LARGE INDECOMPOSABLE MCM MODULES

Graham Leuschke and Roger Wiegand

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Theorem. *Let $(S, \mathbf{n})$ be a Cohen-Macaulay local ring of dimension at least two, and let Z be an indeterminate. Then $R := S[Z]/(Z^2)$ has unbounded Cohen-Macaulay type.*

Proof. We will show that for every $n \geq 2$ there is an indecomposable MCM R-module of rank $2n$. Fix $n \geq 2$, and let W be a free S-module of rank $2n$. Let I be the $n \times n$ identity matrix and J the $n \times n$ nilpotent Jordan block with 1 on the superdiagonal and 0 elsewhere. Let $\{x, y\}$ be part of a minimal generating set for the maximal ideal $\mathbf{m}$ of S, and put $\varphi := xI + yJ$. Finally, put $\psi := \left[\begin{smallmatrix} 0 & \varphi \\ 0 & 0 \end{smallmatrix}\right]$. Noting that $\psi^2 = 0$, we make W into an R-module by letting z act as ψ. Then W is a MCM R-module, and we claim that it is indecomposable.

Suppose $W = U \oplus V$ as R-modules, with $U \neq W$. We want to show that $U = 0$. There is a $2n \times 2n$ idempotent matrix ε such that $U = \varepsilon(W) = \ker(1-\varepsilon)$ and $V = (1-\varepsilon)(W) = \ker(\varepsilon)$. Since U is an R-submodule of W, we have $\psi(U) \subseteq U$, that is, $(1 - \varepsilon)\psi\varepsilon = 0$. Similarly, since $\psi(V) \subset V$, we have $\varepsilon\psi(1 - \varepsilon) = 0$. Combining these two equations, we have

$$\psi\varepsilon = \varepsilon\psi. \tag{1}$$

Write $\varepsilon = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$, where each block is $n \times n$. From (1), we obtain the equation

$$\begin{bmatrix} \gamma x + J\gamma y & \delta x + J\delta y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha x + \alpha J y \\ 0 & \gamma x + \gamma J y \end{bmatrix}. \tag{2}$$

Since $x + \mathbf{n}^2$ and $y + \mathbf{n}^2$ are linearly independent over $k := S/\mathbf{n}$, we get the equations

$$\bar{\gamma} = 0, \qquad \bar{\delta} = \bar{\alpha}, \qquad \bar{J}\bar{\delta} = \bar{\alpha}\bar{J}, \tag{3}$$

where the bars denote reductions modulo $\mathbf{n}$. Therefore $\bar{\varepsilon} = \left[\begin{smallmatrix} \bar{\alpha} & \bar{\beta} \\ 0 & \bar{\alpha} \end{smallmatrix}\right]$, and $\bar{\alpha}\bar{J} = \bar{J}\bar{\alpha}$. Since $\bar{\alpha}$ commutes with the non-derogatory matrix $\bar{J}$, $\bar{\alpha}$ belongs to $k[\bar{J}]$. In particular, $\bar{\alpha}$ is upper-triangular with a constant, say a, on the diagonal.

Since $U \neq W$, Nakayama's lemma implies that $\bar{\epsilon}$ is not surjective, whence $a = 0$. Therefore $\bar{\varepsilon}^{2n} = 0$, and, since $\bar{\varepsilon}^2 = \bar{\varepsilon}$, we have $\bar{\varepsilon} = 0$. By Nakayama's lemma, $1 - \varepsilon$ is surjective and, being idempotent, must be equal to the identity matrix. Thus $U = 0$, as desired. □

References