F' \subseteq F$ and $w \in M_{F'}$, we see, from equations (8), that

$$K_{(F,j,\ell)}(w) = K_{(F',j,\ell)}(w) D_F \tag{11}$$

in $D^{(\mathbf{N}\times\mathbf{N}\times\mathbf{N})}$, for all $j, \ell \in \mathbf{N}$.

There is an internal decomposition

$$g_{(F,j,\ell)}(n_w\mathbf{1}) = g_{(F,j,\ell)}(w) \oplus K_{(F,j,\ell)}(w),$$

for $F \in \mathcal{F}(\Omega), w \in M_F$ and $j, \ell \in \mathbf{N}$. In particular, for $j = \ell = 0$, there is a unique idempotent endomorphism $\varphi_w^{D_F}$ of $g_{(F,0,0)}(n_w\mathbf{1}) \cong D_F^{n_w}$ such that $\mathrm{Im}(\varphi_w^{D_F}) = g_{(F,0,0)}(w)$ and $\ker(\varphi_w^{D_F}) = K_{(F,0,0)}(w)$ for every $F \in \mathcal{F}(\Omega)$ and every $w \in M_F$. A k-basis of $g_{(F,0,0)}(n_w\mathbf{1})$ is given by the elements $v_{(0,0,t)}e_i^F$ with $i \in F \cup \{\infty\}$ and $t = 1, \dots, n_w$; and the idempotent endomorphism $\varphi_w^{D_F}$ maps such an element $v_{(0,0,t)}e_i$ to itself for $t \le w_i$ and to zero for $t > w_i$. Since the elements u_w and the integers n_w depend only on w and not on the set F, we have a commutative square

$$\begin{array}{ccc} g_{(F',0,0)}(n_w\mathbf{1}) & \xrightarrow{\varphi_w^{D_{F'}}} & g_{(F',0,0)}(n_w\mathbf{1}) \\ \downarrow\subseteq & & \downarrow\subseteq \\ g_{(F,0,0)}(n_w\mathbf{1}) & \xrightarrow{\varphi_w^{D_F}} & g_{(F,0,0)}(n_w\mathbf{1}) \end{array} \tag{12}$$

whenever $F' \subseteq F \in \mathcal{F}(\Omega)$ and $w \in M_{F'}$. We have $\left(g_{(F',0,0)}(n_w\mathbf{1})\right) \otimes_{D_{F'}} D_F = \left(g_{(F',0,0)}(n_w\mathbf{1})\right) D_F = g_{(F,0,0)}(n_w\mathbf{1})$ by (9), and now (12) implies that

$$\varphi_w^{D_F} = \varphi_w^{D_{F'}} \otimes_{D_{F'}} D_F, \tag{13}$$

whenever $F' \subseteq F \in \mathcal{F}(\Omega)$ and $w \in W_{F'}$.

Let $Z = \{\, z_i \mid i \in \Omega \,\}$ denote a set of noncommuting indeterminates over k. Fix $F \in \mathcal{F}(\Omega)$, and put $Z_F := \{\, z_i \mid i \in F \,\}$. Let $\omega_F \colon k\langle Z_F\rangle \to D_F$ be the surjective k-algebra homomorphism from the free k-algebra $k\langle Z_F\rangle$ to the semisimple Artinian ring D_F taking z_i to the idempotent e_i^F with support $\{i\}$. Thus we view D_F as a $k\langle Z_F\rangle$-ring$_k$.

We now apply Lemma 3.4 with $W = W_F$ (see §6.2), with $A = A_F$ (see §6.3), and with the positive integers n_w ($w \in W_F$) of §6.2. Let R_F denote the $k\langle Z_F\rangle$-ring$_k$ S provided by Lemma 3.4, and let $\varphi_w^{R_F}$ and $\psi_{(x,y)}^{R_F}$ be the idempotents and isomorphisms provided.

We wish to exploit the universal property of R_F to get a homomorphism $\xi_F\colon R_F \to D_F$ of R-rings$_k$. To do this, we have to produce certain idempotents and certain isomorphisms over D_F. We already have the $|W_F|$ idempotent endomorphisms $\varphi_w^{D_F}$ of the free D_F-modules $g_{(F,0,0)}(n_w\mathbf{1})$ for $w \in W_F$. In order to define, for each $(x,y) \in A_F$, the desired isomorphism $\psi_{(x,y)}^{D_F}\colon x(\varphi_w^{D_F} \mid w \in W_F) \to y(\varphi_w^{D_F} \mid w \in W_F)$, we will embed these projective modules as explicit direct summands of $D^{(\mathbf{N}\times\mathbf{N}\times\mathbf{N})}$. Given $x \in \mathrm{Free}_F$ (see §6.3 for notation), write $x = aI + \sum_{\ell=1}^m b_\ell p_{w_\ell} + \sum_{\ell=1}^n c_\ell q_{w_\ell}$ with $a, b_\ell, c_\ell, \in \mathbf{N}$ and $w_1, \ldots, w_m \in W_F$. We specify that

$$x(\varphi_w^{D_F} \mid w \in W_F) := \quad \Big(g_{(F,0,0)}(a\mathbf{1})\Big) \oplus \Big(\textstyle\bigoplus_{\ell=1}^m \bigoplus_{j=1}^{b_\ell} g_{(F,j,\ell)}(w_\ell)\Big) \oplus \Big(\bigoplus_{\ell=1}^m \bigoplus_{j=1}^{c_\ell} K_{(F,j,\ell)}(w_\ell)\Big)\,. \tag{14}$$

We observe that this is an internal direct sum inside $D_F^{(\mathbf{N}\times\mathbf{N}\times\mathbf{N})}$ and that, for each $i \in F \cup \{\infty\}$, $x(\varphi_w^{D_F} \mid w \in W_F)e_i^F$ has a canonical k-basis consisting of the elements

$$\begin{aligned}
&v_{(0,0,t)}e_i \quad (t = 1, \ldots, a),\\
&v_{(j,\ell,t)}e_i \quad (\ell = 1, \ldots, m,\ j = 1, \ldots, b_\ell,\ t = 1, \ldots, (w_\ell)_i), \quad \text{and}\\
&v_{(j,\ell,t)}e_i \quad (\ell = 1, \ldots, m,\ j = 1, \ldots, c_\ell,\ t = (w_\ell)_i + 1, \ldots, n_{w_\ell}).
\end{aligned}$$

Therefore we have, for each $i \in F \cup \{\infty\}$,

$$\dim_k \Big(x(\varphi_w^{D_F} \mid w \in W_F)e_i^F\Big) = a + \sum_{\ell=1}^m \sum_{j=1}^{b_\ell} (w_\ell)_i + \sum_{\ell=1}^m \sum_{j=1}^{b_\ell} (n_{w_\ell} - (w_\ell)_i).$$

Referring to §6.3, we see that, for each $i \in F \cup \{\infty\}$, the k-dimension of $x(\varphi_w^{D_F} \mid w \in W_F)e_i^F$ is precisely $(T_F(x))_i$, the i^{th} coordinate of $T_F(x)$ in M_F.

Suppose now that $(x,y) \in A_F$ (notation in §6.3). Then $T_F(x) = T_F(y)$, and by the last paragraph see that $x(\varphi_w^{D_F} \mid w \in W_F)e_i^F$ and $y(\varphi_w^{D_F} \mid w \in W_F)e_i^F$ have the same k-dimension for each $i \in F \cup \{\infty\}$. Therefore, for each $i \in F\cup\{\infty\}$, there is a unique order-preserving bijection from the canonical k-basis of $x(\varphi_w^{D_F} \mid w \in W_F)e_i^F$ to that of $y(\varphi_w^{D_F} \mid w \in W_F)e_i^F$. This bijection induces a unique k-linear isomorphism from $x(\varphi_w^{D_F} \mid w \in W_F)e_i^F$ onto $y(\varphi_w^{D_F} \mid w \in W_F)e_i^F$, and, taking the direct sum of these isomorphisms over all $i \in F\cup\{\infty\}$, we obtain the desired D_F-isomorphism $\psi_{(x,y)}^{D_F} : x(\varphi_w^{D_F} \mid w \in W_F) \to y(\varphi_w^{D_F} \mid w \in W_F)$.

Applying Lemma 3.4, with D_F playing the role of the ring T in the lemma, we obtain a unique homomorphism $\xi_F : R_F \to D_F$ of $k\langle Z_F\rangle$-rings$_k$ satisfying the universal property promised by the lemma. Since the map $\omega_F\colon k\langle Z_F\rangle \to D_F$ is surjective, ξ_F is surjective as well by (2).

Next we prove that the isomorphisms $\psi^{D_F}_{(x,y)}$ are compatible with the extensions $D_{F'} \subseteq D_F$, for $F' \subseteq F \in \mathcal{F}(\Omega)$. Suppose $(x,y) \in A_{F'}$ and $F' \subseteq F \in \mathcal{F}(\Omega)$. (Then x and y are non-zero elements of $\mathrm{Free}_{F'}$.) We shall prove that $\psi^{D_F}_{(x,y)}\colon x(\varphi^{D_F}_w \mid w \in W_F) \to y(\varphi^{D_F}_w \mid w \in W_F)$ coincides with $\psi^{D_{F'}}_{(x,y)} \otimes_{D_{F'}} D_F$. Observe that $x(\varphi^{D_{F'}}_w \mid w \in W_{F'}) \otimes_{D_{F'}} D_F = x(\varphi^{D_{F'}}_w \mid w \in W_{F'})D_F = x(\varphi^{D_F}_w \mid w \in W_F)$, by equations (9) and (11); and a similar identification holds for y. (In formula (14), and in the analogous expression for $y(\varphi^{D_F}_w \mid w \in W_F)$, the elements $w_1, \ldots, w_m$ are all in $W_{F'}$, since $x, y \in \mathrm{Free}_{F'}$.) To prove that the two maps coincide, it will suffice to check that the restrictions to $x(\varphi^{D_F}_w \mid w \in W_F)e^F_i$ of the map $\psi^{D_F}_{(x,y)}$ and of the map induced by $\psi^{D_{F'}}_{(x,y)}$ agree for each $i \in F \cup \{\infty\}$.

The first map is induced by an order-preserving bijection between the canonical k-bases of $x(\varphi^{D_F}_w \mid w \in W_F)e^F_i$ and $y(\varphi^{D_F}_w \mid w \in W_F)e^F_i$, while the second is induced by an order-preserving bijection between the canonical k-bases of $x(\varphi^{D_{F'}}_w \mid w \in W_{F'})e^{F'}_i$ and $y(\varphi^{D_{F'}}_w \mid w \in W_{F'})e^{F'}_i$. If $i \in F'$ then $x(\varphi^{D_{F'}}_w \mid w \in W_{F'})e^{F'}_i$ and $x(\varphi^{D_F}_w \mid w \in W_F)e^F_i$ have exactly the *same* canonical k-bases (and similarly for y), so of course the two maps are the same. If, on the other hand, $i \in F \cup \{\infty\} - F'$, we have an order-preserving bijection $v \mapsto ve^F_i$ from the standard basis of $x(\varphi^{D_{F'}}_w \mid w \in W_{F'})e^{F'}_\infty$ to that of $x(\varphi^{D_F}_w \mid w \in W_F)e^F_i$ (and similarly for y), and hence the two maps agree in this case as well. We have proved that

$$\psi^{D_F}_{(x,y)} = \psi^{D_{F'}}_{(x,y)} \otimes_{D_{F'}} D_F, \tag{15}$$

whenever $F' \subseteq F \in \mathcal{F}(\Omega)$ and $(x,y) \in A_{F'}$.

Still assuming that $F' \subseteq F \in \mathcal{F}(\Omega)$, we apply Lemma 3.4 with $R = k\langle Z_{F'}\rangle$, $S = R_{F'}$ and $T = R_F$. We obtain a unique homomorphism $R_{F'} \to R_F$ of R-rings$_k$ such that $\varphi^{R_F}_w = \varphi^{R_{F'}}_w \otimes_{R_{F'}} R_F$ for all $w \in W_{F'}$, and such that $\psi^{R_F}_{(x,y)} = \psi^{R_{F'}}_{(x,y)} \otimes_{R_{F'}} R_F$ for all $(x,y) \in A_{F'}$. From the universal property of the maps $R \to R_F$ and the compatibility conditions (13) and (15), we deduce that the surjections $\xi_F : R_F \to D_F$ are compatible with inclusions. That is, if $F' \subseteq F \in \mathcal{F}(\Omega)$, the following diagram commutes:

$$\begin{array}{ccc} R_{F'} & \xrightarrow{\xi_{F'}} & D_{F'} \\ \downarrow & & \downarrow \\ R_F & \xrightarrow{\xi_F} & D_F. \end{array} \tag{16}$$

Applying the universal property again, we see that the k-algebras R_F with the maps $R_{F'} \to R_F$ form a direct system of k-algebras, and that the ξ_F give a surjective homomorphism of this direct system of k-algebras onto the direct system of k-algebras $\{\, D_F \mid F \in \mathcal{F}(\Omega) \,\}$.

§6.7 The isomorphisms $f_F \colon M_F \to \mathrm{V}(\text{proj-}R_F)$

Let $F \in \mathcal{F}(\Omega)$. By [8, Corollary 2.4.3], every right or left ideal of $k\langle Z_F \rangle$ is free; in particular, $k\langle Z_F \rangle$ is hereditary. By [8, Corollary 1.4.2], every right or left projective $k\langle Z_F \rangle$-module is free. In particular, $\mathrm{V}(\text{proj-}k\langle Z_F \rangle) \cong \mathbf{N}$ via the homomorphism that maps $[k\langle Z_F \rangle]$ to 1. In view of Lemma 6.4 and the presentation of the monoid $\mathrm{V}(\text{proj-}R_F)$ given in Lemma 3.4, we see that there is a unique isomorphism of monoids $f_F \colon M_F \to \mathrm{V}(\text{proj-}R_F)$ taking w to $[\mathrm{Im}(\varphi_w^{R_F})]$ and u_w to $[\ker(\varphi_w^{R_F})]$ (because the elements $(p_w + q_w, n_w I)$ are already in the congruence $\sim_F$ generated by A_F). These isomorphisms are compatible with the inclusion maps in $\mathcal{F}(\Omega)$. Putting everything together, we obtain, for $F' \subseteq F \in \mathcal{F}(\Omega)$, the following commutative diagram of monoid homomorphisms:

$$\begin{array}{ccccccc}
\mathrm{V}(\text{proj-}R_{F'}) & & & \longrightarrow & & & \mathrm{V}(\text{proj-}D_{F'}) \\
& {\scriptstyle f_{F'}} \nwarrow \cong & & & & {\scriptstyle h_{F'}} \nearrow \cong & \\
& & M_{F'} & \longrightarrow & C_{F'} & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & M_F & \longrightarrow & C_F & & \\
& {\scriptstyle f_F} \swarrow \cong & & & & {\scriptstyle h_F} \searrow \cong & \\
\mathrm{V}(\text{proj-}R_F) & & & \longrightarrow & & & \mathrm{V}(\text{proj-}D_F).
\end{array} \tag{17}$$

The arrows in the outer square are the homomorphisms induced by the k-algebra homomorphisms in (16), and the inner square is (5).

§6.8 Localizing the k-algebras R_F

For $F \in \mathcal{F}(\Omega)$, let Σ_F be the set of square matrices over R_F that become invertible over D_F (via the surjective map ξ_F). Let $\lambda_F \colon R_F \to (R_F)_{\Sigma_F}$ be the universal Σ_F-inverting R_F-ring (§3.5), and let $\eta_F \colon (R_F)_{\Sigma_F} \to D_F$ be the unique homomorphism (necessarily a surjection) such that $\eta_F \lambda_F = \xi_F$. By Lemma 3.6, $J((R_F)_{\Sigma_F}) = \ker(\eta_F)$. Thus $(R_F)_{\Sigma_F}/J((R_F)_{\Sigma_F}) \cong D_F$ is a semisimple Artinian ring, and $(R_F)_{\Sigma_F}$ is hereditary by [3, Theorem 5.3]. The monoid homomorphism $\mathrm{V}(\text{proj-}\xi_F) \colon \mathrm{V}(\text{proj-}R_F) \to \mathrm{V}(\text{proj-}D_F)$ factors as $\mathrm{V}(\text{proj-}\xi_F) = \mathrm{V}(\text{proj-}\eta_F) \circ \mathrm{V}(\text{proj-}\lambda_F)$. By [13, Theorem 5.1], $\mathrm{V}(\text{proj-}\lambda_F)$ is an isomorphism for every F.

Given $F' \subseteq F \in \mathcal{F}(\Omega)$, we have a commutative rectangle

$$\begin{array}{ccccc} R_{F'} & \xrightarrow{\lambda_{F'}} & (R_{F'})_{\Sigma_{F'}} & \xrightarrow{\eta_{F'}} & D_{F'} \\ \downarrow & & ?\downarrow ? & & \downarrow \subseteq \\ R_F & \xrightarrow{\lambda_F} & (R_F)_{\Sigma_F} & \xrightarrow{\eta_F} & D_F, \end{array} \tag{18}$$

and we want to fill in the questionable arrow to obtain two commutative squares. If $\alpha \in \Sigma_{F'}$, the image of α in $D_{F'}$ is invertible, and therefore so is its image in D_F. It follows that the image of α in $(R_F)_{\Sigma_F}$ (obtained by passing through R_F) is already invertible. Thus the map $R_{F'} \to (R_F)_{\Sigma_F}$ inverts all matrices in $\Sigma_{F'}$, and by the universal property of the map $\lambda_{F'}$ there is a unique arrow making the left-hand square in (18) commute. Commutativity of the right-hand square now follows from the uniqueness of the factorization of the $\Sigma_{F'}$-inverting map $R_{F'} \to D_F$ through the universal $\Sigma_{F'}$-inverting map $R_{F'} \to (R_{F'})_{\Sigma_{F'}}$. Thus we have a natural commutative diagram

$$\begin{array}{ccc} (R_{F'})_{\Sigma_{F'}} & \xrightarrow{\eta_{F'}} & D_{F'} \\ \downarrow & & \downarrow \subseteq \\ (R_F)_{\Sigma_F} & \xrightarrow{\eta_F} & D_F, \end{array} \tag{19}$$

in which the horizontal arrows are surjective k-algebra homomorphisms. A routine argument (another application of the universal property of the maps λ_F) shows that the k-algebras $(R_F)_{\Sigma_F}$ with the maps $(R_{F'})_{\Sigma_{F'}} \to (R_F)_{\Sigma_F}$ form a direct system of k-algebras, and that the η_F give a homomorphism of this direct system of k-algebras into the direct system of k-algebras $\{ D_F \mid F \in \mathcal{F}(\Omega) \}$. As $\mathrm{V}(\text{proj-}\lambda_F)$ is an isomorphism for every F, diagram (17) becomes

$$\begin{array}{ccccccc} \mathrm{V}(\text{proj-}(R_{F'})_{\Sigma_{F'}}) & & & \longrightarrow & & & \mathrm{V}(\text{proj-}D_{F'}) \\ & \nwarrow \cong & & & & {}_{h_{F'}} \nearrow \cong & \\ & & M_{F'} & \longrightarrow & C_{F'} & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & M_F & \longrightarrow & C_F & & \\ & \swarrow \cong & & & & {}_{h_F} \searrow \cong & \\ \mathrm{V}(\text{proj-}(R_F)_{\Sigma_F}) & & & \longrightarrow & & & \mathrm{V}(\text{proj-}D_F). \end{array} \tag{20}$$

§6.9 Taking the direct limit

Taking the direct limits of diagrams (19) and (20), we get a surjective k-algebra homomorphism $\pi\colon \varinjlim_F (R_F)_{\Sigma_F} \to \bigcup_F D_F$ and a commutative diagram

$$\begin{array}{ccc} \bigcup_F M_F & \longrightarrow & \bigcup_F C_F \\ \downarrow \cong & & \downarrow \cong \\ \varinjlim_F \mathrm{V}(\text{proj-}(R_F)_{\Sigma_F}) & \longrightarrow & \varinjlim_F \mathrm{V}(\text{proj-}D_F). \end{array}$$

Set $R = \varinjlim_F (R_F)_{\Sigma_F}$. Recall (6) that $\bigcup_F M_F = (G \oplus \mathbf{1Z}) \cap \mathbf{N}^\Omega$ and $\bigcup_F C_F = C := \mathbf{N}^{(\Omega)} \oplus \mathbf{1N}$. Also, $\bigcup_F D_F = D := k^{(\Omega)} \oplus k \cdot 1_{k^\Omega}$ (see §6.5). As D is von Neumann regular and $\pi\colon R \to D$ is surjective, $J(R) \subseteq \ker \pi$. For the reverse inclusion, if $x \in \ker \pi = \ker \varinjlim_F \eta_F$, then x is the image of an x' in $\ker \eta_F$ for some F, so that $x' \in J((R_F)_{\Sigma_F})$. Hence $1 + x'$ is invertible in $(R_F)_{\Sigma_F}$, thus $1 + x$ is invertible in R. Therefore the inclusion $\ker \pi \subseteq J(R)$ holds as well, and $J(R) = \ker \pi$. Using Proposition 3.7, we obtain a commutative diagram of monoids and monoid homomorphisms

$$\begin{array}{ccc} (G \oplus \mathbf{1Z}) \cap \mathbf{N}^I & \overset{\subseteq}{\longrightarrow} & C \\ \downarrow \cong & & {\scriptstyle h} \downarrow \cong \\ \mathrm{V}(\text{proj-}R) & \overset{\mathrm{V}(\text{proj-}\pi)}{\longrightarrow} & \mathrm{V}(\text{proj-}D). \end{array}$$

Since $\pi\colon R \to D$ is a surjective homomorphism with kernel $J(R)$, one has that $\mathrm{End}(P_R)/J(\mathrm{End}(P_R)) \cong \mathrm{End}_D(P \otimes_R D)$ for every $P_R \in \text{proj-}R$ (Lemma 2.3); therefore $P_R \in \mathcal{S}_R$ if and only if $P \otimes_R D \in \mathcal{S}_D$. By Lemma 4.4, the projective D-modules with a semilocal endomorphism ring are those corresponding, via the isomorphism $h\colon C \to \mathrm{V}(\text{proj-}D)$, to the elements of $\mathbf{N}^{(\Omega)}$. Since $(G \oplus \mathbf{1Z}) \cap \mathbf{N}^\Omega \cap \mathbf{N}^{(\Omega)} = G \cap \mathbf{N}^{(\Omega)} = M$, it follows that we have a commutative diagram

$$\begin{array}{ccc} M & \overset{T}{\longrightarrow} & \mathbf{N}^{(\Omega)} \\ \downarrow \cong & & \downarrow \cong \\ V(\mathcal{S}_R) & \longrightarrow & \mathrm{V}(\mathcal{S}_D), \end{array}$$

and the proof of the main theorem is complete.

It is interesting to speculate on what further restrictions there might be on the homomorphisms $\mathrm{V}(\mathcal{S}_R) \to \mathrm{V}(\mathcal{S}_{R/J(R)})$ if one assumes that R is commutative. The referee has pointed out that the ring in §4.3 could be replaced by a commutative ring. That is, given any set Ω one can find a commutative ring R such that $\mathrm{V}(\mathcal{S}_R) = \{(0)\}$ but $\mathrm{V}(\mathcal{S}_{R/J(R)}) \cong \mathbf{N}^{(\Omega)}$.

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