# SEMIGROUPS OF MODULES: A SURVEY

ROGER WIEGAND AND SYLVIA WIEGAND

*This paper is dedicated to Carl Faith and Barbara Osofsky, who continue to inspire, enlighten and energize us, as they have for over forty years.*

## 1. Introduction

This paper is a survey of some recent developments in the study of direct-sum decompositions of modules. The idea is to examine direct-sum behavior by looking at semigroups of isomorphism classes of modules. To be precise, let $R$ be a ring, and let $\mathcal{C}$ be a class of modules closed under isomorphism, finite direct sums, and direct summands. Assume there is a set $\mathsf{V}(\mathcal{C}) \subseteq \mathcal{C}$ of representatives; thus each element $M \in \mathcal{C}$ is isomorphic to exactly one element $[M] \in \mathsf{V}(\mathcal{C})$. We make $\mathsf{V}(\mathcal{C})$ into an additive semigroup in the obvious way: $[M] + [N] = [M \oplus N]$. This semigroup encodes information about the way modules in $\mathcal{C}$ decompose as direct sums of other modules in $\mathcal{C}$.

Given a module $M$, we are particularly interested in the class $+(M)$, which consists of modules that are isomorphic to direct summands of direct sums of finitely many copies of $M$. We will see, for example, that if $R$ is a (commutative Noetherian) local ring and $M$ is a finitely generated $R$-module, then $\mathsf{V}(+(M))$ is a finitely generated *Krull* semigroup (defined in Definition 2.4). In the terminology associated with convex polytopes [5], $\mathsf{V}(+(M))$ is a *positive normal affine* semigroup. (See definitions and various characterizations in §2.) Conversely, we have a "realization theorem": given any finitely generated Krull semigroup $H$, there is a one-dimensional local domain $R$ with a finitely generated $R$-module $M$ such that $\mathsf{V}((+M)) \cong H$. These two results show exactly how badly Krull-Schmidt uniqueness can fail for direct-sum decompositions of finitely generated modules over local rings. For example, using the realization theorem, one can find an indecomposable finitely generated module $M$ over a local ring such that $M, M^{(2)}, \dots, M^{(4)}$ have only the obvious direct-sum decompositions, whereas $M^{(5)} \cong V \oplus W$, where $V$ and $W$ are non-isomorphic indecomposables. On the other hand, using the fact that $\mathsf{V}(+(M))$ is a Krull semigroup, one can show, for example, that there is no local ring $R$ with indecomposable finitely generated modules $M$ and $N$ such that $M^{(2)} \cong N^{(3)}$. More subtle restrictions are mentioned in Section 9.

*Date*: July 29, 2008.

Similarly, $\mathsf{V}(+(M))$ is a finitely generated Krull semigroup whenever $M$ is an Artinian right $R$-module (over any ring $R$), and again there is a realization theorem: Every finitely generated semigroup arises in this way.

In Sections 7 and 8 of the paper we discuss non-finitely generated Krull semigroups and realization theorems.

## 2. Krull Semigroups

Here we recall some of the basic terminology in the theory of Krull semigroups. All of our semigroups are written additively and have a neutral element 0. Moreover, we assume always that semigroups are

(1) *commutative* ($x + y = y + x$),
(2) *cancellative* ($x + y = x + z \implies y = z$), and
(3) *reduced* ($x + y = 0 \implies x = y = 0$).

Of course, all of these properties except (2) are automatically satisfied for the semigroups $\mathsf{V}(\mathcal{C})$, and we impose appropriate conditions on the rings and modules to ensure that cancellation holds.

The theory of (commutative, cancellative) semigroups, often called *monoids*, has developed along two almost disjoint paths: first, as part of a successful campaign to divorce the multiplicative theory of integral domains from the underlying additive group [8], [22], [7]; and second, in the remarkable applications of commutative algebra to problems in combinatorics and topology [34, Chapter 1], [5, §6]. The present survey concerns yet another, more recent development—the use of semigroups in module theory. The first two developments tend to use different terminology, and we shall use a blend, occasionally pointing out how concepts from one school fit into the other.

For elements $x$ and $y$ in a semigroup $H$, we say $x$ *divides* $y$, and we write $x \mid y$, provided there is an element $z \in H$ such that $x+z = y$. If necessary, e.g., as in the following definition, we write $x \mid_H y$.

**Definition 2.1.** A semigroup homomorphism $\varphi : H \to K$ is a *divisor* homomorphism provided, for $x, y \in H$, we have $x \mid_H y \iff \varphi(x) \mid_K \varphi(y)$. If $H$ is a subsemigroup of $K$, we say $H$ is a *full* subsemigroup of $K$ provided the inclusion map is a divisor homomorphism, that is, for all $x, y \in H$ we have $x \mid_H y \iff x \mid_K y$.

Since all of our semigroups are reduced and cancellative, divisor homomorphisms are always injective, and thus a divisor homomorphism $H \to K$ embeds $H$ as a full subsemigroup of $K$.

**The free semigroup 2.2.** Let $\mathbb{N}_0$ denote the semigroup of non-negative integers. For an index set (or natural number) $b$, we denote by $\mathbb{N}_0^{(b)}$ the direct sum of $b$ copies of $\mathbb{N}_0$. Thus $\mathbb{N}_0^{(b)}$ is the *free semigroup with basis* $b$. In $\mathbb{N}_0^{(b)}$ we have $x \mid y$ if and only if $x \leq y$ relative to the coordinatewise partial ordering: If $z_i$ denotes the $i^{\text{th}}$ coordinate of an element $z \in \mathbb{N}_0^{(b)}$, then $x \leq y \iff x_i \leq y_i$ for all $i \in b$. Therefore, if $H$ is a full subsemigroup of $\mathbb{N}_0^{(b)}$, we have $x \mid_H y \iff x_i \leq y_i$ for each $i$.

**Examples 2.3.** (1) The subsemigroup $H := \{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{N}_0^{(2)} \mid x \equiv y \pmod 3\}$ is a full subsemigroup of $\mathbb{N}_0^{(2)}$.

(2) Similarly, the subsemigroup $\Gamma$ of $\mathbb{N}_0^{(3)}$ consisting of all the triples $(x, y, z)$ satisfying $x + 4y = 5z$ is a full subsemigroup of $\mathbb{N}_0^{(3)}$.

(3) Examples 2.3.1 and 2.3.2 are instances of the following: If $G$ is an abelian group and $\varphi : \mathbb{N}_0^{(b)} \to G$ is a semigroup homomorphism, then $\varphi^{-1}(0_G)$ is a full subsemigroup of $\mathbb{N}_0^{(b)}$.

**Definition 2.4.** A *Krull* semigroup is a semigroup $H$ admitting a divisor homomorphism $H \to \mathbb{N}_0^{(b)}$ for some (possibly infinite) set $b$. Equivalently, $H$ is isomorphic to a full subsemigroup of a free semigroup.

We sometimes consider a stronger condition on subsemigroups of $\mathbb{N}_0^{(b)}$:

**Definition 2.5.** A subsemigroup $H$ of $\mathbb{N}_0^{(b)}$ is an *expanded* subsemigroup of $\mathbb{N}_0^{(b)}$ provided $H = \mathbb{N}_0^{(b)} \cap L$ for some $\mathbb{Q}$-subspace $L$ of $\mathbb{Q}^{(b)}$.

If $t$ is a positive integer, every $\mathbb{Q}$-subspace $L$ of $\mathbb{Q}^{(t)}$ can be represented as the kernel of a matrix over $\mathbb{Q}$ (take the matrix with a basis for $L^{\perp}$ as the rows). Thus a subsemigroup $H$ of $\mathbb{N}_0^{(t)}$ is an expanded subsemigroup of $\mathbb{N}_0^{(t)}$ if and only if $H$ is the set of non-negative integer solutions to a finite system of homogeneous linear equations over $\mathbb{Z}$. Semigroups arising in this way are sometimes called *diophantine* semigroups.

We pay particular attention to *finitely generated* Krull semigroups. These turn out to be the same gadgets as the positive normal affine semigroups studied in [5]. We review the terminology here. A *positive affine semigroup* is a semigroup isomorphic to a finitely generated subsemigroup of $\mathbb{N}_0^{(t)}$ for some positive integer $t$. (The expression "positive semigroup" means that $x + y = 0$ only if $x = y = 0$, a property that we have tacitly attributed to all semigroups.) The positive affine semigroup $H$ is *normal* provided, for each $x, y \in H$, if $nx \mid ny$ for some positive integer $n$, then $x \mid y$. Equivalently, let $G$ denote the group of differences of $H$ (i.e., the smallest group containing $H$); then $H$ is normal if and only if, for every $g \in G$ and every positive integer $n$, $ng \in H$ implies that $g \in H$.

Recall that a *clutter* (or *antichain*) is a subset $S$ of a poset such that no two elements of $S$ are comparable. A key property of the poset $\mathbb{N}_0^{(t)}$ is that every clutter is finite, cf. [9, Theorem 9.18]. A proof using the Hilbert Basis Theorem can be found in the first chapter of Stanley's book [34]. An alternate approach is given in the next lemma, whose proof is left to the reader:

**Lemma 2.6.** *Consider the following property of a partially ordered set $X$:*

(†) *$X$ has the descending chain condition and every clutter in $X$ is finite.*

*If $X$ and $Y$ are posets satisfying (†), then $X \times Y$ (with coordinate-wise partial ordering) satisfies (†). In particular, since $\mathbb{N}_0$ satisfies (†), so does $\mathbb{N}_0^{(t)}$ for every positive integer $t$.*

**Proposition 2.7.** *The following conditions on a semigroup $H$ are equivalent:*

1. *$H$ is a positive normal affine semigroup.*
2. *$H$ is isomorphic to a full subsemigroup of $\mathbb{N}_0^{(t)}$ for some $t \geq 1$.*
3. *$H$ is isomorphic to an expanded subsemigroup of $\mathbb{N}_0^{(u)}$ for some $u \geq 1$.*
4. *$H$ is a finitely generated Krull semigroup.*
5. *$H \cong G \cap \mathbb{N}_0^{(t)}$ for some $t \geq 1$ and some subgroup $G$ of $\mathbb{Z}^{(t)}$.*

*Proof.* (1) $\implies$ (2) by [5, Exercise 6.1.10, p. 263]. If $H$ is a full subsemigroup of $\mathbb{N}_0^{(t)}$ and $G$ is the subgroup of $\mathbb{Z}^{(t)}$ generated by $H$, it is easy to see that $H = G \cap \mathbb{N}_0^{(t)}$. Therefore (2) implies (5).

Assume (5). Then $H$ is a Krull semigroup, and, by Lemma 2.6, $H$ is generated by its minimal non-zero elements, which are finite in number. Thus (5) implies (4). (Cf. [9, Corollary 9.19].)

To see that (4) implies (1), suppose $H$ is a full subsemigroup of $\mathbb{N}_0^{(b)}$ and that $H$ is finitely generated. There is a finite subset $t$ of $b$ such that $h \subseteq \mathbb{N}_0^{(t)}$. This shows that $H$ is a positive affine semigroup. Since $H$ is a full subsemigroup of $\mathbb{N}_0^{(t)}$, it follows that $H$ is normal.

We have shown that (1), (2), (4) and (5) are equivalent. Obviously (3) implies (2). A proof that (1) implies (3), attributed to Hochster, is outlined in [5, Exercise 6.4.16, p. 290]. □

Here is a simple example that might give the reader some indication of why (2) implies (3). The semigroup $H := \{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{N}_0^{(2)} \mid x \equiv y \pmod 3\}$ of (2.3.1) is *not* the restriction to $\mathbb{N}_0^{(2)}$ of the kernel of a matrix. However, $H$ is isomorphic to $H_1 := \{\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in \mathbb{N}_0^{(3)} \mid x + 2y = 3z\}$.

We remark that for $t > 1$ the semigroup $\mathbb{N}_0^{(t)}$ has subsemigroups that are not finitely generated, cf. [9, Ex. 12, p. 137]. On the other hand, all finitely generated commutative semigroups are finitely presented, cf. [9, Theorem 9.28].

We now return to the examples and properties of Krull semigroups alluded to in §1.

**Atoms and decompositions 2.8.** Recall that an *atom* of a semigroup is a non-zero element that cannot be written as a sum of two non-zero elements. If $H$ is a full subsemigroup of $\mathbb{N}_0^{(b)}$, the atoms of $H$ are just the minimal non-zero elements, with respect to the coordinatewise ordering on $\mathbb{N}_0^b$. For example, the atoms of the semigroup $\Gamma$ of (2.3.2) are $a := (1,1,1), b := (5,0,1)$ and $c := (0,5,4)$. For $n \le 4$, the decomposition of $na$ as a sum of atoms is unique, whereas $5a = b + c$.

**Remark 2.9.** In a Krull semigroup $H$ one cannot have atoms $x$ and $y$ such that $2x = 3y$. To see this, we may assume that $H$ is a full subsemigroup of some free semigroup $\mathbb{N}_0^{(b)}$. Such a relation would imply that $y \le x$ in the coordinatewise ordering on $\mathbb{N}_0^{(b)}$. But then $y \mid x$ and, since $x$ is an atom, $y$ must equal $x$. Then $2x = 3x$, and cancellation implies that $x = 0$, a contradiction.

## 3. Projective modules over semilocal rings

For folks in the non-commutative world, a ring $R$ is *semilocal* provided $R/\operatorname{J}(R)$ is semisimple Artinian, where $\operatorname{J}(R)$ is the Jacobson radical. In this section we adhere to that terminology. Let us denote by $\operatorname{mod}(R)$, respectively, $\operatorname{pmod}(R)$ the class of finitely generated right $R$-modules, respectively, finitely generated projective right $R$-modules. The function $\operatorname{pmod}(R) \longrightarrow \operatorname{pmod}(R/\operatorname{J}(R))$ taking $P$ to $P/P\operatorname{J}(R)$ gives a semigroup homomorphism $\mathsf{V}(\operatorname{pmod}(R)) \xrightarrow{\iota} \mathsf{V}(\operatorname{pmod}(R/\operatorname{J}(R)))$. (Of course $\operatorname{pmod}(R/\operatorname{J}(R)) = \operatorname{mod}(R/\operatorname{J}(R))$.) In fact, $\iota$ is a divisor homomorphism [20]. For, suppose $P, Q \in \operatorname{pmod}R$ and $[P/P\operatorname{J}(R)] \mid [Q/Q\operatorname{J}(R)]$. There is then a surjective homomorphism $f : Q/Q\operatorname{J}(R) \twoheadrightarrow P/P\operatorname{J}(R)$. We can lift $f$ to a homomorphism $g : Q \to P$. By Nakayama's lemma, $g$ is surjective, and thus $[P] \mid [Q]$.

Now every finitely generated $R/\operatorname{J}(R)$-module is uniquely (up to isomorphism) a direct sum of simple modules, and thus $\mathsf{V}(\operatorname{pmod}R/\operatorname{J}(R))$ is the free semigroup generated by the finitely many isomorphism classes of minimal right ideals of $R$. By Proposition 2.7, $\mathsf{V}(\operatorname{pmod}R)$ is a finitely generated Krull semigroup.

The main theorem of [16] says that every finitely generated Krull semigroup can be realized in this way. In fact, every full embedding $H \hookrightarrow \mathbb{N}_0^{(t)}$ can be realized as the map $\mathsf{V}(\operatorname{pmod}(R)) \xrightarrow{\iota} \mathsf{V}(\operatorname{pmod}(R/\operatorname{J}(R)))$, as long as some element of $H$ maps to an element $x = (x_1 \dots x_t) \in \mathbb{N}_0^{(t)}$ with each $x_i > 0$. We call such an element of $\mathbb{N}_0^{(t)}$ *strictly positive.* (Notice that $\iota([R])$ is strictly positive, since every simple module occurs as a direct summand of $R/\operatorname{J}(R)$.) The next remark shows that we can always assume that a given subsemigroup $H$ of $\mathbb{N}_0^{(t)}$ contains a strictly positive element. We can, if necessary, replace the embedding $H \hookrightarrow \mathbb{N}_0^{(t)}$ with the composite $H \hookrightarrow \mathbb{N}_0^{(t)} \overset{\pi}{\twoheadrightarrow} \mathbb{N}_0^{(b)}$, where $b$ is a proper subset of $\{1, \dots, t\}$ and $\pi$ is the projection.

**Remark 3.1.** *Suppose $H$ is a subsemigroup of $\mathbb{N}_0^{(t)}$. Then there exists a subset $b \subseteq \{1, \dots, t\}$ such that*

(1) *If* $(g_1, \dots, g_t) \in H$ *and* $i \notin b$*, then* $g_i = 0$*; and*
(2) *There is an element* $(h_1, \dots, h_t) \in H$ *such that* $h_i > 0$ *for all* $i \in b$*.*

*Proof.* Choose an element $h \in H$ with the largest number of positive coordinates, and let $b = \{i \mid h_i > 0\}$. If there were an element $g \in H$ violating (1), then $g + h$ would have more positive coordinates that $h$, a contradiction. □

**Theorem 3.2** (Facchini and Herbera [16])**.** *Let* $H$ *be a finitely generated Krull semigroup, and let* $H \overset{\varphi}{\hookrightarrow} \mathbb{N}_0^{(t)}$ *be a full embedding. Assume there is an element* $h \in H$ *such that* $\varphi(h)$ *is strictly positive. Let* $k$ *be any field. Then there exist a semilocal, right hereditary* $k$*-algebra* $R$ *and a commutative diagram of semigroups*

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & \mathbb{N}_0^{(t)} \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow \\ \mathsf{V}(\mathrm{pmod}(R)) & \xrightarrow{\iota} & \mathsf{V}(\mathrm{pmod}(R/\,\mathrm{J}(R))) \end{array}$$

*with the following properties:*

(1) $\alpha$ *and* $\beta$ *are isomorphisms.*
(2) $\alpha(h) = [R]$*.*

The proof of this result is very intricate and uses some surprising constructions due to George Bergman [4] and a localization technique due to P. M. Cohn [10]. We sketch the proof of a similar result, Theorem 5.1, in §5. We close this section with an amusing application from [16].

**Example 3.3** ([16, Example 5.8])**.** *Let* $r$ *and* $s$ *be arbitrary integers, both greater than* 1*. There is a semilocal, right hereditary ring* $R$ *with the following properties:*

(1) $R^{(s)}$ *has, up to isomorphism, exactly two decompositions as a direct sum of indecomposable right* $R$*-modules: the obvious one, as a direct sum of* $s$ *copies of* $R$*; and as a direct sum of* $r$ *pairwise non-isomorphic indecomposable modules.*
(2) *For each natural number* $t < s$*,* $R^{(t)}$ *has only the obvious decomposition.*

*Proof.* Consider the subsemigroup $H$ of $\mathbb{N}_0^{(r)}$ generated by $h := (1, 1, \dots, 1)$, $g_1 := (s, 0, \dots, 0)$, $g_2 := (0, s, 0, \dots, 0)$, $\dots$, $g_r := (0, \dots, 0, s)$. One can check that $H$ is a full subsemigroup of $\mathbb{N}_0^{(r)}$ (cf. [16, Lemma 5.5]) and that $\{h, g_1, g_2, \dots g_r\}$ is the set of atoms of $H$. Notice that $sh = g_1 + \cdots + g_r$ and that if $1 \le t < s$ then $th$ has only the obvious decomposition as a sum of atoms. Applying Theorem 3.2 to the inclusion $\varphi : H \to \mathbb{N}_0^{(r)}$, we obtain an isomorphism $H \to \mathsf{V}(\mathrm{pmod}(R))$ taking the strictly positive element $h$ to $[R]$. The desired relations and non-relations in $\mathsf{V}(\mathrm{pmod}(R))$ then follow from those in $H$. □

## 4. Finitely generated modules over commutative local rings

Let's switch gears completely. In this section, "local ring" means "commutative Noetherian local ring". If $R$ is local with maximal ideal $\mathfrak{m}$, we let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of $R$. Since direct-sum cancellation holds for finitely generated modules over local rings [13], we can consider the semigroup $\mathsf{V}(\mathrm{mod}(R))$. There is a homomorphism of semigroups $\mathsf{V}(\mathrm{mod}(R)) \xrightarrow{\iota} \mathsf{V}(\mathrm{mod}(\widehat{R}))$ taking $[M]$ to $[\widehat{M}] = [\widehat{R} \otimes_R M]$. By the Krull-Schmidt uniqueness theorem for finitely generated modules over complete local rings, $\mathsf{V}(\mathrm{mod}(\widehat{R}))$ is a free semigroup with basis $\{[V] \mid V$ is an indecomposable finitely generated $R$-module$\}$. The next result is an easy consequence of results in Guralnick's paper [21], but we give a different proof modeled after the proof of the same result for discrete valuation rings in [32]

**Proposition 4.1.** *The homomorphism $\iota : \mathsf{V}(\mathrm{mod}(R)) \longrightarrow \mathsf{V}(\mathrm{mod}(\widehat{R}))$ is a divisor homomorphism. In particular, $\mathsf{V}(\mathrm{mod}(R))$ is a Krull semigroup.*

*Proof.* (Cf. [37]) Suppose $M$ and $N$ are finitely generated $R$-modules and that $[\widehat{M}] \mid [\widehat{N}]$. Choose $\widehat{R}$-homomorphisms $\phi : \widehat{M} \to \widehat{N}$ and $\psi : \widehat{N} \to \widehat{M}$ such that $\psi\phi = 1_{\widehat{N}}$. Since $H := \mathrm{Hom}_R(M,N)$ is a finitely generated $R$-module it follows that $\widehat{H} = \widehat{R} \otimes_R H = \mathrm{Hom}_{\widehat{R}}(\widehat{M}, \widehat{N})$. Therefore $\phi$ can be approximated to any order by an element of $H$, that is, $\phi$ can be approximated modulo any given power of $\widehat{\mathfrak{m}}$. In fact, order 1 suffices: Choose $f \in \mathrm{Hom}_R(M,N)$ such that $\widehat{f} - \phi \in \widehat{\mathfrak{m}}\widehat{H}$. Similarly, there exists $g \in \mathrm{Hom}_R(N,M)$ with $\widehat{g} - \psi \in \widehat{\mathfrak{m}}\,\mathrm{Hom}_{\widehat{R}}(\widehat{N}, \widehat{M})$. Then the image of $\widehat{g}\widehat{f} - 1_{\widehat{X}}$ is in $\widehat{\mathfrak{m}}\widehat{X}$, and now Nakayama's lemma implies that $\widehat{g}\widehat{f}$ is surjective, and therefore an isomorphism. It follows that $\widehat{g}$ is a split surjection (with splitting map $(\widehat{g}\widehat{f})^{-1}$). By faithful flatness $g$ is a split surjection. ☐

**Corollary 4.2.** *Let $M$ be a finitely generated module over a local ring $R$. Then $\mathsf{V}(+(M))$ is a finitely generated Krull semigroup.*

*Proof.* Write $\widehat{M} = V_1^{(e_1)} \oplus \cdots \oplus V_t^{(e_t)}$, where the $V_i$ are pairwise non-isomorphic indecomposable $\widehat{R}$-modules and the $e_i$ are positive integers. The homomorphism $\iota$ carries $\mathsf{V}(+(M))$ into $\mathsf{V}(+(\widehat{M}))$. But $\mathsf{V}(+(\widehat{M}))$ is a free semigroup, with basis $\{[V_1], \ldots, [V_t]\}$. Thus $\mathsf{V}(+(M)$ is isomorphic to a full subsemigroup of $\mathbb{N}_0^{(t)}$. ☐

Let $\xi : \mathsf{V}(+(M) \to \mathsf{V}(+(\widehat{M}))$ be the semigroup homomorphism induced by $\iota$. Again, we have a realization theorem (cf. [38]):

**Theorem 4.3.** *Let $t$ be a positive integer, and let $H$ be an expanded subsemigroup of $\mathbb{N}_0^{(t)}$. Assume that $H$ contains a strictly positive element $h$. Then there exist a one-dimensional local domain $R$, a finitely generated torsion-free $R$-module $M$, and a commutative diagram* *of semigroups*

$$\begin{array}{ccc} H & \xrightarrow{\subseteq} & \mathbb{N}_0^{(t)} \\ \alpha\Big\downarrow & & \beta\Big\downarrow \\ \mathsf{V}(+(M)) & \xrightarrow{\xi} & \mathsf{V}(+(\widehat{M})) \end{array},$$

*where*

(1) *$\alpha$ and $\beta$ are isomorphisms, and*
(2) *$\alpha(h) = [M]$.*

*Moreover, $R$ can be chosen to be a localization of a finitely generated $\mathbb{Q}$-algebra.*

*Sketch of the proof.* The idea is to build a domain with lots of analytic branches, so that $\widehat{R}$ has lots of minimal prime ideals, say $P_1, \dots, P_s$. One also arranges things so that each branch $\widehat{R}/P_i$ has infinite Cohen-Macaulay type. A very technical construction then shows that $\widehat{R}$ has indecomposable finitely generated torsion-free modules with arbitrary rank functions. That is, if $(r_1, \dots, r_s)$ is any non-trivial $s$-tuple of non-negative integers, then there is an indecomposable finitely generated $\widehat{R}$-module $N$ such that $N_{P_i} \cong \widehat{R}_{P_i}^{(r_i)}$. Then one uses the fact, due to Levy and Odenthal [30], that a finitely generated $\widehat{R}$-module is extended from an $R$-module if and only if its rank is constant.

This last fact implies that the map $\xi$ in Theorem 4.3 above embeds $\mathsf{V}(+(M))$ as an expanded subsemigroup of $\mathsf{V}(+(\widehat{M}))$. Therefore Theorem 4.3 would be false if "expanded" were replaced by "full" in the hypotheses. In other words, we cannot use one-dimensional rings to realize a given *full* embedding. If, however, we pass to two-dimensional rings, we get a completely general realization theorem, though the ring that does the realizing is much less tractable.

**Theorem 4.4.** *Let $t$ be a positive integer, and let $H$ be a full subsemigroup of $\mathbb{N}_0^{(t)}$. Assume that $H$ contains a strictly positive element $h$. Then there exist a two-dimensional local unique factorization domain $R$, a finitely generated reflexive $R$-module $M$, and a commutative diagram of semigroups*

$$\begin{array}{ccc} H & \xrightarrow{\subseteq} & \mathbb{N}_0^{(t)} \\ \alpha\Big\downarrow & & \beta\Big\downarrow \\ \mathsf{V}(+(M)) & \xrightarrow{\xi} & \mathsf{V}(+(\widehat{M})) \end{array},$$

*where*

(1) *$\alpha$ and $\beta$ are isomorphisms, and*
(2) *$\alpha(h) = [M]$.*

*Proof.* Let $G$ be the subgroup of $\mathbb{Z}^{(t)}$ generated by $H$, and write $\mathbb{Z}^{(t)}/G = C_1 \oplus \cdots \oplus C_s$, where each $C_i$ is a cyclic group. Then $\mathbb{Z}^{(t)}/G$ can be embedded in $(\mathbb{R}/\mathbb{Z})^{(s)}$.

Choose any positive integer $d$ such that $(d-1)(d-2) \geq s$. Let $C$ be a smooth plane curve of degree $d$ in $\mathbb{P}^2_{\mathbb{C}}$, and let $A$ be the homogeneous coordinate ring for the embedding $C \subseteq \mathbb{P}^2_{\mathbb{C}}$. Let $\mathfrak{P}$ be the homogeneous maximal ideal of $A$, and let $B$ be the $\mathfrak{P}$-adic completion of the local ring $A_{\mathfrak{P}}$. Then $B$ is a local integrally closed domain of dimension 2, and a computation (cf. [38, p. 94]) shows that the divisor class group $C\ell(B)$ of $B$ contains a copy of $(\mathbb{R}/\mathbb{Z})^{(s)}$. Since $\mathbb{Z}^{(t)}/G$ embeds in $(\mathbb{R}/\mathbb{Z})^{(s)}$, there is a homomorphism $\varphi : \mathbb{Z}^{(t)} \to C\ell(B)$ with $\ker(\varphi) = G$. Let $\{e_1, \ldots, e_t\}$ be the standard basis of $\mathbb{Z}^{(t)}$. For each $i \leq t$, write $\varphi(e_i) = [L_i]$, where $L_i$ is a divisorial ideal of $B$ representing the divisor class of $\varphi(e_i)$.

Next we use Heitmann's amazing theorem [25], which implies that $B$ is the completion of some local unique factorization domain $R$. For each element $m = (m_1, \ldots, m_t) \in \mathbb{N}_0^{(t)}$, we let $\beta(m)$ be the isomorphism class of the $B$-module $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$. The divisor class of this module is $m_1[L_1] + \cdots + m_t[L_t] = \varphi(m_1, \ldots, m_t)$. By [36, (1.5)] (cf. also [33, Proposition 3]), the module $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$ is the completion of an $R$-module if and only if its divisor class is trivial, that is, if and only if $m \in G \cap \mathbb{N}_0^{(t)}$. But $m \in G \cap \mathbb{N}_0^{(t)} = H$, since $H$ is a full subsemigroup of $\mathbb{N}_0^{(t)}$. Therefore $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$ is the completion of an $R$-module if and only if $m \in H$. If $m \in H$, we let $\alpha(m)$ be the isomorphism class of a module whose completion is isomorphic to $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$. In particular, by choosing a module $M$ such that $[M] = \alpha(h)$, we get the desired commutative diagram. □

Returning briefly to one-dimensional rings, we ask whether there is a fixed one-dimensional local ring $R$, with reduced completion $\widehat{R}$, that can realize *every* finitely generated Krull semigroup as $\mathsf{V}(+(M))$ for a suitable finitely generated $R$-module $M$. Karl Kattchee [26] has shown that the answer is no. The key idea is to look at the *embedding dimension* of a Krull semigroup $H$, which is the smallest $t$ such that $H$ can be embedded as an expanded subsemigroup of $\mathbb{N}_0^{(t)}$. By the *rank* of a semigroup $H$, we mean the $\mathbb{Q}$-dimension of $\mathbb{Q} \otimes_{\mathbb{Z}} G$, where $G$ is the group of differences of $H$. Kattchee shows that, for every $R$-module $M$, the embedding dimension of $\mathsf{V}(+(M))$ is strictly less than $t + \operatorname{rank}(+(M))$, where $t$ is the number of minimal prime ideals of $\widehat{R}$. In particular, if $\widehat{R}$ has at most $t$ minimal prime ideals and $H$ is a finitely generated Krull semigroup of rank 3 such that the embedding dimension of $H$ is more than $2 + t$, then $H$ cannot be realized as a semigroup of the form $+(M)$. He then constructs, for any $n$, a finitely generated Krull semigroup having rank 3 and embedding dimension at least $n$.

## 5. Artinian modules

In a 1932 paper [29], Krull asked whether Artinian modules satisfy Krull-Schmidt uniqueness. In other words, is the decomposition of an Arinian right module into a direct sum of finitely many indecomposables necessarily unique up to isomorphism and rearrangement of the summands? The answer is "yes" if $R$ is either commutative or right Noetherian, cf. [35]. A negative answer for general rings was finally given by Facchini, Herbera, Levy and Vámos in 1995 [17]. They showed, roughly speaking, that any failure of Krull-Schmidt demonstrated by finitely generated modules over a commutative, semilocal Noetherian ring can be replicated using Artinian modules. We use their approach, along with Theorem 4.4, to show that every finitely generated Krull semigroup is isomorphic to $\mathsf{V}(+(M))$, for a suitable Artinian module $M$. This result is best possible, since, as we point out in Proposition 5.2, $\mathsf{V}(+(M))$ is a Krull semigroup for any Artinian right module $M$ over any ring.

In this section and the next, we return to the conventions of §3. In particular, a ring $R$ is semilocal provided $R/\operatorname{J}(R)$ is Artinian. We begin by proving a another version of Theorem 3.2. The "Moreover" statement is what allows us to get results about Artinian modules.

**Theorem 5.1** ([38, Theorem 4.1]). *Let $H$ be a finitely generated Krull semigroup, and let $H \overset{\varphi}{\hookrightarrow} \mathbb{N}_0^{(t)}$ be a full embedding. Assume there is an element $h \in H$ such that $\varphi(h)$ is strictly positive. Then there exist a semilocal ring $E$ and a commutative diagram of semigroups*

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & \mathbb{N}_0^{(t)} \\ \gamma \big\downarrow \cong & & \delta \big\downarrow \cong \\ \mathsf{V}(\operatorname{pmod}(E)) & \xrightarrow{\iota} & \mathsf{V}(\operatorname{pmod}(E/\operatorname{J}(E))) \end{array}$$

*with the following properties:*

(1) *$\gamma$ and $\delta$ are isomorphisms.*
(2) *$\gamma(h) = [E]$.*

*Moreover, $E$ can be taken to be a module-finite algebra over a commutative Noetherian local ring.*

*Proof.* Choose $R$ and $M$ as in Theorem 4.4. Then $E := \operatorname{End}_R(M)$ is a module-finite $R$-algebra. There is a well-known isomorphism $\eta : \mathsf{V}(+(M)) \to \mathsf{V}(\operatorname{pmod}(E))$ taking $[N]$ to $[\operatorname{Hom}_R(M, N)]$. The inverse map takes $[P]$ to $[P \otimes_E M]$. (Cf. [12].) Let $\widehat{E} = \widehat{R} \otimes_R E = \operatorname{End}_{\widehat{R}}(\widehat{M})$. Put $J := \operatorname{J}(E)$. Then $\widehat{J} := \widehat{R} \otimes_R J$ is the Jacobson radical of $\widehat{E}$, and since $E/J$ has finite length as an $R$-module the natural map $E/J \to \widehat{E}/\widehat{J}$ is an isomorphism. Also, since $\widehat{E}$ is module-finite over the complete Noetherian local ring $\widehat{R}$, the natural map

$\mathsf{V}(\mathrm{pmod}(\widehat{E})) \to \mathsf{V}(\mathrm{pmod}(\widehat{E}/\widehat{J}))$ is an isomorphism. The desired commutative diagram is obtained by splicing these natural isomorphisms onto the diagram provided by Theorem 4.4:

$$\begin{array}{ccc} H & \xrightarrow{\subseteq} & \mathbb{N}_0^{(t)} \\ \alpha\downarrow\cong & & \beta\downarrow\cong \\ \mathsf{V}(+(M)) & \xrightarrow{\xi} & \mathsf{V}(+(\widehat{M})) \\ \eta\downarrow\cong & & \widehat{\eta}\downarrow\cong \\ \mathsf{V}(\mathrm{pmod}(E)) & \longrightarrow & \mathsf{V}(\mathrm{pmod}(\widehat{E})) \\ \downarrow & & \downarrow\cong \\ \mathsf{V}(\mathrm{pmod}(E/J) & \xrightarrow{\cong} & \mathsf{V}(\mathrm{pmod}(\widehat{E}/\widehat{J})) \end{array}$$

□

**Proposition 5.2.** *Let $R$ be any ring and $M$ an Artinian right $R$-module. Then $\mathsf{V}(+(M))$ is a finitely generated Krull semigroup.*

*Proof.* A result due to Camps and Dicks [6] says that $E := \mathrm{End}_R(M)$ is semilocal. Therefore, as we showed at the beginning of §3, $\mathsf{V}(\mathrm{pmod}(E))$ is a finitely generated Krull semigroup. The isomorphism $\eta : \mathsf{V}(+(M)) \to \mathsf{V}(\mathrm{pmod}(E))$ in the proof of Theorem 5.1 completes the proof. □

Here is the realization theorem, this time in terms of Artinian modules:

**Theorem 5.3.** *Let $H$ be a finitely generated Krull semigroup. There exist a ring $R$ and an Artinian cyclic right module $M$ such that $\mathsf{V}(+(M)) \cong H$.*

*Proof.* By Theorem 5.1 there exists a module-finite algebra $E$, over a commutative Noetherian local ring, such that $H \cong \mathsf{V}(\mathrm{pmod}(E))$. By [17, Corollary 1.3], there exist a ring $R$ and an Artinian cyclic right $R$-module $M$ such that $E \cong \mathrm{End}_R(M)$. Once again, the isomorphism $\eta : \mathsf{V}(+(M)) \to \mathsf{V}(\mathrm{pmod}(E))$ completes the proof. □

## 6. Torsion-free abelian groups

The complicated direct-sum behavior of finite-rank torsion-free abelian groups has received a lot of attention (cf. [1] for an excellent survey). Even the torsion-free $\mathbb{Z}_{(p)}$-modules of finite rank have a very rich structure with respect to direct-sum decompositions (although they *do* satisfy direct-sum cancellation, [1, (8.1), (8.4), (8.8)]). A realization theorem, published by Yakovlev in 2000, illustrates this richness:

**Theorem 6.1** ([39]). *Let $H$ be a finitely generated Krull semigroup and let $p$ be a prime number. Then there is a finite-rank torsion-free $\mathbb{Z}_{(p)}$-module $G$ such that* $\mathsf{V}(+(G)) \cong H$. □

Of course the module $G$ in Theorem 6.1 is not finitely generated!

## 7. Modules with semilocal endomorphism rings

In this section we give a realization theorem for Krull semigroups that are not necessarily finitely generated. We begin with a theorem due to A. Facchini that tells us what to "realize":

**Theorem 7.1** ([14]). *Let $R$ be a ring and $\mathcal{C}$ a class of modules closed under isomorphism, finite direct sums, and direct summands. Assume there is a set $\mathsf{V}(\mathcal{C}) \subseteq \mathcal{C}$ of representatives and that $\operatorname{End}_R(M)$ is semilocal for each $M \in \mathcal{C}$. Then $\mathsf{V}(\mathcal{C})$ is a Krull semigroup .*

The realization theorem below is due to Facchini and R. Wiegand. Perhaps surprisingly, the modules that realize a given Krull semigroup can be taken to be finitely generated and projective. Given a ring $R$, let $\mathcal{S}(R)$ be the class of finitely generated projective right $R$-modules whose endomorphism rings are semilocal.

**Theorem 7.2** ([19]). *Let $H$ be a Krull semigroup , and let $\varphi : H \to \mathbb{N}_0^{(b)}$ be a divisor homomorphism from $H$ to a free semigroup . Let $k$ be any field. Then there exist a $k$-algebra $R$ and a commutative diagram of semigroups*

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & \mathbb{N}_0^{(b)} \\ \alpha \big\downarrow \cong & & \beta \big\downarrow \cong \\ \mathsf{V}(\mathcal{S}(R)) & \xrightarrow{\tau} & \mathsf{V}(\mathcal{S}(R/\operatorname{J}(R))) \end{array} ,$$

*in which $\alpha$ and $\beta$ are isomorphisms and $\tau$ is the natural map induced by change of rings.*

The proof is extremely technical and, like Theorem 3.2, makes heavy use of Bergman's constructions [4] and Cohn's inversive localization [10]. The rings that are constructed are frightfully non-commutative. Indeed, this is unavoidable, in view of the following observation due to Facchini:

**Theorem 7.3** ([15, Theorem 4.2]). *Let $R$ be a commutative ring. If* $\mathsf{V}(\operatorname{pmod}(R))$ *is a Krull semigroup, then it is a free semigroup.*

## 8. The semigroup of *all* finitely generated modules

In this section the word "local" means "commutative local and Noetherian". Suppose $(R, \mathfrak{m})$ is a one-dimensional local ring. We let $\mathcal{F}(R)$ be the class of finitely generated modules that are free on the punctured spectrum. Thus $M \in \mathcal{F}(R)$ if and only if $M_P$ is a free $R_P$-module for each minimal prime ideal $P$. Of course, if $R$ is reduced then $\mathcal{F}(R) = \mathrm{mod}(R)$. In this section we give complete sets of invariants for the semigroups $\mathsf{V}(\mathcal{F}(R))$, where $R$ ranges over one-dimensional local rings. The results we discuss are due to Facchini, Hassler, Karr, Klingler and R. Wiegand and appear in two papers, [18] (the Cohen-Macaulay case) and [24] (the general case).

We recall that a local ring $(R, \mathfrak{m}, k)$ is *Dedekind-like* provided $R$ is one-dimensional and reduced, the integral closure $\overline{R}$ of $R$ in its total quotient ring is generated by at most 2 elements as an $R$-module, and $\mathfrak{m}$ is the Jacobson radical of $\overline{R}$. In [27] and [28], Klingler and Levy show, essentially, that Dedekind-like rings and their homomorphic images are exactly the Noetherian local rings of positive dimension whose finite-length modules have tame representation type. Our description of the semigroup $\mathsf{V}(\mathrm{mod}(R))$ for a Dedekind-like ring $R$ depends crucially on the description of all finitely generated modules in [28]. Although the the description in [28] does not include the special situation when $\widehat{R}/\widehat{\mathfrak{m}}$ is a purely inseparable field of degree 2 over $R/\mathfrak{m}$, this difficulty has no bearing on our computations. We outline the procedure in this case below the statement of Theorem 8.1.

Fix a positive integer $q$ and an infinite cardinal $\tau$. We put $\mathcal{H}(0, \tau) := \mathbb{N}_0^{(\tau)}$, the free semigroup of rank $\tau$. If $q > 0$, let $B$ be any $q \times \tau$ integer matrix such that each element of $\mathbb{Z}^{(q)}$ occurs $\tau$ times as a column of $B$. We let

$$\mathcal{H}(q, \tau) := \mathbb{N}_0^{(\tau)} \cap \ker(B : \mathbb{Z}^{(\tau)} \to \mathbb{Z}^{(q)}).$$

Not surprisingly, the isomorphism class of the semigroup $\mathcal{H}(q, \tau)$ does not depend on how the columns are arranged, as long as each column is repeated $\tau$ times, cf. [18, Lemmas 1.1 and 2.1]. The semigroups $\mathcal{H}(q, \tau)$ show up in the form $\mathsf{V}(\mathcal{F}(R))$ if $R$ is non Dedekind-like. For some Dedekind-like rings, we obtain a different semigroup: Let $E$ be the $1 \times \aleph_0$ matrix $\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & \dots \end{bmatrix}$, and put

$$\mathcal{H}_1 := \mathbb{N}_0^{\aleph_0} \cap \ker(E : \mathbb{Z}^{(\aleph_0)} \to \mathbb{Z})$$

For a one-dimensional local ring $(R, \mathfrak{m}, k)$, we define the *splitting number* $\mathrm{spl}(R)$ to be the difference $|\mathrm{Spec}(\widehat{R})| - |\mathrm{Spec}(R)|$. Thus, for example, $\mathrm{spl}(R) = 0$ means that the natural map $\mathrm{Spec}(\widehat{R}) \to \mathrm{Spec}(R)$ is bijective. It is interesting that the divisor class group $C\ell(\mathsf{V}(\mathcal{F}(R)))$ depends only on the splitting number of $R$. (See, for example, [8] or [22] for the definitions and basic properties of the divisor class group of a Krull semigroup. It measures in some sense the lack of unique factorization. For example, a Krull semigroup is free if and only if it has unique factorization, if and only if its divisor class group is trivial.)

**Theorem 8.1** ([18, Theorem 2.2], [24, Theorem 6.3]). *Suppose $(R, \mathfrak{m}, k)$ is a one-dimensional commutative Noetherian local ring. Let $q := \operatorname{spl}(R)$ be the splitting number of $R$, and let $\tau = \tau(R) = |k| \cdot \aleph_0$.*

1. *If $R$ is not a homomorphic image of a Dedekind-like ring, then $\mathsf{V}(\mathcal{F}(R)) \cong \mathcal{H}(q, \tau)$.*
2. *If $R$ is a discrete valuation ring, then $\mathsf{V}(\mathcal{F}(R)) = \mathsf{V}(\operatorname{mod}(R)) \cong \mathbb{N}_0^{(\aleph_0)}$.*
3. *If $R$ is a homomorphic image of a Dedekind-like ring but is not a discrete valuation ring, and if $q = 0$, then $\mathsf{V}(\mathcal{F}(R)) = \mathsf{V}(\operatorname{mod}(R)) \cong \mathbb{N}_0^{(\tau)}$.*
4. *If $R$ is a homomorphic image of a Dedekind-like ring and $q > 0$, then $q = 1$ and $\mathsf{V}(\mathcal{F}(R)) = \mathsf{V}(\operatorname{mod}(R)) \cong \mathbb{N}_0^{(\tau)} \oplus \mathcal{H}_1$.*

*In every case, $C\ell(\mathsf{V}(\mathcal{F}(R))) \cong \mathbb{Z}^{(q)}$.*

In the special case where $\widehat{R}/\widehat{\mathfrak{m}}$ is purely inseparable over $R/\mathfrak{m}$, the completion $\widehat{R}$ is a domain, and consequently $\mathsf{V}(\operatorname{mod}(R)) \cong \mathsf{V}(\operatorname{mod}(\widehat{R}))$. Even though the $R$-modules have so far defied classification, one can show in this case that there are $\tau$ pairwise non-isomorphic indecomposables in this case, and hence that $\mathsf{V}(\operatorname{mod}(R)) \cong \mathbb{N}_0^{(\tau)}$.

It would be nice to have a theorem describing the semigroup of maximal Cohen-Macaulay (equivalently, finitely generated and torsion-free) modules over a one-dimensional reduced ring. The reason we do not yet have such a theorem is that we do not yet know exactly which ranks can occur for maximal Cohen-Macaulay modules over a complete, reduced one-dimensional local ring. If we know that each analytic branch has infinite Cohen-Macaulay type, however, we can appeal to the following result, proved by Andrew Crabbe:

**Theorem 8.2** (Crabbe, [11]). *Let $(R, \mathfrak{m}, k)$ be a reduced, one-dimensional, complete local ring with minimal prime ideals $P_1, \dots, P_s$. Assume that $R/P_1$ has infinite Cohen-Macaulay type. Let $(r_1, \dots, r_s)$ be an arbitrary non-trivial $s$-tuple of non-negative integers, and assume that $r_1 \geq r_i$ for $i = 2, \dots, s$. Put $\tau = \tau(R) := |k| \cdot \aleph_0$. Then there is a family of $\tau$ pairwise non-isomorphic indecomposable maximal Cohen-Macaulay $R$-modules $M$ such that $M_{P_i} \cong R_{P_i}^{(r_i)}$ for each $i = 1, \dots, s$.*

Now let $\mathcal{CM}(R)$ denote the class of maximal Cohen-Macaulay $R$-modules, together with the zero-module. Using Crabbe's result, we can make minor modifications in the proof of [18, Theorem 2.2] to obtain the following description of the semigroup $\mathsf{V}(\mathcal{CM}(R))$:

**Corollary 8.3.** *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local ring whose completion $\widehat{R}$ is reduced. Assume that $R/P$ has infinite Cohen-Macaulay type for each minimal prime ideal $P$ of $R$. Let $q = \operatorname{spl}(R)$, and put $\tau = \tau(R) := |k| \cdot \aleph_0$. Then $\mathsf{V}(\mathcal{CM}(R)) \cong \mathcal{H}(q, \tau)$.*

At the other extreme, when $R$ has finite Cohen-Macaulay type, Nick Baeth [2] has given a detailed description of the semigroups $\mathcal{CM}(R)$, in the case where $R$ contains a field of characteristic different from $2, 3, 5$ and the residue field $k$ is perfect. He and Melissa Luckas have worked out most of his computations in the non-equicharacteristic, non-perfect case, without assumptions on the characteristic of the residue field [3].

## 9. The divisor sequence of a Krull semigroup

Given a non-zero element $x$ in a Krull semigroup $H$, we let $a_n = a_n(x)$ be the number of distinct atoms of $H$ that divide $nx$. We are interested in the behavior of the *divisor sequence* $(a_1(x), a_2(x), a_3(x), \dots)$ associated to $x$. For example, for the element $a$ in the example of (2.8), the sequence is $(1, 1, 1, 1, 3, 3, 3, 3, 3, \dots)$. For an arbitrary element $x$ in a Krull semigroup, the sequence satisfies the following three properties:

(1) $a_i \leq a_{i+1}$ for all $n$ (non-decreasing).
(2) There is an $N$ such that $a_n = a_N$ for $n \geq N$ (eventually constant).
(3) If $a_n = 1$, then $a_{n+1} \neq 2$.

To verify (3), suppose $nx$ is divisible by only one atom, say $y$, and that $(n+1)x$ is divisible by exactly two atoms $y$ and $z$. Since $x$ is a sum of atoms, and thus is divisible by *some atom*, and since every divisor of $x$ is also a divisor of $nx$, we see that $x$ must be divisible only by $y$. Thus $x = ry$ for some $r > 0$. With $s = (n+1)r$, we have $sy = (n+1)x = ty + uz$, with $u > 0$. After cancelling some $y$'s, we get $vy = uz$, with $v > 0$. An argument similar to that of Remark 2.9 forces $y = z$.

At this point it is not known whether every sequence satifying these three properties actually arises as the divisor sequence associated to some element of a Krull semigroup. If $x$ is an atom, however, some restrictions apply, in addition to the obvious one that $a_1(x) = 1$. In fact, Hassler [23] has shown that the sequence $(1, 3, 3, 4, 4, 4, 4, \dots)$ cannot occur for an atom. Terri Moore [31] has recently obtained several additional restrictions, in the case where the element $x$ is an atom. Conversely, she has shown that any given non-decreasing, eventually constant sequence $(a_1, a_2, a_3, \dots)$ with $a_1 = 1$ and with no difference $a_{n+1} - a_n$ equal to 1, actually occurs as the divisor sequence of an atom in a suitable Krull semigroup.

## References

[1] D. Arnold, Finite Rank Torsion Free Abelian Groups and Rings, Springer-Verlag, vol. 931, Berlin, 1982.
[2] N. Baeth, *A Krull-Schmidt theorem for one-dimensional rings with finite Cohen-Macaulay type*, J. Pure Appl. Algebra **208** (2007), 923–940.
[3] N. Baeth and M. Luckas, *Bounds for indecomposable torsion-free modules*, preprint.
[4] G. M. Bergman, *Coproducts and some universal ring constructions,* Trans. Amer. Math. Soc. **200** (1974), 33–88.
[5] W. Bruns and J. Herzog, Cohen-Macaulay Rings, Cambridge Stud. Adv. Math. vol. 39, Cambridge University Press, Cambridge, 1993.

[6] R. Camps and W. Dicks, *On semilocal rings*, Israel J. Math. **81** (1993), 203–211.

[7] S. Chapman and A. Geroldinger, *Krull domains and monoids, their sets of lengths, and associated combinatorial problems*, in "Factorization in integral domains", Lecture Notes in Pure and Appl. Math. vol. 189, Dekker, New York, 1997, 73–112.

[8] L. Chouinard, *Krull semigroups and divisor class groups*, Canadian J. Math. **19** (1981), 1459–1468.

[9] A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Vol. II, Math. Surveys 7, Amer. Math. Soc., Providence, RI, 1967.

[10] P. M. Cohn, *Inversive localization in Noetherian rings*, Comm. Pure Appl. Math. **26** (1973), 679–691.

[11] A. Crabbe, oral communication.

[12] A. Dress, *On the decomposition of modules*, Bull. Amer. Math. Soc. **74** (1969), 984–986.

[13] E. G. Evans, Jr., *Krull-Schmidt and cancellation over local rings*, Pacific J. Math. **46** (1973),115–121.

[14] A. Facchini, *Direct sum decompositions of modules, semilocal endomorphism rings, and Krull monoids,* J. Algebra **256** (2002), 280–307.

[15] A. Facchini, *Krull Monoids and Their Application in Module Theory*, in "Algebras, Rings and their representations", A. Facchini, K. Fuller, C. M. Ringel and C. Santa-Clara, eds., World Scientific, Hackensack, NJ, 2006, 53–71.

[16] A. Facchini and D. Herbera, *Projective modules over semilocal rings*, Algebra and its Applications, D. K. Huynh, S. K. Jain and S. R. López-Permouth, eds., Contemp. Math. **259** (2000), 181–198.

[17] A. Facchini, D. Herbera, L. S. Levy and P. Vámos, *Krull-Schmidt fails for Artinian modules*, Proc. Amer. Math. Soc. **123** (1995), 3587–3592.

[18] A. Facchini, W. Hassler, L. Klingler and R. Wiegand, *Direct-sum decompositions over one-dimensional Cohen-Macaulay local rings*, in "Multiplicative Ideal Theory in Commutative Algebra: a tribute to the work of Robert Gilmer", J. Brewer, S. Glaz, W. Heinzer, eds., Springer, 2006.

[19] A. Facchini and R. Wiegand, *Direct-sum decompositions of modules with semilocal endomorphism rings*, J. Algebra **274** (2004), 689–707.

[20] K. R. Fuller and W. A. Shutters, *Projective modules over non-commutative semilocal rings*, Tôhoku Math. J. **27** (1975), 303–311.

[21] R. Guralnick, *Lifting homomorphisms of modules*, Illinois J. Math. **29** (1985), 153–156.

[22] F. Halter-Koch, *Halbgruppen mit Divisorentheorie*, Expo. Math. **8** (1990) 27–66.

[23] W. Hassler, *private communication.*

[24] W. Hassler, R. Karr, L. Klingler and R. Wiegand, *Big indecomposable modules and direct-sum relations*, Illinois J. Math. **41** (2007), 99–122.

[25] R. Heitmann, *Characterizations of completions of unique factorization domains*, Trans. Amer. Math. Soc. **337** (1993), 379–387.

[26] K. Kattchee, *Monoids and direct-sum decompositions over local rings*, J. Algebra **256** (2002), 51–65.

[27] L. Klingler and L. S. Levy, *Representation Type of Commutative Noetherian Rings I: Local Wildness*, Pacific J. Math. **200** (2001), 345–386.

[28] L. Klingler and L. S. Levy, *Representation Type of Commutative Noetherian Rings II: Local Tameness*, Pacific J. Math. **200** (2001), 387–483.

[29] W. Krull, *Matrizen, Moduln und verallgemeinerte Abelsche Gruppen im Bereich der Ganzen algebraischen Zahlen*, Heidelberger Akad. Wiss. **2** (1932), 13–38.

[30] L. S. Levy and C. Odenthal, *Package deal theorems and splitting orders in dimension* 1, Trans. Amer. Math. Soc. **348** (1996), 3457–3503.

[31] T. Moore, Ph.D. dissertation, University of Nebraska, in preparation.

[32] I. Reiner and K. W. Roggenkamp, Integral Representations, Lecture Notes in Math. vol. 744, 1979.

[33] C. Rotthaus, D, Weston and R. Wiegand, *Indecomposable Gorenstein modules of odd rank*, J. Algebra **214** (1999), 122–127.

[34] R. P. Stanley, Combinatorics and Commutative Algebra, Second Edition, Birkhäuser, Boston, 1996.

[35] R. B. Warfield, *A Krull-Schmidt theorem for infinite sums of modules*, Proc. Amer. MAth. Soc. **22** (1969), 460–465.

[36] D. Weston, *On descent in dimension two and non-split Gorenstein modules*, J. Algebra **118** (1988), 263–275.

[37] R. Wiegand, *Failure of Krull-Schmidt for direct sums of copies of a module*, in "Advances in Commutative Ring Theory", D. Dobbs, M. Fontana and S.-E. Kabbaj, eds., Dekker, New York, 1999.

[38] R. Wiegand, *Direct-sum decompositions over local rings*, J. Algebra **240** (2001), 83–97.

[39] A. V. Yakovlev, *On direct decompositions of $p$-adic groups* (Russian), Algebra i Analiz **12** (2000), no. 6, 217–223; translation in *St. Petersburg Math. J.* **12** (2001), no. 6, 1043–1047.

Roger Wiegand, Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130

*E-mail address*: `rwiegand@math.unl.edu`

Sylvia Wiegand, Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130

*E-mail address*: `swiegand@math.unl.edu`