Direct-sum decompositions over one-dimensional Cohen-Macaulay local rings

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This article is dedicated to Robert Gilmer, a pioneer of multiplicative ideal theory. Thanks to his fundamental contributions and his leadership, the theory has become a major mathematical enterprise, with connections to many areas of mathematics. Here we explore one of these connections.

1 Introduction and Terminology

The theory of commutative cancellative monoids grew out of the multiplicative ideal theory of integral domains. Many problems in multiplicative arithmetic become more clearly focused when one strips away the additive structure of an integral domain and looks only at the multiplicative monoid of non-zero elements. Krull monoids, introduced in [4], and their divisor class groups have provided perhaps the most fertile ground for investigation. It is known [17]

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that an integral domain R is a Krull domain if and only if its multiplicative monoid $R - \{0\}$ is a Krull monoid. Moreover, the divisor class group of R agrees with the divisor class group of the monoid $R - \{0\}$. In particular, the divisor class group of a Krull domain depends only on the multiplicative monoid of non-zero elements.

Commutative monoids arise also in representation theory. Given any class $\mathcal{C}$ of finitely generated modules, one can form the set $\mathrm{V}(\mathcal{C})$ consisting of isomorphism classes $[M]$ of modules $M \in \mathcal{C}$. If $\mathcal{C}$ contains (0) and is closed under finite direct sums, the set $\mathrm{V}(\mathcal{C})$ becomes a commutative monoid under the operation $[M]+[N] = [M \oplus N]$. If, moreover, $\mathcal{C}$ is closed under direct summands, the monoid $\mathrm{V}(\mathcal{C})$ carries detailed information about the direct-sum behavior of modules in $\mathcal{C}$, e.g., whether or not the Krull-Remak-Schmidt theorem holds, and, when it does not, how badly it fails.

Our goal in this paper is to give a complete set of invariants for the monoid $\mathrm{V}(R\text{-mod})$, where $(R, \mathfrak{m}, k)$ is a one-dimensional reduced Noetherian local ring and R-mod is the class of all finitely generated R-modules. In order to state our results precisely and place them in context, we recall some terminology.

1.1 Krull monoids

All of our monoids are assumed to be commutative. We always use additive notation, that is, $+$ is the operation in our monoids, and 0 is the identity element. We will always assume that 0 is the only invertible element, that is, $x + y = 0 \implies x = y = 0$. Further, we shall restrict our attention to *cancellative* monoids: $x + z = y + z \implies x = y$. Note that the monoid $\mathrm{V}(R\text{-mod})$, for $(R, \mathfrak{m}, k)$ a commutative Noetherian local ring, is cancellative [5].

If x and y are elements of an additive monoid H we write $x \leq y$ to indicate that $x + z = y$ for some (necessarily unique) $z \in H$. A monoid homomorphism $\phi : H_1 \to H_2$ is a *divisor* homomorphism provided $x \leq y$ in $H_1 \iff \phi(x) \leq \phi(y)$ in H_2. In this case, it follows from our standing assumptions that ϕ is injective. A monoid is *free* provided it is isomorphic to the direct sum $\mathbb{N}^{(\Lambda)}$ of copies of the additive monoid $\mathbb{N} := \{0, 1, 2, \dots\}$ for some (possibly infinite) index set Λ. Of course, given elements x and y in the free monoid $\mathbb{N}^{(\Lambda)}$, we have $x \leq y$ if and only if $x_\lambda \leq y_\lambda$ for each $\lambda \in \Lambda$, that is, $\leq$ is the product partial ordering for free monoids. A monoid H is a *Krull* monoid (see, e.g. [11]) provided there exists a divisor homomorphism from H to a free monoid.

It turns out [11, Theorem 23.4] that every Krull monoid H actually has a *divisor theory*, that is, a divisor homomorphism $\phi : H \to F$ such that

(1) F is a free monoid, and
(2) every element of F is the greatest lower bound of some finite set of elements in $\phi(H)$.

We note [11, Theorem 20.4] that a divisor theory of a Krull monoid is unique up to canonical isomorphism, that is, if $\phi : H \to F$ and $\phi' : H \to F'$ are two divisor theories, then there exists a unique isomorphism $\psi : F \to F'$ with $\phi' = \psi\phi$.

Given a divisor theory $\phi : H \to F = \mathbb{N}^{(\Lambda)}$, we have an induced homomorphism $\mathcal{Q}(\phi) : \mathcal{Q}(H) \to \mathcal{Q}(F)$, where $\mathcal{Q}(H)$ and $\mathcal{Q}(F) \cong \mathbb{Z}^{(\Lambda)}$ are the quotient groups of H and F, respectively. The *divisor class group* $\mathrm{Cl}(H)$ is, by definition, the cokernel of $\mathcal{Q}(\phi)$. Krull monoids are *atomic*, that is, every non-zero element is a sum of atoms (elements that cannot be written as a sum of two non-zero elements). The representation of an element as a sum of atoms is, in general, highly non-unique (cf. Theorem 4.1).

It is easy to see that the following conditions on a Krull monoid H are equivalent:

(1) H is free.
(2) Every non-zero element of H has a unique (up to a permutation) representation as a sum of atoms (that is, H is *factorial*).
(3) $\mathrm{Cl}(H) = 0$.

Thus the class group is, in some sense, a measure of the deviation of H from being factorial.

Let H be a Krull monoid, and let $\phi : H \to \mathbb{N}^{(\Lambda)}$ be a divisor theory. Given a divisor class $\alpha \in \mathrm{Cl}(H)$, that is, a coset of $\mathrm{im}(\mathcal{Q}(\phi))$ in $\mathbb{Z}^{(\Lambda)}$, we put

$$\mathfrak{A}(\alpha) = \{x \in \alpha \,|\, x \text{ is an atom of } \mathbb{N}^{(\Lambda)}\}.$$

(Of course the atoms of $\mathbb{N}^{(\Lambda)}$ are just the "unit vectors", with 1 in a single coordinate and 0's elsewhere.) The following result is [11, Theorem 23.5]:

Lemma 1.1 *Let H_1 and H_2 be Krull monoids. Then $H_1 \cong H_2$ if and only if there is a group isomorphism $\theta : \mathrm{Cl}(H_1) \to \mathrm{Cl}(H_2)$ such that $|\mathfrak{A}(\alpha)| = |\mathfrak{A}(\theta(\alpha))|$ for each $\alpha \in \mathrm{Cl}(H_1)$.*

Recall that an arbitrary ring S (not necessarily commutative) with Jacobson radical $\mathrm{J}(S)$ is said to be *semilocal* provided $S/\mathrm{J}(S)$ is semisimple Artinian. Let R be an arbitrary ring (not necessarily commutative or Noetherian), and let $\mathcal{C}$ be a class of right R-modules closed under isomorphism, finite direct sums, and direct summands. Assume that $\mathrm{End}_R(M)$ is semilocal for every $M \in \mathcal{C}$. Then $\mathrm{V}(\mathcal{C})$ is a Krull monoid [7, Theorem 3.4].

The following proposition gives further examples, outside the main context of this paper, in which $\mathrm{V}(\mathcal{C})$ is a Krull monoid. Recall that a *torsion-free* R-module is a module M such that for all $r \in R$, $x \in M$, $rx = 0 \implies r \in \mathfrak{z}(R)$ or $x = 0$, where $\mathfrak{z}(R)$ denotes the set of zerodivisors of R.

Proposition 1.2 *Let R be a commutative ring (not necessarily Noetherian).*

(1) *If R is semilocal, the monoid* $\mathrm{V}(R\text{-mod})$ *is a Krull monoid.*

(2) *Let R be a one-dimensional commutative Noetherian local ring with no non-zero nilpotents, and let $\mathcal{C}$ be the class of all torsion-free R-modules of finite rank (torsion-free modules M satisfying* $\dim_{R_P}(M_P) < \infty$ *for every minimal prime ideal P of R). Then* $\mathrm{V}(\mathcal{C})$ *is a Krull monoid.*

Proof. (1) If M is a finitely generated R-module, then $\mathrm{End}_R(M)$ is semilocal (cf. [23, Lemma 2.3] or [8, Proposition 3.2]). Therefore it is possible to apply [7, Theorem 3.4].

(2) Let $P_1, \dots, P_s$ be the minimal primes of R. The total quotient ring K of R is the direct product $R_{P_1} \times \cdots \times R_{P_s}$, where the localizations R_{P_i} are the quotient fields of the domains R/P_i. The R-module K is an injective envelope of R [20, Proposition 1.6]. By [19, Theorem 1], the R/P_i-module $R_{P_i}/(R/P_i)$ is Artinian. Therefore every homomorphic image of the R/P_i-module R_{P_i} has finite Goldie dimension. It follows that the same is true for every homomorphic image of the injective R-module K. From [8, Corollary 5.8], we see that every torsion-free R-module M of finite rank is isomorphic to an R-submodule of K^n for some n, and hence $\mathrm{End}_R(M)$ is semilocal [8, Corollary 5.8]. By [7, Theorem 3.4], $\mathrm{V}(\mathcal{C})$ is a Krull monoid.

Now let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with $\mathfrak{m}$-adic completion $\widehat{R}$. The homomorphism $[M] \mapsto [\widehat{M}]$ is a divisor homomorphism from $\mathrm{V}(R\text{-mod})$ to $\mathrm{V}(\widehat{R}\text{-mod})$, [24, Proposition 1.2]. By the Krull-Remak-Azumaya-Schmidt theorem [6] and [22, Lemma 13] (see also [21]), $\mathrm{V}(\widehat{R}\text{-mod})$ is free. Thus we find again that $\mathrm{V}(R\text{-mod})$ is a Krull monoid. Given a finitely generated R-module M one can form the monoid $+(M)$ consisting of isomorphism classes of modules that are direct summands of direct sums of finitely many copies of M. This is always a finitely generated Krull monoid [24], and the main result of [24] is that *every* finitely generated Krull monoid arises in this way. In fact, given any finitely generated Krull monoid H, there exist a one-dimensional Noetherian local domain R (essentially of finite type over the field of rational numbers) and a finitely generated torsion-free R-module M such that $+(M) \cong H$. This result motivated us to seek a characterization of the Krull monoids that arise in the form $\mathrm{V}(\mathfrak{C}(R))$, where $\mathfrak{C}(R)$ is the class of finitely generated torsion-free modules M over a Noetherian local ring R. Alas, we were unsuccessful, for reasons that we discuss briefly in Section 5. We were, however, able to classify the monoids $\mathrm{V}(R\text{-mod})$, for $(R, \mathfrak{m}, k)$ a reduced Noetherian local ring of dimension one. Our main theorem is that these monoids depend, up to isomorphism, only on the *splitting number* $\mathrm{spl}(R) := |\,\mathrm{Spec}(\widehat{R})| - |\,\mathrm{Spec}(R)|$, the cardinality of the residue field k, and whether or not R is Dedekind-like (defined below). We show, in all cases, that $\mathrm{Cl}(\mathrm{V}(R\text{-mod})) \cong \mathbb{Z}^{(\mathrm{spl}(R))}$.

1.2 Dedekind-like rings

A commutative Noetherian local ring $(R, \mathfrak{m}, k)$ is *Dedekind-like* [14, Definition 2.5] provided R is one-dimensional and reduced, the integral closure $\overline{R}$ of R

is generated by at most 2 elements as an R-module, and $\mathfrak{m}$ is the Jacobson radical of $\overline{R}$. In a recent series of papers [14, 15, 16] L. Klingler and L. S. Levy classified all finitely generated indecomposable R-modules up to isomorphism over essentially every Dedekind-like ring. The exceptional Dedekind-like rings for which their classification has not yet been worked out are those where $\overline{R}/\mathfrak{m}$ is a purely inseparable field extension of k of degree 2. (An example is the ring $F+XK[[X]]$, where K/F is a purely inseparable field extension of degree 2.) For convenience, we will refer to these rings as *exceptional Dedekind-like rings*.

2 The Main Theorem

Fix a positive integer q, and let B be a $q \times \aleph_0$ matrix whose columns are an enumeration of $\mathbb{Z}^{(q)}$. (The order does not matter. There are $2^{\aleph_0}$ matrices B of this type, one for each permutation of the $\aleph_0$ elements of $\mathbb{Z}^{(q)}$.) Next, let τ be an infinite cardinal, and let $\mathfrak{B} = \mathfrak{B}(q,\tau)$ be the $q \times \tau\aleph_0$ matrix consisting of τ copies of B arranged "horizontally". Of course $\tau\aleph_0 = \tau$, so we can regard $\mathfrak{B}$ as a homomorphism $\mathbb{Z}^{(\tau)} \to \mathbb{Z}^{(q)}$. We define $\mathfrak{H}(q,\tau) := \mathbb{N}^{(\tau)} \cap \ker(\mathfrak{B}(q,\tau))$. Finally, we put $\mathfrak{H}(0,\tau) = \mathbb{N}^{(\tau)}$. These are the monoids we will obtain as $\mathrm{V}(R\text{-mod})$ for the rings that are not Dedekind-like. By the next lemma and Lemma 1.1, it follows that the isomorphism class of $\mathfrak{H}(q,\tau)$ is independent of all choices:

Lemma 2.1 *Let q be a non-negative integer, let τ be an infinite cardinal, and let $\mathfrak{D}$ be a $q \times \tau$ matrix with entries in $\mathbb{Z}$. Assume that, for each $\delta \in \mathbb{Z}^{(q)}$, $\mathfrak{D}$ has τ distinct columns that coincide with δ (equivalently, $\mathfrak{D} = \mathfrak{B}(q,\tau)$ up to column permutations). Let $H = \mathbb{N}^{(\tau)} \cap \ker(\mathfrak{D} : \mathbb{Z}^{(\tau)} \to \mathbb{Z}^{(q)}) \cong \mathfrak{H}(q,\tau)$. Then:*

(1) $\mathfrak{D} : \mathbb{Z}^{(\tau)} \to \mathbb{Z}^{(q)}$ *is surjective.*
(2) *The inclusion $H \hookrightarrow \mathbb{N}^{(\tau)}$ is a divisor theory.*
(3) $\ker(\mathfrak{D}) = \mathcal{Q}(H)$.
(4) $\mathrm{Cl}(H) = \mathbb{Z}^{(q)}$.
(5) $|\mathfrak{A}(\alpha)| = \tau$ *for each $\alpha \in \mathrm{Cl}(H)$.*

Proof. Since every vector δ of $\mathbb{Z}^{(q)}$ occurs as a column of $\mathfrak{D}$, (1) is clear. To prove (2), let β be an arbitrary element of $\mathbb{N}^{(\tau)}$. Select distinct elements $t, u \in \tau - \mathrm{Supp}(\beta)$ such that the t^{th} and u^{th} columns of $\mathfrak{D}$ both coincide with $-\mathfrak{D}\beta$. Letting e_t and e_u denote the unit vectors of $\mathbb{N}^{(\tau)}$ with supports $\{t\}$ and $\{u\}$, respectively, we see that $\beta + e_t$ and $\beta + e_u$ are both in H. Clearly β is the greatest lower bound of $\beta + e_t$ and $\beta + e_u$ in the monoid $\mathbb{N}^{(\tau)}$.

The inclusion "$\supseteq$" in (3) is clear. For the reverse inclusion, let $\alpha \in \ker(\mathfrak{D})$, and write $\alpha = \beta - \gamma$, with $\beta, \gamma \in \mathbb{N}^{(\tau)}$. Choose, as in the proof of (2), a column index $t \in \tau$ such that $\beta + e_t \in H$. Then $\gamma + e_t = \beta + e_t - \alpha \in \mathbb{N}^{(\tau)} \cap \ker(\mathfrak{D}) = H$. Therefore $\alpha = (\beta + e_t) - (\gamma + e_t) \in \mathcal{Q}(H)$. This proves (3), and assertion (4) follows immediately from (1), (2) and (3).

For (5), let $\alpha \in \mathrm{Cl}(H) = \mathbb{Z}^{(q)}$. By hypothesis, there are τ column indices t such that the t^{th} column coincides with α. Each e_t is then an atom of $\mathbb{N}^{(\tau)}$ in the divisor class α.

For some Dedekind-like rings, we will obtain another monoid. Let E be the $1\times\aleph_0$ matrix $[\,1\ {-1}\ 1\ {-1}\ 1\ {-1}\ \cdots]$, and put $\mathfrak{H}_1 := \mathbb{N}^{(\aleph_0)} \cap \ker(E : \mathbb{Z}^{(\aleph_0)} \to \mathbb{Z})$.

In order to include Cohen-Macaulay rings with non-zero nilpotents in our theorem, we introduce the class $\mathfrak{F}(R)$ of *generically free* modules—finitely generated R-modules M such that M_P is R_P-free for every minimal prime ideal P. Of course, if R is reduced, each R_P is a field, so $\mathfrak{F}(R) = R$-mod. Moreover, if R is one-dimensional, Noetherian and reduced, then R is automatically Cohen-Macaulay. If $P_1, \dots, P_s$ are the minimal prime ideals of R and $M_{P_i} \cong R_{P_i}^{(r_i)}$ for each i, we call $(r_1, \dots, r_s)$ the *rank* of M.

Theorem 2.2 (Main Theorem) *Suppose $(R, \mathfrak{m}, k)$ is a one-dimensional Noetherian Cohen-Macaulay local ring. Let $q := \mathrm{spl}(R)$ be the splitting number of R, and let $\tau = \tau(R) = \max\{|k|, \aleph_0\}$.*

(1) *If R is not Dedekind-like, then $\mathrm{V}(\mathfrak{F}(R)) \cong \mathfrak{H}(q, \tau)$.*
(2) *If R is a discrete valuation ring, then $\mathrm{V}(\mathfrak{F}(R)) = \mathrm{V}(R\text{-mod}) \cong \mathbb{N}^{(\aleph_0)}$.*
(3) *If R is Dedekind-like but not a discrete valuation ring, and if $q = 0$, then $\mathrm{V}(\mathfrak{F}(R)) = \mathrm{V}(R\text{-mod}) \cong \mathbb{N}^{(\tau)}$.*
(4) *If R is Dedekind-like and $q > 0$, then $q = 1$ and $\mathrm{V}(\mathfrak{F}(R)) = \mathrm{V}(R\text{-mod}) \cong \mathbb{N}^{(\tau)} \oplus \mathfrak{H}_1$.*

In every case, $\mathrm{Cl}(\mathrm{V}(\mathfrak{F}(R))) \cong \mathbb{Z}^{(q)}$.

The proof will be an easy consequence of the following four lemmas.

Lemma 2.3 *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Noetherian Cohen-Macaulay local ring, and let $\tau = \max\{|k|, \aleph_0\}$. Then $|\,\mathrm{V}(\mathfrak{F}(R))| \le \tau$.*

Proof. An easy induction argument shows that every finite-length R-module has cardinality at most τ. Now, for each positive integer ℓ, let $\mathfrak{M}_\ell$ be the class of modules of length at most ℓ. We claim that $|\,\mathrm{V}(\mathfrak{M}_\ell)| \le \tau$ for each ℓ. Since $|\,\mathrm{V}(\mathfrak{M}_1)| = 2$, we may assume that $\ell \ge 2$ and proceed by induction. Given a module M of length ℓ, we can choose a short exact sequence $0 \to A \to M \to B \to 0$, in which A and B are strictly shorter than M. By the induction hypothesis there are at most τ choices for A and for B. Moreover, $\mathrm{Ext}^1_R(B, A)$, being a module of finite length, has cardinality at most τ. Thus the claim follows. Letting ℓ vary, we see that there are at most τ non-isomorphic modules of finite length.

Next, let K be the total quotient ring of R, that is, the localization of R with respect to the complement of the union of the associated primes $P_1, \dots, P_s$ of R. As R is Cohen-Macaulay, the P_i are exactly the minimal primes of R, and thus $K = R_{P_1} \times \cdots \times R_{P_s}$. Let S_i be the image of the natural homomorphism $R \to R_{P_i}$ (inclusion followed by projection).

Now let M be a torsion-free, generically free R-module of rank $(r_1, \ldots, r_s)$. Then we have an embedding $M \to K \otimes_R M \cong R_{P_1}^{(r_1)} \times \cdots \times R_{P_s}^{(r_s)}$. Clearing denominators, we see that there is an injective R-homomorphism $j : M \to F := S_1^{(r_1)} \times \cdots \times S_s^{(r_s)}$, with $\operatorname{coker}(j)$ a torsion R-module (hence of finite length). Thus M is the kernel of a homomorphism $F \to T$, where T is of finite length. There are only countably many choices for F and only τ choices for T, by the first paragraph of the proof. Since $\operatorname{Hom}_R(F, T)$ has finite length, it has cardinality at most τ, and it follows that there are at most τ choices for M.

Finally, let M be an arbitrary generically free module, and let T be the torsion submodule of M. Then T has finite length and M/T is generically free, so there are at most τ possibilities for T and for M/T. Since $\operatorname{Ext}_R^1(M/T, T)$ has finite length, its cardinality is at most τ, and it follows that there are at most τ possibilities for M, up to isomorphism.

The following theorem is a sharpening the main theorem of [12]. In the next section we will give a sketch of the additional arguments needed to prove the version given here.

Lemma 2.4 *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Noetherian Cohen-Macaulay local ring, and assume that R is not Dedekind-like. Let $P_1, \ldots, P_s$ be the minimal prime ideals of R, let $(r_1, \ldots, r_s)$ be an s-tuple of non-negative integers, and let $\tau = \max\{|k|, \aleph_0\}$. Then there are τ pairwise non-isomorphic indecomposable generically free R-modules M of rank $(r_1, \ldots, r_s)$.*

The next result, which tells us how $\mathrm{V}(\mathfrak{F}(R))$ sits inside $\mathrm{V}(\mathfrak{F}(\widehat{R}))$, is a special case of [18, Theorem 3.4].

Lemma 2.5 *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Noetherian Cohen-Macaulay local ring with $\mathfrak{m}$-adic completion $\widehat{R}$. Let K and L be the total quotient rings of R and $\widehat{R}$, respectively. The following conditions on a generically free $\widehat{R}$-module N are equivalent:*

(1) *There is a finitely generated R-module M (necessarily generically free) such that $N \cong \widehat{R} \otimes_R M$.*
(2) *There is a finitely generated K-module W such that $L \otimes_{\widehat{R}} N \cong L \otimes_K W$.*
(3) $\operatorname{rank}_{\widehat{R}_P}(N_P) = \operatorname{rank}_{\widehat{R}_Q}(N_Q)$ *whenever P and Q are minimal primes of $\widehat{R}$ lying over the same prime of R.*

In [15] Klingler and Levy classify all finitely generated modules over non-exceptional Dedekind-like rings. The next lemma distills the aspects of this classification that we shall need in our proof of the Main Theorem. The key property we need for Dedekind-like *domains*—that there are $\max\{|k|, \aleph_0\}$ non-isomorphic finite-length modules—holds even for exceptional Dedekind-like rings. Since the classification in [15] does not apply to exceptional Dedekind-like rings, we include a sketch of this fact.

Lemma 2.6 *Let $(R, \mathfrak{m}, k)$ be a local Dedekind-like ring. Then R is reduced with at most two minimal prime ideals. Put $\tau := \max\{|k|, \aleph_0\}$.*

(1) *If R is not a discrete valuation ring, then R has τ pairwise non-isomorphic indecomposable finite-length modules.*
(2) *Suppose R is not a domain. For each pair (a, b) of non-negative integers, let $G(a, b)$ be the cardinality of the set of isomorphism classes of finitely generated indecomposable R-modules of rank (a, b). Then $G(0, 0) = \tau$; $G(a, b) = \aleph_0$ if $(a, b) \in \{(0, 1), (1, 0), (1, 1)\}$; and $G(a, b) = 0$ otherwise.*

Proof. To prove (1) we may assume that R is $\mathfrak{m}$-adically complete, since R and its completion $\widehat{R}$ have exactly the same finite-length modules. Since the finite-length R-modules $R/\mathfrak{m}^n$ are indecomposable and pairwise non-isomorphic, we may assume that k is infinite. If the normalization $\overline{R}$ is *not* a discrete valuation ring (i.e., R is *strictly split* in the terminology of [15]), we appeal to Section 7 of [15], in particular, [15, Theorem 7.3, (ii)]. Even if we choose the "blocking matrix" L to be 1×1, we still obtain $|k|$ pairwise non-isomorphic "block-cycle" indecomposables, and these are all of finite length. (Cf. [15, Section 6] for the connection between matrices and modules.)

Still assuming that R is complete and k is infinite, we now suppose that $\overline{R}$ is a discrete valuation ring properly containing R. Then $F := \overline{R}/\mathfrak{m}$ is a (possibly inseparable) field extension of degree 2 over k. Write $\mathfrak{m} = \overline{R}\pi$, where $\pi \in \mathfrak{m}$. Given $u \in \overline{R}^{\times}$ (the group of units of $\overline{R}$), put $M_u := R/R\pi u$, an indecomposable R-module of finite length. Suppose $u, v \in \overline{R}^{\times}$ and $M_u \cong M_v$. We claim that u and v are in the same coset of $\overline{R}^{\times}/R^{\times} = F^{\times}/k^{\times}$. Since $|F^{\times}/k^{\times}| = |k|$, this will complete the proof of (1).

Choose an R-isomorphism $\phi : M_u \to M_v$. Since $R \twoheadrightarrow M_u$ and $R \twoheadrightarrow M_v$ are projective covers, we can lift ϕ to an automorphism $\Phi : R \to R$. Then Φ is multiplication by some element $t \in R^{\times}$, with $Rt\pi u = R\pi v$. Write $\pi v = st\pi u$, with $s \in R$. Then $v = stu$, and $st \notin \mathfrak{m}$ (lest $v \in \mathfrak{m}$). Thus $v \in R^{\times}u$, as desired.

Now we prove (2). By (1) and Lemma 2.3, $G(0, 0) = \tau$. Also, if $(a, b) \notin \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, then $G(a, b) = 0$ by [16, Lemma 16.2 and Corollary 16.4]. To complete the proof, we need only show that $G(0, 1), G(1, 0)$ and $G(1, 1)$ are all countably infinite. Since both R and $\widehat{R}$ have exactly two minimal primes, Lemma 2.5 implies that every finitely generated $\widehat{R}$ module is extended from a finitely generated R-module. Therefore we may again assume that R is complete. The desired result now follows from the classification given in [15, Section 3].

Proof of the Main Theorem. Suppose first that $q = 0$. Then every minimal prime ideal of R has a unique minimal prime of $\widehat{R}$ lying over it. By Lemma 2.5, the homomorphism $\mathrm{V}(\mathfrak{F}(R)) \to \mathrm{V}(\mathfrak{F}(\widehat{R}))$ is bijective. Therefore $\mathrm{V}(\mathfrak{F}(R))$ is a free monoid, and $\mathrm{Cl}(\mathrm{V}(\mathfrak{F}(R))) = 0$. To complete the proof in cases (2) and (3), and in the case $q = 0$ of (1), we need only show that the number of isomorphism classes of indecomposable generically free modules is $\aleph_0$ if R

is a discrete valuation ring and τ otherwise. If R is a discrete valuation ring, the indecomposable modules are $R/\mathfrak{m}^n, 1 \le n \le \infty$. If R is not a discrete valuation ring, we quote Lemmas 2.3, 2.4 and 2.6 (Part 1).

Next, assume that $q > 0$ and R is Dedekind-like. Then R is a domain and $\widehat{R}$ has exactly two minimal primes. Using part (2) of Lemma 2.6, we let $A_t,\ B_n,\ C_n,\ D_n\ (t \in \tau,\ n \ge 1)$ be the indecomposable finitely generated $\widehat{R}$-modules of rank $(0,0),(0,1),(1,0),(1,1)$, respectively. By Lemma 2.5, an $\widehat{R}$-module $\bigoplus_t A_t^{(a_t)} \oplus \bigoplus_n B_n^{(b_n)} \oplus \bigoplus_n C_n^{(c_n)} \oplus \bigoplus_n D_n^{(d_n)}$ is extended from an R-module if and only if $\sum_n (b_n - c_n) = 0$, so it follows immediately that $\mathrm{V}(\mathfrak{F}(R)) = \mathrm{V}(R\text{-mod}) \cong \mathbb{N}^{(\tau)} \oplus \mathfrak{H}_1$. It is easy to see that the inclusion $\mathrm{V}(R\text{-mod}) \hookrightarrow \mathrm{V}(\widehat{R}\text{-mod})$ is a divisor theory, and it follows that $\mathrm{Cl}(\mathrm{V}(R\text{-mod})) \cong \mathbb{Z}$, completing the proof in case (4).

Finally, assume that $q > 0$ and R is not Dedekind-like. Then $\widehat{R}$ is not Dedekind-like either [16, Lemma 11.8]. By Lemma 2.4, there are τ non-isomorphic indecomposable generically free $\widehat{R}$-modules. Let $P_1, \dots, P_s$ be the minimal prime ideals of R, and let $Q_{i,1}, \dots, Q_{i,t_i}$ be the primes of $\widehat{R}$ lying over P_i for $i = 1, \dots, s$. Then $q := \mathrm{spl}(R) = t_1 + \cdots + t_s - s = (t_1 - 1) + \cdots + (t_s - 1)$. Let $\mathfrak{L}$ be the $q \times \tau$ matrix whose columns are indexed by the indecomposable finitely generated $\widehat{R}$-modules, and whose columns are given by the following scheme: Let W be an indecomposable $\widehat{R}$ module, of rank $r_{i,j}$ at $Q_{i,j}$. The column indexed by the module W is then the transpose of the array

$$[r_{1,1} - r_{1,2}, \dots, r_{1,1} - r_{1,t_1}, \dots\dots, r_{s,1} - r_{s,2}, \dots, r_{s,1} - r_{s,t_s}].$$

Using Lemma 2.5, we see that $\mathrm{V}(\mathfrak{F}(R)) \cong H := \mathbb{N}^{(\tau)} \cap \ker(\mathfrak{L} : \mathbb{Z}^{(\tau)} \to \mathbb{Z}^{(q)})$.

Next, we verify the hypotheses of Lemma 2.1 for the matrix $\mathfrak{L}$. Let α be an arbitrary element of $\mathbb{Z}^{(q)}$. We want to show that $\mathfrak{L}$ has τ columns that coincide with α. Write

$$\alpha := [a_{1,2}, \dots, a_{1,t_1}, \dots\dots, a_{s,2}, \dots, a_{s,t_s}]^{\mathrm{tr}}.$$

Choose a positive integer b greater than each entry of α. Put $r_{i,1} := b,\ i = 1, \dots, s$, and put $r_{i,j} := b - a_{i,j},\ i = 1, \dots, s,\ j = 2, \dots, t_i$. By Lemma 2.4 there are τ indecomposable generically free $\widehat{R}$-modules having rank $r_{i,j}$ at the prime $Q_{i,j}$. The column of $\mathfrak{L}$ corresponding to such a W is precisely α, as desired. Now Lemma 2.1 completes the proof of the Main Theorem.

3 Proof of Lemma 2.4

The main theorem of [12] produces infinitely many indecomposable generically free modules of prescribed rank, over any one-dimensional Noetherian Cohen-Macaulay local ring $(R, \mathfrak{m}, k)$ that is not Dedekind-like. If k is uncountable, we need to modify that proof, in order to obtain $|k|$ non-isomorphic indecomposables. The proof of [12, Theorem 1.2] is divided into two cases. We give a careful sketch of the modifications needed in the first case ([12, Theorem 2.3], called the ramified case); we give a brief summary of the modifications needed in the second case ([12, Theorem 2.4], called the unramified case). For consistency with the notation of [12], all matrices and homomorphisms in this section act on the *right*.

Let $(R, \mathfrak{m}, k)$ be a one-dimensional Noetherian Cohen-Macaulay local ring which is not Dedekind-like, and let $\overline{R}$ be the normalization of R. In [12, Theorem 2.3], we suppose that there is a local ring $(\Omega, \mathfrak{n}, k)$ with $R \subsetneq \Omega \subseteq \overline{R}$, such that $\mathfrak{m}$ is the conductor of R in Ω, $\mathfrak{m} \subsetneq \mathfrak{n}$, and Ω is generated by 2 elements as an R-module. Let $P_1, \dots, P_t$ be the minimal prime ideals of R, and suppose that $r_1, \dots, r_t$ are non-negative integers. We want to build a family $(M_\kappa)_{\kappa \in k}$ of indecomposable, pairwise non-isomorphic, finitely generated R-modules, each with rank $(r_1, \dots, r_t)$. Let $Q_1, \dots, Q_t$ be the minimal prime ideals of Ω. Pick $\delta \in \mathfrak{n} - (\mathfrak{m} \cup Q_1 \dots, \cup Q_t)$, where $\delta^2 \in \mathfrak{m}$, as described in the first paragraph of the proof of [12, Theorem 2.3], and let $\bar{\delta}$ denote the image of δ in $\Omega/\mathfrak{m}$.

The following construction of the module X is described in detail in the third and fourth paragraphs of the proof of [12, Theorem 2.3]; we recall the main steps of the construction. For each $i, 1 \le i \le t$, choose an element $\lambda_i \in \mathcal{Q}(R/P_i)^\times \cap R$. Fixing an integer $n \ge \max\{r_1, \dots, r_t\}$, for each index j, $1 \le j \le n$, set $C_j = \{1 \le i \le t \mid r_i \ge j\}$, and define $w_{C_j} = \delta^8 \sum_{i \notin C_j} \lambda_i \in \delta^8 \Omega$. Put $X_1 = \Omega/w_{C_1}\Omega \oplus \dots \oplus \Omega/w_{C_n}\Omega$, $X_2 = (\Omega/\delta^4\mathfrak{m})^{(n)}$, $X_3 = (\Omega/\delta^2\mathfrak{m})^{(n)}$, and $X_4 = (\Omega/\mathfrak{m})^{(n)}$, and let $X = X_1 \oplus \dots \oplus X_4$.

The next step is to define an R-module S by the following pullback square (see [12, (2.3.2)]):

$$\begin{array}{ccc} S & \xrightarrow{\subseteq} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\nu} \\ k^{(4n)} & \xrightarrow{A} & (\Omega/\mathfrak{m})^{(4n)} \end{array} \tag{1}$$

Here the elements of $k^{(4n)}$ are viewed as row vectors, subjected to right multiplication by the block matrix

$$A := \begin{pmatrix} I & 0 & 0 & 0 \\ \bar{\delta}I & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & \bar{\delta}I & \bar{\delta}I & I \end{pmatrix},$$

where I is the $n \times n$ identity matrix over k.

As in [12, (2.3.3)], let $\sigma = \left(\begin{smallmatrix} 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \end{smallmatrix}\right) : (\Omega/\mathfrak{m})^{2n} \to X$ denote the injective Ω-homomorphism, where $\sigma_2 : (\Omega/\mathfrak{m})^{(n)} \to X_2$ and $\sigma_3 : (\Omega/\mathfrak{m})^{(n)} \to X_3$ are given by multiplication by δ^4 and δ^2, respectively. Next, let $B_\kappa : k^{(2n)} \to (\Omega/\mathfrak{m}\Omega)^{(2n)}$ be right multiplication by the matrix $B_\kappa = \left(\begin{smallmatrix} I & \bar{\delta}(\kappa I + H) \\ 0 & I \end{smallmatrix}\right)$ where I is the $n \times n$ identity matrix, and H is the $n \times n$ Jordan block

$$H = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Finally, we define $M_\kappa := S/\operatorname{im}(\tau_\kappa)$, where $\tau_\kappa = B_\kappa \sigma$. Note that the matrix B in [12, (2.3.4)] is equal to B_0. This is the only modification in the construction of the indecomposable modules we make. By adding κI to H, we "parameterize" the matrix B by the residue field k.

By exactly the same argument as in the paragraph after [12, (2.3.5)] it follows that the rank of M_κ is $(r_1, \dots, r_t)$. To see that the M_κ are indecomposable and pairwise non-isomorphic, suppose $f : M_\kappa \to M_{\kappa'}$ is an R-homomorphism. Note that $\operatorname{im}(\tau_\kappa)$ and $\operatorname{im}(\tau_{\kappa'})$ are contained in $\mathfrak{m}S$, and an easy argument shows that the R-submodules $\operatorname{im}(B_\kappa)$ and $\operatorname{im}(B_{\kappa'})$ of $(\Omega/\mathfrak{m})^{(2n)}$ contain no non-zero Ω-submodules of $(\Omega/\mathfrak{m})^{(2n)}$. It follows that S is a *separated cover* of M_κ and $M_{\kappa'}$ [15, Lemma 4.9]. As described in the proof of [12, Theorem 2.3], f lifts to an R-homomorphism $\theta : S \to S$, and θ extends to an Ω-homomorphism $\theta' : X \to X$. Exactly as in the given proof, one shows that θ' modulo the radical of Ω is right multiplication by a block upper-triangular matrix over k with four identical diagonal $n \times n$ blocks Δ.

Since the map $\theta : S \to S$ induces the R-homomorphism f from $M_\kappa = S/\operatorname{im}(\tau_\kappa)$ to $M_{\kappa'} = S/\operatorname{im}(\tau_{\kappa'})$, it follows that $(\operatorname{im}(\tau_\kappa))\theta \subseteq \operatorname{im}(\tau_{\kappa'})$. Therefore θ can be lifted to an R-homomorphism $\tilde\theta : k^{(2n)} \to k^{(2n)}$ such that $\tau_\kappa \theta = \tilde\theta \tau_{\kappa'}$. Moreover, since B_κ is invertible, $\operatorname{im}(B_\kappa \sigma)$ generates $\operatorname{im}(\sigma)$ as an Ω-submodule of X. Since the map $\theta' : X \to X$ extends θ, it follows that $(\operatorname{im}(\sigma))\theta' \subseteq \operatorname{im}(\sigma)$. Therefore θ' lifts to an Ω-homomorphism $\tilde\theta' : (\Omega/\mathfrak{m})^{(2n)} \to (\Omega/\mathfrak{m})^{(2n)}$ such that $\sigma\theta' = \tilde\theta'\sigma$. These maps yield a commutative cube:

$$\begin{array}{ccccccc}
k^{(2n)} & & & \xrightarrow{B_{\kappa'}} & & & (\Omega/\mathfrak{m})^{(2n)} \\
& \nwarrow \tilde\theta & & & & \tilde\theta' \nearrow & \\
& & k^{(2n)} & \xrightarrow{B_\kappa} & (\Omega/\mathfrak{m})^{(n)} & & \\
\tau_{\kappa'} \downarrow & & \tau_\kappa \downarrow & & \downarrow \sigma & & \downarrow \sigma \\
& & S & \xrightarrow{\subseteq} & X & & \\
& \theta \swarrow & & & & \searrow \theta' & \\
S & & & \xrightarrow{\subseteq} & & & X
\end{array}$$

By arguing as at the end of the proof of [12, Theorem 2.3], we see that the identity $\tilde\theta B_{\kappa'} = B_\kappa \tilde\theta'$ yields

$$\Delta(\kappa' I + H) = (\kappa I + H)\Delta. \tag{2}$$

On the one hand, if $\kappa = \kappa'$, then (2) implies that $\Delta H = H\Delta$, and the argument proceeds exactly as in the last two paragraphs of the proof of [12, Theorem 2.3], showing that Δ is upper-triangular with constant diagonal. If, now, f is a non-surjective idempotent endomorphism of M_κ, then $\operatorname{im}(f)$ is contained in $\mathfrak{m}M_\kappa$, and hence f must be zero. This shows that M_κ is indecomposable. On the other hand, if $\kappa \neq \kappa'$, then an inductive argument shows that (2) implies $\Delta = 0$. Then, by Nakayama's Lemma, we can conclude that θ' is not surjective, so θ is not surjective either, since $\theta' = \theta \otimes \mathrm{id}_X$. This implies that f is not surjective, because any surjective f lifts to a surjective θ [15, Theorem 4.12]. Therefore M_κ and $M_{\kappa'}$ are not isomorphic, and the proof of Lemma 2.4 is complete in the ramified case.

If the special hypothesis of [12, Theorem 2.3] does not hold, that is, if there does not exist a local ring $(\Omega, \mathfrak{n}, k)$ with $R \subsetneq \Omega \subseteq \overline{R}$, such that $\mathfrak{m}$ is the conductor of R in Ω, $\mathfrak{m} \subsetneq \mathfrak{n}$, and Ω is generated by 2 elements as an R-module, then [12, Proposition 2.2] proves that R is reduced, its normalization $\overline{R}$ is finitely generated as an R-module, $\mathfrak{m}$ is the Jacobson radical of $\overline{R}$, $\mathfrak{m}$ is the conductor of R in $\overline{R}$, and $\dim_k(\overline{R}/\mathfrak{m}) \geq 3$. This case is handled in [12, Theorem 2.4], where it is further divided into three subcases. The modifications needed to produce, for each possible rank $(r_1, ..., r_t)$, a family of $|k|$ indecomposable, pairwise non-isomorphic, finitely generated R-modules are quite similar to those sketched above for the ramified case in [12, Theorem 2.3]. Here we only mention the changes needed from that sketch, leaving the details to the interested reader.

In each of the three subcases of [12, Theorem 2.4], we begin by finding an appropriate ring Ω between R and $\overline{R}$ over which to work. We continue by defining an Ω-module X and then an R-module S defined by the conductor square (1), for the choice of matrix A as given in [12, Theorem 2.4]. In each of the three subcases, we define an injection $\sigma : (\Omega/\mathfrak{m})^{2n} \to X$, parameterize the matrix B in [12, Theorem 2.4] by $\kappa \in k$ to get B_κ, and let $M_\kappa = S/\operatorname{im}(B_\kappa \sigma)$. The matrices B_κ, in the three subcases, are defined as follows:

(1) **Basic case.** Set $B_\kappa = \begin{pmatrix} \kappa I + H & I \\ \bar{u}^2 I & \bar{u} I \end{pmatrix}$, where $1, \bar{u}, \bar{u}^2$ are units of $\Omega/\mathfrak{m}$ linearly independent over k.
(2) **Inseparable case.** Set $B_\kappa = \begin{pmatrix} \kappa I + H & I \\ \bar{v} I & \bar{u} I \end{pmatrix}$, where $1, \bar{u}, \bar{v}, \bar{u}\bar{v}$ are units of $\Omega/\mathfrak{m}$ linearly independent over k.
(3) **Small residue field case.** Set $B_\kappa = \bar{e}_1 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \bar{e}_2 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + \bar{e}_3 \begin{pmatrix} I & \kappa I + H \\ 0 & I \end{pmatrix}$, where $\bar{e}_1, \bar{e}_2, \bar{e}_3$ are primitive idempotents in the ring $\Omega/\mathfrak{m}$.

In each of these cases, the proof proceeds as in [12, Theorem 2.4], with changes similar to those made in the ramified case described above.

4 Sets of lengths

Let H be an atomic monoid and $a \in H$. The set

$$\mathsf{L}(a) = \mathsf{L}_H(a) = \{n \in \mathbb{N} \mid a \text{ is a sum of } n \text{ atoms of } H\}$$

is called the *set of lengths* of a, and $\mathfrak{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ is called the *system of sets of lengths* of H. Sets of lengths and related invariants (e.g. the elasticity of an atomic monoid) are frequently studied objects in the theory of non-unique factorizations in integral domains and monoids. The reader is referred to [1], [3], [9] and [10] for recent results in this area of research. For certain classes of integral domains and monoids (e.g. for finitely generated monoids, orders in algebraic number fields, congruence monoids in Dedekind domains, and certain higher-dimensional algebras over finite fields), sets of lengths have the following special structure: they are, up to bounded initial and final segments, a union of arithmetical progressions with bounded distance. The interested reader is referred to the monograph [10] for details. In strong contrast to these finiteness results, F. Kainrath proved the following theorem on sets of lengths of Krull monoids with infinite class group:

Theorem 4.1 *Let H be a Krull monoid with infinite class group G such that $|\mathfrak{A}(g)| > 0$ for all $g \in G$. Then, for any non-empty finite subset $L \subseteq \mathbb{N}_{\geq 2} := \{n \in \mathbb{N} \mid n \geq 2\}$, there exists $h \in H$ such that $\mathsf{L}(h) = L$.*

Proof. See [13].

Here we consider the system of sets of lengths of the monoids $\mathrm{V}(\mathfrak{F}(R))$:

Theorem 4.2 *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local Cohen-Macaulay ring.*

(1) *If* $\mathrm{spl}(R) = 0$*, then* $\mathrm{V}(\mathfrak{F}(R))$ *is factorial (that is, free).*
(2) *If* $\mathrm{spl}(R) > 0$ *and R is Dedekind-like, then* $\mathrm{V}(\mathfrak{F}(R))$ *is half factorial (that is, $|L(h)| = 1$ for all $h \in \mathrm{V}(\mathfrak{F}(R))$) but not factorial.*
(3) *If* $\mathrm{spl}(R) > 0$ *and R is not Dedekind-like, then*

$$\mathfrak{L}(\mathrm{V}(\mathfrak{F}(R))) = \{L \subseteq \mathbb{N}_{\geq 2} \mid L \textit{ is non-empty and finite}\}.$$

Proof. That $\mathrm{V}(\mathfrak{F}(R))$ is half factorial when $\mathrm{spl}(R) > 0$ and R is Dedekind-like follows, for example, from [16, Corollary 16.4(i)]. The rest of the theorem follows easily from Lemma 2.1, our Main Theorem, and Theorem 4.1.

5 Torsion-free modules

In this section we shall consider the monoid of finitely generated torsion-free modules in the case that $\widehat{R}$ has no non-zero nilpotents.

Setup. Let $(R, \mathfrak{m}, k)$ denote a one-dimensional Noetherian local ring whose completion $\widehat{R}$ has no non-zero nilpotents. We denote by $\mathrm{V}(\mathfrak{C}(R))$ the monoid of all isomorphism classes of finitely generated torsion-free R-modules. Let $p_1, \ldots, p_s$ denote the minimal primes of R. Then the set of minimal primes of $\widehat{R}$ is a disjoint union $\mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_s$, where each $P \in \mathfrak{P}_i$ contracts to p_i. We denote the rank function by $\mathrm{rank}_R : \mathrm{V}(\mathfrak{C}(R)) \longrightarrow \mathbb{N}^{(s)}$.

The main difficulty we encountered in trying to describe the monoid $\mathrm{V}(\mathfrak{C}(R))$ is that the analogue of Lemma 2.4 can fail. For example, let $R := \mathbb{C}[[X,Y]]/(Y^4 - X^5)Y$. Then R has two minimal primes $P_1 := (y^4 - x^5)$ and $P_2 := (y)$. We claim that there is no indecomposable torsion-free R-module of rank $(0,s)$ if $s \geq 2$. For suppose M is a torsion-free R-module with rank $(0,s)$, with $s \geq 2$. Then $M_{P_1} = 0$. Also $(P_2)_{P_2} = 0$, so P_2M vanishes at both minimal primes. Since P_2M is torsion-free, it follows that $P_2M = 0$, that is, M is a module over the discrete valuation ring R/P_2. Since $s \geq 2$, M decomposes. On the other hand, it follows from [24, Lemma 2.2] that for every pair (r,s) of non-negative integers with $0 < r \geq s$ there is an indecomposable torsion-free R-module of with rank (r,s). We have been unable to determine exactly which rank functions can occur for indecomposable finitely generated torsion-free R-modules. For example, is there one of rank $(1,2)$? Are there infinitely many?

A related problem, whose answer seems to depend on which ranks can occur, is to determine when the divisor homomorphism $\mathrm{V}(\mathfrak{C}(R)) \to \mathrm{V}(\mathfrak{C}(\widehat{R}))$ is actually a divisor theory. Suppose, for example, that R is a domain and that $\widehat{R}$ has 2 minimal primes P_1 and P_2. Assume that $\widehat{R}/P_1$ and $\widehat{R}/P_2$ are both discrete valuation rings. Then $S := \widehat{R}/P_1 \times \widehat{R}/P_2$ is the integral closure $\widehat{R}$. Since S is 2-generated as an $\widehat{R}$-module, $(1,0)$, $(0,1)$ and $(1,1)$ are the only possible ranks of indecomposable $\widehat{R}$-modules (by Bass's decomposition theorem, [2, Section 7]). Furthermore, there is only one isomorphism class of $\widehat{R}$-modules having rank $(1,0)$ (and the same for $(0,1)$). Thus it is easy to see that $\mathrm{V}(\mathfrak{C}(R)) \to \mathrm{V}(\mathfrak{C}(\widehat{R}))$ is not a divisor theory in this case.

If R is a domain whose completion $\widehat{R}$ has two minimal primes P_1, P_2, with exactly *one* of the $\widehat{R}/P_i$ a discrete valuation domain (essentially the case discussed in the paragraph after the **Setup**), we have been unable to determine whether or not the map $\mathrm{V}(\mathfrak{C}(R)) \to \mathrm{V}(\mathfrak{C}(\widehat{R}))$ is a divisor theory. The following theorem (in conjunction with the negative result in the preceding paragraph) gives the answer in every other case:

Theorem 5.1 *Suppose R is as in the* **Setup** *and that at least one of the following conditions is satisfied:*

(1) *The number of minimal primes of $\widehat{R}$ is different from 2.*
(2) *R is not a domain, and $\widehat{R}$ has exactly 2 minimal primes.*
(3) *R is a domain, $\widehat{R}$ has exactly 2 minimal primes P_1 and P_2 and neither $\widehat{R}/P_1$ nor $\widehat{R}/P_2$ is a discrete valuation domain.*

Then the natural homomorphism $\mathrm{V}(\mathfrak{C}(R)) \to \mathrm{V}(\mathfrak{C}(\widehat{R}))$ is a divisor theory.

Proof. Denote by $P_1, \ldots, P_t$ the minimal primes of $\widehat{R}$, and let $[A] \in \mathrm{V}(\mathfrak{C}(\widehat{R}))$. We have to display $[A]$ as the greatest lower bound of finitely many torsion-free $\widehat{R}$-modules which are extended from R-modules. If R and $\widehat{R}$ have the same number of minimal primes, then A itself is extended, by Lemma 2.5. Thus the theorem holds in case (2) and also if $t = 1$.

Suppose next that $t \geq 3$. For $1 \leq i \leq t$ let $U_i \in \mathrm{V}(\mathfrak{C}(R))$ denote the indecomposable $\widehat{R}$-module

$$\widehat{R}/(P_1 \cap \cdots \cap P_{i-1} \cap P_{i+1} \cap \cdots \cap P_t).$$

Then the j-th component of $\mathrm{rank}_{\widehat{R}}(U_i)$ is 1 if $j \neq i$ and 0 if $j = i$. Let E_i denote the $\widehat{R}$-module $\widehat{R}/P_i$. The rank of E_i is the i-th unit vector of $\mathbb{N}^{(t)}$. An easy argument shows that there exist $l_i, m_i \in \mathbb{N}$ such that

$$B := A \oplus U_1^{(l_1)} \oplus \cdots \oplus U_t^{(l_t)} \quad \text{and} \quad C := A \oplus E_1^{(m_1)} \oplus \cdots \oplus E_t^{(m_t)}$$

have constant rank. Now the modules B and C are extended, by Lemma 2.5. Since the U_i, E_j are pairwise non-isomorphic, we see that $[A]$ is the greatest lower bound of $[B]$ and $[C]$, as desired. This completes the proof in case (1).

Finally, we suppose that condition (3) is satisfied. For $i = 1, 2$, let U_i be a non-principal ideal of $\widehat{R}/P_i$, and put $E_i = \widehat{R}/P_i$. Exactly as in the case $t \geq 3$, we use the four pairwise non-isomorphic indecomposable torsion-free modules U_i, E_j to represent $[A]$ as the greatest lower bound of two extended modules.

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