ASCENT OF MODULE STRUCTURES, VANISHING OF EXT, AND EXTENDED MODULES

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ABSTRACT. Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be commutative Noetherian local rings, and let $\varphi : R \to S$ be a flat local homomorphism such that $\mathfrak{m}S = \mathfrak{n}$ and the induced map on residue fields $R/\mathfrak{m} \to S/\mathfrak{n}$ is an isomorphism. Given a finitely generated R-module M, we show that M has an S-module structure compatible with the given R-module structure if and only if $\operatorname{Ext}^i_R(S, M)$ is finitely generated as an R-module for each $i \geq 1$.

We say that an S-module N is *extended* if there is a finitely generated R-module M such that $N \cong S \otimes_R M$. Given a short exact sequence $0 \to N_1 \to N \to N_2 \to 0$ of finitely generated S-modules, with two of the three modules N_1, N, N_2 extended, we obtain conditions forcing the third module to be extended. We show that every finitely generated module over the Henselization of R is a direct summand of an extended module, but that the analogous result fails for the $\mathfrak{m}$-adic completion.

INTRODUCTION

Suppose $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are commutative Noetherian local rings and $\varphi: R \to S$ is a flat local homomorphism with the property that the induced homomorphism $R/\mathfrak{m} \to S/\mathfrak{m}S$ is bijective. We consider questions of ascent and descent of modules between R and S: (1) Given a finitely generated R-module M, when does M have an S-module structure that is compatible with the R-module structure via φ? (2) Given a finitely generated S-module N, is there a finitely generated R-module M such that N is S-isomorphic to $S \otimes_R M$, or (3) S-isomorphic to a direct summand of $S \otimes_R M$?

In Section 1 we make some general observations about homomorphisms $R \to S$ satisfying the condition $R/\mathfrak{m} = S/\mathfrak{m}S$. We show that if a compatible S-module structure exists, then it arises in an obvious way: The natural map $M \to S \otimes_R M$ is an isomorphism. (One example to keep in mind is that of a finite-length module M when $S = \widehat{R}$, the $\mathfrak{m}$-adic completion.) Moreover, if $R \to S$ is flat, then M has a compatible S-module structure if and only if $S \otimes_R M$ is finitely generated as an R-module. In Section 2 we prove, assuming that $R \to S$ is flat, that M has a compatible S-module structure if and only if $\operatorname{Ext}^i_R(S, M)$ is finitely generated as an R-module for $i = 1, \ldots, \dim_R(M)$. Theorem 2.5 summarizes the main results of the first two sections. In Section 3 we address questions (2) and (3) and show

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that (3) always has an affirmative answer when S is the Henselization, but not necessarily when S is the $\mathfrak{m}$-adic completion.

1. Ascent of module structures

Throughout this section $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are Noetherian local rings and $\varphi\colon R \to S$ is a ring homomorphism. We consider the following condition on φ:

(†) The induced homomorphism $R/\mathfrak{m} \to S/\mathfrak{m}S$ is bijective.

This condition is equivalent to the following: (i) $\mathfrak{m}S = \mathfrak{n}$ and (ii) $R + \mathfrak{n} = S$. (Condition (ii) just says that $R \to S$ induces an isomorphism on residue fields.)

Familiar examples include the $\mathfrak{m}$-adic completion $R \to \widehat{R}$, the Henselization $R \to R^{\mathrm{h}}$, and the natural map $R \twoheadrightarrow S = R/I$, when I is a proper ideal of R.

From (i), it follows immediately that $\mathfrak{m}^t S = \mathfrak{n}^t$ for all t. Similarly, the next result shows that (ii) carries over to powers (though here we need *both* (i) and (ii), as is shown by the example $\mathbb{C}[[T^2, T^3]] \subseteq \mathbb{C}[[T]]$).

Lemma 1.1. *If $\varphi\colon R \to S$ satisfies (†), then $R + \mathfrak{n}^t = S$ for each $t \geq 1$.*

Proof. By choosing a composition series, we see that every S-module of finite length has (the same) finite length as an R-module. In particular, S/n^{t+1} has finite length and therefore is finitely generated as an R-module. We have

$$\frac{R + \mathfrak{n}^t}{\mathfrak{n}^{t+1}} + \mathfrak{m}\frac{S}{\mathfrak{n}^{t+1}} = \frac{R + \mathfrak{n}^t + \mathfrak{m}S}{\mathfrak{n}^{t+1}} = \frac{R + \mathfrak{n}}{\mathfrak{n}^{t+1}} = \frac{S}{\mathfrak{n}^{t+1}}.$$

Nakayama's lemma implies that $(R + \mathfrak{n}^t)/\mathfrak{n}^{t+1} = S/\mathfrak{n}^{t+1}$. $\square$

The next result is an indispensable tool for several of our proofs.

Proposition 1.2. *Assume $\varphi\colon R \to S$ satisfies (†). Let M and N be S-modules, with ${}_SN$ finitely generated. Then* $\operatorname{Hom}_R(M, N) = \operatorname{Hom}_S(M, N)$.

Proof. We'll show that $\operatorname{Hom}_R(M, N) \subseteq \operatorname{Hom}_S(M, N)$, since the reverse inclusion is obvious. Let $f \in \operatorname{Hom}_R(M, N)$. Given $x \in M$ and $s \in S$, we want to show that $f(sx) = sf(x)$. Since ${}_SN$ is finitely generated, it will suffice to show that $f(sx) - sf(x) \in \mathfrak{n}^t N$ for each $t \geq 1$.

Fix an integer $t \geq 1$, and note the following relations:

$$f(\mathfrak{n}^t M) + \mathfrak{n}^t N = f(\mathfrak{m}^t M) + \mathfrak{n}^t N \subseteq \mathfrak{m}^t N + \mathfrak{n}^t N = \mathfrak{n}^t N$$

Use (1.1) to choose an element $r \in R$ such that $r - s \in \mathfrak{n}^t$. Then we have

$$f(sx) - sf(x) = f(sx) - f(rx) + rf(x) - sf(x) = f(sx - rx) + (r - s)f(x).$$

It follows that $f(sx) - sf(x)$ is in

$$f((s - r)M) + (r - s)N \subseteq f(\mathfrak{n}^t M) + \mathfrak{n}^t N \subseteq \mathfrak{n}^t N. \quad \square$$

Corollary 1.3. *Let $\varphi: R \to S$ be a flat local homomorphism satisfying (†), and let M be a finitely generated S-module. Then M is indecomposable as an R-module if and only if it is indecomposable as an S-module.*

Proof. We know that M is indecomposable as an R-module if and only $\operatorname{End}_R(M)$ has no nontrivial idempotents, and similarly over S. The equality $\operatorname{End}_R(M) = \operatorname{End}_S(M)$ from Proposition 1.2 now yields the desired result. □

For any ring homomorphism $\varphi : R \to S$, every S-module acquires an R-module structure via φ. We want to understand when the reverse holds: Given an R-module M, often assumed to be finitely generated, when does M have an S-module structure $(s, x) \mapsto s \circ x$ that is compatible with the R-module structure, that is, $rx = \varphi(r) \circ x$, for $r \in R$ and $x \in M$? When this happens, we will say simply that ${}_RM$ *has a compatible* S*-module structure.* We are particularly interested in the case where the S-module structure is *unique*.

Lemma 1.4. *Assume $\varphi: R \to S$ satisfies (†). Let N be a finitely generated S-module, and let V be an R-submodule of N. Then ${}_RV$ has at most one compatible S-module structure. In detail: If V has an S-module structure $(s, v) \mapsto s \circ v$ that is compatible with the R-module structure on V inherited from the S-module structure $(s, n) \mapsto s \cdot n$ on N, then $s \circ v = s \cdot v$ for all $s \in S$ and $v \in V$.*

Proof. Let $s \in S$ and $v \in V$ be given. As before, we fix an integer $t \geq 1$ and choose $r \in R$ such that $r - s \in \mathfrak{n}^t$. Note the following relations:

$$\mathfrak{n}^t \circ V = (\mathfrak{m}^t S) \circ V = \mathfrak{m}^t \circ (S \circ V) = \mathfrak{m}^t \circ V = \mathfrak{m}^t \cdot V \subseteq \mathfrak{m}^t \cdot N = \mathfrak{n}^t \cdot N$$

It follows that we have

$$s \circ v - s \cdot v = s \circ v - r \circ v + r \cdot v - s \cdot v = (s - r) \circ v + (r - s) \cdot v \in \mathfrak{n}^t \circ V + \mathfrak{n}^t \cdot V \subseteq \mathfrak{n}^t \cdot N.$$

Since t was chosen arbitrarily, we conclude that $s \circ v = s \cdot v$. □

1.5. Proposition and Notation. *Assume $\varphi: R \to S$ satisfies (†). Let M be an R-module (not necessarily finitely generated) that is an R-submodule of some finitely generated S-module N. Let $\mathcal{V}(M)$ be the set of R-submodules of M that have an S-module structure compatible with their R-module structure. Then $\mathcal{V}(M)$ is exactly the set of S-submodules of N that are contained in M. The set $\mathcal{V}(M)$ has a unique maximal element $\mathrm{V}(M)$. Moreover, we have $\mathrm{V}(M) = \{x \in M \mid Sx \subseteq M\} = \{x \in N \mid Sx \subseteq M\}$.*

Proof. The first assertion is clear from (1.4). It follows that $\mathcal{V}(M)$ is closed under sums. Since N is a Noetherian S-module, the other assertions follow easily. □

Although $\mathrm{V}(M)$ is defined only when M can be embedded as an R-submodule of some finitely generated S-module N, its definition is intrinsic. Thus the submodule $\mathrm{V}(M)$ of M does not depend on the choice of the module N or the R-embedding $M \hookrightarrow N$. (See Corollary 1.7 for another intrinsic characterization of $\mathrm{V}(M)$.)

Proposition 1.6. *Assume $\varphi: R \to S$ satisfies (†), and let L be an S-module (not necessarily finitely generated). Let M be an R-submodule of some finitely generated S-module, and let $\mathrm{V}(M)$ be as in (1.5). Then the natural injection $\mathrm{Hom}_R(L, \mathrm{V}(M)) \to \mathrm{Hom}_R(L, M)$ is an isomorphism.*

Proof. Let $g \in \mathrm{Hom}_R(L, M)$, and let W be the image of f. We want to show that $W \subseteq \mathrm{V}(M)$. Let h be the composition $L \xrightarrow{g} M \hookrightarrow N$, where N is some finitely generated S-module containing M as an R-submodule. By (1.2), the map h is S-linear, so $W = h(L)$ is an S-submodule of N. Therefore we have $W \subseteq \mathrm{V}(M)$. □

Corollary 1.7. *Assume $\varphi: R \to S$ satisfies (†). Let M be an R-submodule of a finitely generated S-module. The following natural maps are isomorphisms:*

$$\mathrm{V}(M) \xrightarrow{\cong} \mathrm{Hom}_S(S, \mathrm{V}(M)) \xrightarrow{=} \mathrm{Hom}_R(S, \mathrm{V}(M)) \xrightarrow{\cong} \mathrm{Hom}_R(S, M)$$

It follows that $\mathrm{V}(M)$ is exactly the image of the natural map $\varepsilon: \mathrm{Hom}_R(S, M) \to M$ taking ψ to $\psi(1)$. In particular, if M is finitely generated as an R-module, so is $\mathrm{Hom}_R(S, M)$. □

The next result contains the first part of our answer to Question (1) from the Introduction.

Theorem 1.8. *Assume $\varphi: R \to S$ satisfies (†), and let M be a finitely generated R-module. The following conditions are equivalent:*

(1) *M has a compatible S-module structure.*
(2) *The natural map $\iota: M \to S \otimes_R M$ (taking x to $1 \otimes x$) is bijective.*
(3) *The natural map $\varepsilon: \mathrm{Hom}_R(S, M) \to M$ (taking ψ to $\psi(1)$) is bijective.*

If, in addition, φ is flat, these conditions are equivalent to the following:

(4) *$S \otimes_R M$ is finitely generated as an R-module.*

Proof. The implications (2) $\Longrightarrow$ (1), (3) $\Longrightarrow$ (1), and (2) $\Longrightarrow$ (4) are clear. Assume (1), and let $(s, x) \mapsto s \cdot x$ be a compatible S-module structure on M. To prove (2), we note that the module $S \otimes_R M$ has two compatible S-module structures—the one coming from multiplication in S and the one coming from the S-module structure on M. Moreover, with the first structure, $S \otimes_R M$ is finitely generated over S. By (1.4) the two S-module structures must be the same. In particular, for $s \in S$ and $x \in M$ we have $s \otimes x = s(1 \otimes x) = 1 \otimes (s \cdot x)$. Therefore the multiplication map $\mu: S \otimes_R M \to M$ (taking $s \otimes x$ to $s \cdot x$) is the inverse of ι.

Still assuming (1), we prove (3). Since M is finitely generated as an S-module, (1.2) tells us that $\mathrm{Hom}_R(S, M) = \mathrm{Hom}_S(S, M)$. Therefore the map $M \to \mathrm{Hom}_S(S, M)$ taking $x \in M$ to the map $s \mapsto s \cdot x$ is the inverse of ε.

(4) $\Longrightarrow$ (2). Assume that φ is flat. By (4), the S-module $S \otimes_R S \otimes_R M$ is finitely generated for the S-action on the first variable; therefore its two S-module structures (obtained by letting S act on each of the first two factors) are the same, by (1.4). In particular, $s \otimes t \otimes x = st \otimes 1 \otimes x$ for $s, t \in S$ and $x \in M$. Therefore the map $S \otimes_R S \otimes_R M \to S \otimes_R M$ taking $s \otimes t \otimes x$ to $st \otimes x$ is the inverse of $1 \otimes \iota: S \otimes_R M \to S \otimes_R S \otimes_R M$. By faithful flatness, ι is an isomorphism. □

In light of Corollary 1.7, we see that the conditions in the previous result are not equivalent to $\operatorname{Hom}_R(S, M)$ being finitely generated as an R-module, even when φ is flat. In the next section, we will show that the "right" condition is that $\operatorname{Ext}^i_R(S, M)$ be finitely generated for $i = 1, \ldots, \dim_R(M)$.

Next we revisit Theorem 1.8 from a slightly different perspective:

Theorem 1.9. *Let $\varphi\colon R \to S$ be a flat local homomorphism satisfying (†), and let M be a finitely generated S-module. The following conditions are equivalent:*

(1) *M is finitely generated as an R-module.*
(2) *The natural map $\iota_M\colon M \to S \otimes_R M$ (taking x to $1 \otimes x$) is bijective.*
(3) *$S \otimes_R M$ is finitely generated as an R-module.*

In particular, if S has a faithful module that is finitely generated as an R-module, then φ is an isomorphism.

Proof. The implication (1) $\Longrightarrow$ (2) is in Theorem 1.8. Suppose (2) holds. The R-module $S \otimes_R M$ has two S-modules structures and, by (2), is finitely generated with respect to the S-action on the second factor. By Lemma 1.4, the two structures agree, and $S \otimes_R M$ is finitely generated with respect to the S-action on the first factor. By faithfully flat descent, M is finitely generated over R. Using (2) again, we get (3).

If (3) holds, then $S \otimes_R M$ is *a fortiori* finitely generated for the action of S on the first factor. Again using faithfully flat descent, we get (1).

To prove the last statement, suppose N is a faithful S-module that is finitely generated as an R-module. Let $x_1, \ldots, x_t$ generate N as an S-module, and define $\alpha : S \to N^t$ by $1 \mapsto (x_1, \ldots x_t)$. The kernel of α is the intersection of the annihilators of the x_i, and this intersection is (0) since N is faithful. Thus S embeds in N^t and therefore is finitely generated as an R-module. Now we put $M = S$ in (2) and note that $\varphi \otimes_R S\colon R \otimes_R S \to S \otimes_R S$ is the composition $R \otimes_R S \xrightarrow{\cong} S \xrightarrow{\iota_S} S \otimes_R S$. Therefore $\varphi \otimes_R S$ is an isomorphism, and by faithful flatness φ must be an isomorphism. □

Proposition 1.10. *Assume $\varphi\colon R \to S$ satisfies (†). The following conditions are equivalent:*

(1) *R has a compatible S-module structure.*
(2) *φ is an R-split monomorphism.*
(3) *S is a free R-module.*
(4) *φ is a bijection.*

Proof. The implication (4) $\Longrightarrow$ (3) is clear.

(1) $\Longrightarrow$ (4). From (1.8) we conclude that the map $\iota\colon R \to S \otimes_R R$ is bijective, and it follows that φ is the composition of two bijections: $R \xrightarrow{\iota} S \otimes_R R \to S$.

(2) $\Longrightarrow$ (1). Let $\pi\colon S \to R$ be an R-homomorphism such that $\pi\varphi = 1_R$. The composition $\varphi\pi\colon S \to S$ is S-linear by (1.2), so $\varphi(R) = \varphi\pi(S)$ is an S-module, and (1) follows.

(3) $\Longrightarrow$ (2). Let B be a basis for S as an R-module. Write $1 = \sum_{i=1}^n r_i b_i$ where the r_i are in R and the b_i are distinct elements of B. If each r_i were in $\mathfrak{m}$, we would have $1 \in \mathfrak{m}S = \mathfrak{n}$, contradiction. Thus we may assume that r_1 is a unit of R. Let $\pi : S \to R$ be the R-homomorphism taking b_1 to r_1^{-1} and $b \in B - \{b_1\}$ to 0. Then $\pi\varphi = 1_R$, and we have (2). □

Now we focus on flat homomorphisms satisfying (†). (In this context *every* finitely generated R-module can be embedded in a finitely generated S-module, namely $S \otimes_R M$. Thus $\mathrm{V}(M)$ is always defined.) Every finite-length R-module has a compatible S-module structure. (This follows from (1.12) below, by induction on the length, since $R/\mathfrak{m} = S/\mathfrak{m}S$.) There are other examples:

Example 1.11. Let R be a local ring and P a non-maximal prime ideal such that R/P is $\mathfrak{m}$-adically complete (e.g, $R = (\mathbb{C}[X]_{(X)})[[Y]]$ and $P = (X)$). Then R/P has a compatible $\widehat{R}$-module structure. Indeed, the map $R/P \to \widehat{R}/P\widehat{R}$ is bijective.

As we shall see in (1.13), the behavior of prime ideals tells the whole story. The following lemma is clear from the five-lemma and criterion (2) of (1.8):

Lemma 1.12. *Let $\varphi\colon R \to S$ be a flat local homomorphism satisfying* (†)*, and let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of finitely generated R-modules. Then M has a compatible S-module structure if and only if M' and M'' have compatible S-module structures. □

Theorem 1.13. *Let $\varphi\colon R \to S$ be a flat local homomorphism satisfying* (†)*, and let M be a finitely generated R-module. The following conditions are equivalent:*

(1) *M has a compatible S-module structure.*
(2) *$S = R + PS$ (equivalently, R/P has a compatible S-module structure), for every $P \in \mathrm{Min}_R(M)$.*
(3) *$S = R + PS$ (equivalently, R/P has a compatible S-module structure), for every $P \in \mathrm{Supp}_R(M)$.*

Proof. The condition $S = R + PS$ just says that the injection $R/P \hookrightarrow S \otimes_R (R/P)$ is an isomorphism; now (1.8) justifies the parenthetical comments. If (1) holds and $P \in \mathrm{Min}_R(M)$, then there is an injection $R/P \hookrightarrow M$, so (1.12) with $M' = R/P$ yields (2). Assume (2). Given $P \in \mathrm{Supp}_R(M)$ we have $P \supseteq Q$ for some $Q \in \mathrm{Min}_R(M)$. Then $R/Q \twoheadrightarrow R/P$, and (3) follows from (1.12). Assuming (3), choose a prime filtration $M = M_0 \subset \cdots \subset M_t$ with $M_i/M_{i-1} \cong R/P_{i-1}$ with $P_i \in \mathrm{Spec}(R)$, $i = 1 \ldots, t$. Then $P_i \in \mathrm{Supp}_R(M)$ for each i, and now (1) follows from (1.12). □

Let $\varphi\colon (R, \mathfrak{m}, k) \hookrightarrow (S, \mathfrak{n}, l)$ be a flat local homomorphism. Recall that φ is *separable* if the "diagonal" morphism $S \otimes_R S \to S$ splits as $S \otimes_R S$ modules (cf. [DI]). If, further, φ is essentially of finite type, then φ is said to be an *étale* extension of R (cf. [Iv]). An étale extension φ is a *pointed étale neighborhood* of R if $k = l$. It is easy to see that $\mathfrak{m}S = \mathfrak{n}$ whenever φ is an étale extension; thus pointed étale neighborhoods satisfy condition (†). The R-isomorphism classes of pointed étale neighborhoods form a direct system, and the Henselization $R \to R^{\mathrm{h}}$ is the direct limit of them.

Corollary 1.14. *Let R be a local ring and M a finitely generated R-module. The following conditions are equivalent.*

(1) *M admits an R^{h}-module structure that is compatible with its R-module structure via the natural inclusion $R \to R^{\mathrm{h}}$.*

(2) *For each* $P \in \operatorname{Supp}_R(M)$, *the ring* R/P *is Henselian.*
(3) *For each* $P \in \operatorname{Min}_R(M)$, *the ring* R/P *is Henselian.*
(4) *The ring* $R/\operatorname{Ann}_R(M)$ *is Henselian.* $\square$

Corollary 1.15. *Let* R *be a local ring. The following conditions are equivalent.*

(1) R *is Henselian.*
(2) *For each* $P \in \operatorname{Spec}(R)$, *the ring* R/P *is Henselian.*
(3) *For each* $P \in \operatorname{Min}(R)$, *the ring* R/P *is Henselian.* $\square$

2. Vanishing of Ext

Our goal in this section is to add a fifth condition equivalent to the conditions in Theorem 1.8, namely, that $\operatorname{Ext}^i_R(S, M) = 0$ for $i > 0$. Here $R \to S$ is a flat local homomorphism satisfying (†) and M is a finitely generated R-module. Moreover, we will obtain a sixth equivalent condition, namely, that $\operatorname{Ext}^i_R(S, M)$ is finitely generated over R for $i = 1, \ldots, \dim_R(M)$. Since our proof uses complexes we will review the basic yoga here.

2.1. Notation and conventions. An *R-complex* is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots$$

such that $\partial^X_{n-1}\partial^X_n = 0$ for each integer n; the nth *homology module* of X is $\mathrm{H}_n(X) := \operatorname{Ker}(\partial^X_n)/\operatorname{Im}(\partial^X_{n+1})$. A complex X is *bounded* if $X_n = 0$ for $|n| >> 0$, *bounded above* if $X_n = 0$ for $n >> 0$, and *homologically finite* if its total homology module $\mathrm{H}(X) = \oplus_n \mathrm{H}_n(X)$ is a finitely generated R-module.

Let X, Y be R-complexes. The *Hom complex* $\operatorname{Hom}_R(X, Y)$ is the R-complex defined as

$$\operatorname{Hom}_R(X, Y)_n = \prod_p \operatorname{Hom}_R(X_p, Y_{p+n})$$

with nth differential $\partial^{\operatorname{Hom}_R(X,Y)}_n$ given by

$$\{f_p\} \mapsto \{\partial^Y_{p+n} f_p - (-1)^n f_{p-1}\partial^X_p\}.$$

A *morphism* $X \to Y$ is an element $f = \{f_p\} \in \operatorname{Hom}_R(X, Y)_0$ such that $\partial^Y_p f_p = f_{p-1}\partial^X_p$ for all p, that is, an element of $\operatorname{Ker}(\partial^{\operatorname{Hom}_R(X,Y)}_0)$.

A morphism of complexes $\alpha\colon X \to Y$ induces homomorphisms on homology modules $\mathrm{H}_n(\alpha)\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$, and α is a *quasi-isomorphism* when each $\mathrm{H}_n(\alpha)$ is bijective. The symbol "$\simeq$" indicates a quasi-isomorphism.

2.2. Base change. Let $\varphi\colon R \to S$ be a flat homomorphism. For any R-complex X, the flatness of φ provides natural S-module isomorphisms

$$\mathrm{H}_i(S \otimes_R X) \cong S \otimes_R \mathrm{H}_i(X)$$

for each integer i.

2.3. A connection with condition (†)**.** Let $\varphi\colon (R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ be a flat local ring homomorphism, and write $\overline{\varphi}\colon k\to S/\mathfrak{m}S$ for the induced ring homomorphism. Let $X\not\simeq 0$ be an R-complex such that each homology module $\mathrm{H}_i(X)$ is a finite-dimensional k-vector space, and let r_i denote the vector-space dimension of $\mathrm{H}_i(X)$. (In our applications we will consider the case $X=K^R$, the Koszul complex on a minimal system of generators for $\mathfrak{m}$. By [BH, (1.6.5)], the homology $\mathrm{H}(K^R)$ is annihilated by $\mathfrak{m}$, and so each $\mathrm{H}_i(K^R)$ is a finite-dimensional k-vector space. Note that $K^R\not\simeq 0$ since $\mathrm{H}_0(K^R)\cong k$.) Define $\omega\colon X\to S\otimes_R X$ by the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \omega\ } & S\otimes_R X \\ \cong\searrow & & \nearrow_{\varphi\otimes_R X} \\ & R\otimes_R X & \end{array}$$

where the southeast arrow represents the standard isomorphism. We have a commutative diagram of k-linear homomorphisms

$$\begin{array}{ccccccc} \mathrm{H}_i(X) & \xrightarrow{\mathrm{H}_i(\omega)} & \mathrm{H}_i(S\otimes_R X) & \xrightarrow{\ \cong\ } & S\otimes_R \mathrm{H}_i(X) & \xrightarrow{\ \cong\ } & S\otimes_R k^{(r_i)} \\ \cong\big\downarrow & & & & & & \cong\big\downarrow \\ k^{(r_i)} & & & \xrightarrow{\overline{\varphi}^{(r_i)}} & & & (S/\mathfrak{m}S)^{(r_i)} \end{array}$$

Therefore the morphism ω is a quasi-isomorphism if and only if $\overline{\varphi}$ is an isomorphism, that is, if and only if $\varphi\colon R\to S$ satisfies the condition (†) of Section 1.

The following result is contained in [FSW, (5.3)].

Proposition 2.4. *Let X and Y be R-complexes such that $\mathrm{H}_n(X)$ and $\mathrm{H}_n(Y)$ are finitely generated R-modules for each n. Let $\alpha\colon X\to Y$ be a morphism. Assume that P is a bounded complex of finitely generated projective R-modules such that $P\not\simeq 0$ and $\mathrm{Hom}_R(P,\alpha)$ is a quasi-isomorphism. Then α is a quasi-isomorphism.* □

We can now put the finishing touch on Theorem 1.8:

Main Theorem 2.5. *Let $\varphi\colon R\to S$ be a ring homomorphism satisfying (†), and let M be a finitely generated R-module. The following conditions are equivalent:*

(1) *M has a compatible S-module structure.*
(2) *The natural map $\iota\colon M\to S\otimes_R M$ (taking x to $1\otimes x$) is bijective.*
(3) *The natural map $\varepsilon\colon \mathrm{Hom}_R(S,M)\to M$ (taking ψ to $\psi(1)$) is bijective.*

If, in addition, φ is flat, these conditions are equivalent to the following:

(4) *$S\otimes_R M$ is finitely generated as an R-module.*
(5) *$\mathrm{Ext}^i_R(S,M)$ is a finitely generated R-module for $i=1,\dots,\dim_R(M)$.*
(6) *$\mathrm{Ext}^i_R(S,M)=0$ for all $i>0$.*

Proof. The equivalences (1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(3) are in Theorem 1.8, as is (3)$\Longleftrightarrow$(4) when φ is flat. The implication (6) $\Longrightarrow$ (5) is trivial, so it remains to assume that φ is flat and prove (5) $\Longrightarrow$ (3) and (1) $\Longrightarrow$ (6).

(5) $\implies$ (3). Assume that $\operatorname{Ext}^i_R(S,M)$ is finitely generated over R for $i=1,\dots,\dim_R(M)$. We first show that $\operatorname{Ext}^i_R(S,M)=0$ for each $i>\dim_R(M)$. Let P be an R-projective resolution of S, and set $R'=R/\operatorname{Ann}_R(M)$. The fact that M is an R'-module yields the first isomorphism in the following sequence:

$$\operatorname{Hom}_R(P,M)\cong\operatorname{Hom}_R(P,\operatorname{Hom}_{R'}(R',M))\cong\operatorname{Hom}_{R'}(P\otimes_R R',M) \tag{2.5.1}$$

The second isomorphism is Hom-tensor adjointness. Of course we have isomorphisms $\mathrm{H}_n(P\otimes_R R')\cong\operatorname{Tor}^R_n(S,R')$, so the flatness of φ yields $\mathrm{H}_n(P\otimes_R R')=0$ for $n>0$. Therefore the complex $P\otimes_R R'$ is an R'-projective resolution of $S':=S\otimes_R R'$. Since S' is flat over R', we have $\operatorname{pd}_{R'}(S')\leq\dim(R')$ by a result of Gruson and Raynaud [RG, Seconde Partie, Thm. (3.2.6)], and Jensen [J, Prop. 6]. Therefore $\operatorname{Ext}^n_{R'}(S',M)=0$ for each $n>\dim(R')=\dim_R(M)$. This yields the vanishing in the next sequence, for $n>\dim_R(M)$:

$$\operatorname{Ext}^n_R(S,M)\cong\mathrm{H}_{-n}(\operatorname{Hom}_R(P,M))\cong\mathrm{H}_{-n}(\operatorname{Hom}_{R'}(P\otimes_R R',M))\cong\operatorname{Ext}^n_{R'}(S',M)=0$$

The first isomorphism is by definition; the second one is from (2.5.1); and the third one is from the fact, already noted, that $P\otimes_R R'$ is an R'-projective resolution of $S'=S\otimes_R R'$.

Let I be an R-injective resolution of M. From Corollary 1.7, it follows that $\operatorname{Hom}_R(S,M)$ is a finitely generated R-module. Since $\operatorname{Ext}^n_R(S,M)=0$ for $i>\dim_R(M)$ and $\operatorname{Ext}^n_R(S,M)$ is finitely generated over R for $1\leq n\leq\dim_R(M)$, the complex $\operatorname{Hom}_R(S,I)$ is homologically finite over R.

Consider the evaluation morphism $\alpha\colon\operatorname{Hom}_R(S,I)\to I$ given by $f\mapsto f(1)$. To verify condition (3), it suffices to show that α is a quasi-isomorphism. Indeed, assume for the rest of this paragraph that α is a quasi-isomorphism. It is straightforward to show that the map $\mathrm{H}_0(\alpha)\colon\mathrm{H}_0(\operatorname{Hom}_R(S,I))\to\mathrm{H}_0(I)$ is equivalent to the evaluation map $\varepsilon\colon\operatorname{Hom}_R(S,M)\to M$. The quasi-isomorphism assumption implies that ε is an isomorphism, and so condition (3) holds.

We now show that α is a quasi-isomorphism. Let $\mathbf{x}=x_1,\dots,x_m$ be a minimal generating sequence for $\mathfrak{m}$. The flatness of φ conspires with the condition $\mathfrak{m}S=\mathfrak{n}$ to imply that $\varphi(\mathbf{x})=\varphi(x_1),\dots,\varphi(x_m)$ is a minimal generating sequence for $\mathfrak{n}$. Let $K^R=K^R(\mathbf{x})$ and $K^S=K^S(\varphi(\mathbf{x}))$ denote the respective Koszul complexes, and note that we have $\operatorname{rank}_R(K^R_i)=\operatorname{rank}_S(K^S_i)=r:=\binom{m}{i}$. Let $e_{i,1},\dots,e_{i,r}$ be an R-basis for K^R_i, and let $f_{i,1},\dots,f_{i,r}$ be the naturally corresponding S-basis for K^S_i. The construction yields a natural isomorphism of S-complexes $\beta\colon K^R\otimes_R S\to K^S$ taking $e_{i,j}\otimes 1$ to $f_{i,j}$. On the other hand let $K^\varphi\colon K^R\to K^S$ be given by $e_{i,j}\mapsto f_{i,j}$. By (2.3), the flatness of φ and condition (†) work together to show that K^φ is a quasi-isomorphism.

The source and target of the morphism $\alpha\colon\operatorname{Hom}_R(S,I)\to I$ are both homologically finite R-complexes, so it suffices to verify that the induced morphism

$$\operatorname{Hom}_R(K^R,\alpha)\colon\operatorname{Hom}_R(K^R,\operatorname{Hom}_R(S,I))\to\operatorname{Hom}_R(K^R,I)$$

is a quasi-isomorphism; see Proposition 2.4. This isomorphism is verified by the following commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_R(K^R \otimes_R S, I) & \xleftarrow[\simeq]{\operatorname{Hom}_R(\beta,I)} & \operatorname{Hom}_R(K^S, I) \\ {\scriptstyle (*)}\downarrow\cong & & {\scriptstyle \operatorname{Hom}(K^\varphi,I)}\downarrow\simeq \\ \operatorname{Hom}_R(K^R, \operatorname{Hom}_R(S,I)) & \xrightarrow{\operatorname{Hom}_R(K^R,\alpha)} & \operatorname{Hom}_R(K^R, I) \end{array}$$

wherein the isomorphism (*) is Hom-tensor adjointness. The morphism $\operatorname{Hom}_R(\beta, I)$ is a quasi-isomorphism because I is a bounded-above complex of injective R-modules and β is a quasi-isomorphism. (See, e.g., the proof of [Wei, (2.7.6)].) The same reasoning shows that $\operatorname{Hom}(K^\varphi, I)$ is a quasi-isomorphism. From the commutativity of the diagram, it follows that $\operatorname{Hom}_R(K^R, \alpha)$ is a quasi-isomorphism as well.

(1) $\Longrightarrow$ (6). Assume that M has a compatible S-module structure. Let J be an S-injective resolution of M. We first show that J is also an R-injective resolution of M. It suffices to show that each J_i is injective as an R-module. The fact that each J_i is an S-module yields the first natural isomorphism in the following sequence:

$$\operatorname{Hom}_R(-, J_i) \cong \operatorname{Hom}_R(-, \operatorname{Hom}_S(S, J_i)) \cong \operatorname{Hom}_S(- \otimes_R S, J_i)$$

The second isomorphism is from Hom-tensor adjointness. Because the functors $- \otimes_R S$ and $\operatorname{Hom}_S(-, J_i)$ are exact, the same is true of the composite $\operatorname{Hom}_S(- \otimes_R S, J_i)$, and so also for the isomorphic functor $\operatorname{Hom}_R(-, J_i)$. Thus each J_i is injective as an R-module, and so J is an R-injective resolution of M.

Now, consider the following sequence of natural isomorphisms:

$$J \cong \operatorname{Hom}_S(S, J) \cong \operatorname{Hom}_S(S, \operatorname{Hom}_R(R, J)) \cong \operatorname{Hom}_R(S \otimes_R R, J) \cong \operatorname{Hom}_R(S, J) \qquad (2.5.2)$$

For $n \neq 0$, the previous paragraph yields the first isomorphism in the next sequence:

$$\operatorname{Ext}^n_R(S, M) \cong \mathrm{H}_{-n}(\operatorname{Hom}_R(S, J)) \cong \mathrm{H}_{-n}(J) = 0$$

The second isomorphism is from (2.5.2), and the vanishing follows from the fact that J is an injective resolution of M and $n \neq 0$. $\square$

Corollary 2.6. *Let R be a local ring and $\mathfrak{a} \subset R$ an ideal.*

1. *The $\mathfrak{a}$-adic completion $\widehat{R}^{\mathfrak{a}}$ is R-projective if and only if R is $\mathfrak{a}$-adically complete.*
2. *The Henselization R^h is R-projective if and only if R is Henselian.*
3. *If $R \to R'$ is a pointed étale neighborhood and R' is R-projective, then $R = R'$.*

Proof. Suppose $S := \widehat{R}^{\mathfrak{a}}$ is R-projective. Putting $M = R$ in Theorem 2.5 and using Proposition 1.10, we see that $R = S$. This proves (1), and the proofs of (2) and (3) are essentially the same. $\square$

We conclude this section with several examples showing the necessity of the hypotheses of Theorem 2.5 with respect to the implications (5) $\Longrightarrow$ (1) and (6) $\Longrightarrow$ (1). The examples depend on the following addendum to Proposition 1.10, in which we no longer assume condition (†).

Proposition 2.7. *Let $\varphi: A \to B$ be an arbitrary homomorphism of commutative rings. The following conditions are equivalent:*

(1) *The A-module A has a B-module structure $(b, a) \mapsto b \circ a$ such that*

$$a_1 a_2 = \varphi(a_1) \circ a_2 \text{ for all } a_1, a_2 \in A. \tag{i}$$

(2) *A is a ring retract of B, that is, there is a ring homomorphism $\psi: B \to A$ such that*

$$\psi\varphi(a) = a \text{ for each } a \in A. \tag{ii}$$

These conditions imply that φ is an A-split injection.

Proof. Assuming (1), we define a function $\psi: B \to A$ by by putting $\psi(b) := b \circ 1_A$ for each $b \in B$. Condition (ii) follows immediately from (i). Also, given $a \in A$ and $b \in B$ we have $\psi(ab) = \psi(\varphi(a)b) = (\varphi(a)b) \circ 1_A = \varphi(a) \circ (b \circ 1_A)$, by associativity of the B-module structure. Condition (i) implies $\varphi(a) \circ (b \circ 1_A) = a(b \circ 1_A) = a\psi(b)$, so ψ is A-linear. This shows that φ is an A-split injection.

Still assuming (1), let $b_1, b_2 \in B$. By associativity of the B-module structure, we have

$$\psi(b_1 b_2) = (b_1 b_2) \circ 1_A = b_1 \circ (b_2 \circ 1_A) = b_1 \circ \psi(b_2). \tag{iii}$$

On the other hand, the A-linearity of ψ yields $\psi(b_1)\psi(b_2) = \psi(b_1\varphi(\psi(b_2)))$. By (iii), this implies $\psi(b_1\varphi(\psi(b_2))) = b_1 \circ \psi\varphi\psi(b_2) = b_1 \circ \psi(b_2)$. Thus $\psi(b_1)\psi(b_2) = b_1 \circ \psi(b_2)$, and so (iii) implies that ψ is a ring homomorphism.

For the converse, assume (2), and set $b \circ a := \psi(b\varphi(a))$ for all $a \in A$ and $b \in B$. One checks readily the equalities $(b_1 b_2) \circ a = a\psi(b_1 b_2) = b_1 \circ (b_2 \circ a)$ for $b_i \in B$ and $a \in A$. Thus we have defined a legitimate B-module structure on A. The verification of (i) is easy and left to the reader. □

Our first example shows why we need to assume that the induced map between the residue fields of R and S is an isomorphism in the implications (5) $\Longrightarrow$ (1) and (6) $\Longrightarrow$ (1) of Theorem 2.5.

Example 2.8. Let $\varphi: K \to L$ be a proper field extension. Then φ is a flat local homomorphism and $\mathfrak{m}_K L = \mathfrak{m}_L$ (but the induced map $K/\mathfrak{m}_K \to L/\mathfrak{m}_L$ is not an isomorphism). If we take $M = R$, then conditions (5) and (6) of Theorem 2.5 are satisfied, but (1) is not. Indeed, suppose (1) holds. Proposition 2.7 provides a field homomorphism $\psi: L \to K$ such that $\psi\varphi$ is the identity map on K. Since ψ is necessarily injective, it follows that ψ and φ are reciprocal isomorphisms, contradiction. □

The next example shows the necessity of the condition $\mathfrak{m}S = \mathfrak{n}$ for the implications (5) $\Longrightarrow$ (1) and (6) $\Longrightarrow$ (1) in Theorem 2.5.

Example 2.9. Let k be a field and $p \geq 2$ an integer. Set $R = k[[X^p]]$ and $S = k[[X]]$, and let $\varphi: R \to S$ be the inclusion map. Again, we put $M = R$. Then φ is a local homomorphism inducing an isomorphism on residue fields (but $\mathfrak{m}_R S \neq \mathfrak{m}_S$). Since S is a free R-module (with basis $\{1, X, \dots, X^{p-1}\}$), conditions (5) and (6) are satisfied. Suppose, by way of contradiction, that (1) is satisfied. Using Proposition 2.7, we get a ring homomorphism $\psi: S \to R$ such that $\psi\varphi$ is the identity map on R. Putting $z := \psi(X)$, we see that $X^p = \psi(z^p) \in \mathfrak{m}_R^p$, an obvious contradiction.

Similarly, let R be a regular local ring of characteristic $p > 0$. Take $S = R$ and assume that R is F-finite, that is, that the Frobenius endomorphism $\varphi: R \to S$ is module-finite. (This holds, for example, if R is a power series ring over a perfect field.) As an R-module, S is flat by [K], and therefore free. Thus conditions (5) and (6) hold. Assume $k := R/\mathfrak{m}_R$ is perfect and that $\dim(R) > 0$. Then φ induces an isomorphism on residue fields, and essentially the same argument as above shows that condition (1) fails.

The next two examples show why we need φ to be flat for the implications (5) $\implies$ (1) and (6) $\implies$ (1), respectively. Note that the homomorphism φ satisfies (†) in both examples and has finite flat dimension in Example 2.10.

Example 2.10. Let R be a local ring with $\operatorname{depth}(R) \geq 1$ and fix an R-regular element $x \in \mathfrak{m}$. We consider the natural surjection $\varphi: R \to R/(x)$. It is straightforward to show $\operatorname{Ext}^1_R(R/(x), R) \cong R/(x)$ and $\operatorname{Ext}^n_R(R/(x), R) = 0$ when $n \neq 1$. In particular, each $\operatorname{Ext}^n_R(R/(x), R)$ is finitely generated over R. Suppose (1) holds, and let $\psi: S \to R$ be the retraction promised by Proposition 2.7. Then $x = \psi\varphi(x) = 0$, contradition.

Example 2.11. Let R be a local Artinian Gorenstein ring with residue field $k \neq R$. We consider the natural surjection $\varphi: R \to k$. Because R is self-injective, we have $\operatorname{Ext}^n_R(k, R) = 0$ when $n \neq 0$. Thus conditions (5) and (6) of Theorem 2.5 hold. As in Example 2.10, we see easily that (1) fails.

3. Extended modules

Let $\varphi: (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism. Given a finitely generated S-module N, we say that N is *extended* provided there is an R-module M such that $S \otimes_R M \cong N$ as S-modules. By faithfully flat descent, such a module M, if it exists, is unique up to R-isomorphism and is necessarily finitely generated.

We begin with a "two-out-of-three" principle, which is well known when $S = \widehat{R}$. The proof in general seems to require a different approach from the proof in that special case. The following notation will be used in the proof: Given a ring A and A-modules U and V, we write $U \mid_A V$ to indicate that U is isomorphic to a direct summand of V.

Proposition 3.1. *Let $\varphi: R \to S$ be a flat local homomorphism. Let N_1 and N_2 be finitely generated S-modules, and put $N = N_1 \oplus N_2$. If two of the modules N_1, N_2, N are extended, so is the third.*

Proof. We begin with a claim: If M_1 and M are finitely generated R-modules, and if $S \otimes_R M_1 \mid_S S \otimes_R M$, then $M_1 \mid_R M$. To prove the claim, write $S \otimes_R M \cong (S \otimes_R M_1) \oplus U$. We assume, temporarily, that R is Artinian. By [Wi98, (1.2)] we know, at least, that there is some positive integer r such that $M_1 \mid_R M^{(r)}$ (a suitable direct sum of copies of M). Write $M_1 \cong \oplus_{i=1}^s V_i$ where each V_i is indecomposable. We proceed by induction on

s. Since $V_1 \mid_R M^{(r)}$, the Krull-Remak-Schmidt theorem (for finite-length modules) implies that $V_1 \mid_R M$, say, $V_1 \oplus W \cong M$. This takes care of the base case $s = 1$. For the inductive step, assume $s > 1$ and set $W_1 = \oplus_{i=2}^s V_i$. We have $V_1 \oplus W \cong M$ and $M_1 \cong V_1 \oplus W_1$, and hence

$$(S \otimes_R V_1) \oplus (S \otimes_R W) \cong S \otimes_R M \cong (S \otimes_R M_1) \oplus U \cong (S \otimes_R V_1) \oplus (S \otimes_R W_1) \oplus U.$$

Direct-sum cancellation [Ev] implies $(S \otimes_R W) \cong (S \otimes_R W_1) \oplus U$. The inductive hypothesis, applied to the pair W_1, W, now implies that $W_1 \mid_R W$; therefore $M_1 \mid_R M$. This completes the proof of the claim when R is Artinian.

In the general case, let t be an arbitrary positive integer, and consider the flat local homomorphism $R/\mathfrak{m}^t \to S/\mathfrak{m}^t S$. By the Artinian case, $M_1/\mathfrak{m}^t M_1 \mid_{R/\mathfrak{m}^t} M/\mathfrak{m}^t M$. Now we apply Corollary 2 of [G] to conclude that $M_1 \mid_R M$, as desired.

Having proved our claim, we now complete the proof of the proposition. If N_1 and N_2 are extended, clearly N is extended. Assuming N_1 and N are extended, we will prove that N_2 is extended. (The third possibility will then follow by symmetry.) Let $N_1 \cong S \otimes_R M_1$ and $N \cong S \otimes_R M$. Thus $S \otimes_R M_1 \mid_S S \otimes_R M$, and by the claim there is an R-module M_2 such that $M_1 \oplus M_2 \cong M$. Now $N_1 \oplus (S \otimes_R M_2) \cong S \otimes_R M \cong N_1 \oplus N_2$, and, by direct-sum cancellation [Ev], we have $S \otimes_R M_2 \cong N_2$. □

In the language of monoids, (3.1) says that the homomorphism $[M] \mapsto [S \otimes_R M]$ between the monoids of isomorphism classes of finitely generated modules (over R and over S) is a *divisor homomorphism*. The same condition comes up in [BH], in the context of affine semigroups: a subsemigroup H of an affine semigroup G is *full* provided $a + b = c$ in G, with $a, c \in H$, implies $b \in H$.

There is a "two-out-of-three" principle for short exact sequences as well, though some restrictions apply. Variations on this theme have been used in the literature, e.g., in [CPST], [LO], [Wes].

Proposition 3.2. *Let $\varphi\colon R \to S$ be a flat local homomorphism satisfying (†), and consider an exact sequence of finitely generated S-modules $0 \to N_1 \to N \to N_2 \to 0$.*

(1) *Assume that N_1 and N_2 are extended. If $\mathrm{Ext}_S^1(N_2, N_1)$ is finitely generated as an R-module, then N is extended.*
(2) *Assume that N and N_2 are extended. If $\mathrm{Hom}_S(N, N_2)$ is finitely generated as an R-module, then N_1 is extended.*
(3) *Assume that N_1 and N are extended. If $\mathrm{Hom}_S(N_1, N)$ finitely generated as an R-module, then N_2 is extended.*

Proof. For (1), let $N_i = S \otimes_R M_i$ where the M_i are finitely generated R-modules. We have natural homomorphisms $\mathrm{Ext}_R^1(M_2, M_1) \xrightarrow{\alpha} S \otimes_R \mathrm{Ext}_R^1(M_2, M_1) \xrightarrow{\beta} \mathrm{Ext}_S^1(N_2, N_1)$. The map β is an isomorphism because φ is flat, M_2 is finitely generated and R is Noetherian. Therefore $S \otimes_R \mathrm{Ext}_R^1(M_2, M_1)$ is finitely generated as an R-module, and now Theorem 1.8 ((4) $\Longrightarrow$ (2)) says that α is an isomorphism. This means that the given exact sequence of S-modules is isomorphic to $S \otimes_R \mathbf{M}$ for some exact sequence of R-modules $\mathbf{M} = (0 \to M_1 \to M \to M_2 \to 0)$. Clearly, this implies $S \otimes_R M \cong N$.

To prove (2), let $N \cong S \otimes_R M$ and $N_2 \cong S \otimes_R M_2$, where M and M_2 are finitely generated R-modules. Essentially the same proof as in (1), but with Hom in place of Ext, shows that the given homomorphism $N \to N_2$ comes from a homomorphism $f\colon M \to M_2$. Then $M_1 \cong S \otimes_R \operatorname{Ker}(f)$.

For (3), we let $N_1 \cong S \otimes_R M_1$ and $N \cong S \otimes_R M$; we deduce that the given homomorphism $N_1 \to N$ comes from a homomorphism $g\colon M_1 \to M$. Then $N_2 \cong S \otimes_R \operatorname{Coker}(g)$. □

Here is a simple application of Part (1) of Proposition 3.2 (cf. [LO] and [Wi01] for much more general results):

Proposition 3.3. *Let $(R, \mathfrak{m})$ be a one-dimensional local ring whose $\mathfrak{m}$-adic completion $S = \widehat{R}$ is a domain. Then every finitely generated S-module is extended.*

Proof. Given a finitely generated S-module N, let $\{x_1, \ldots, x_n\}$ be a maximal S-linearly independent subset of N. The submodule F generated by the x_i is free and therefore extended. The quotient module N/F is torsion and hence of finite length. Therefore N/F is extended. Since $\operatorname{Ext}^1_S(N/F, F)$ has finite length, the module N is extended, by (3.2). □

Notice that Part (1) of Proposition 3.2 applies also when N_2 is free on the punctured spectrum. For in this case $\operatorname{Ext}^1_S(N_2, N_1)$ has finite length over S and therefore is finitely generated as an R-module. A more subtle condition that forces $\operatorname{Ext}^1_S(N_2, N_1)$ to have finite length is that there are only finitely many isomorphism classes of modules X fitting into a short exact sequence $0 \to N_1 \to X \to N_2 \to 0$. (Cf. [CPST, (4.1)].)

Of course not every module over the completion, or over the Henselization, is extended. Suppose, for example, that $R = \mathbb{C}[X, Y]_{(X,Y)}/(Y^2 - X^3 - X^2)$, the local ring of a node. Then R is a domain, but $\widehat{R} \cong \mathbb{C}[[U, V]]/(UV))$, which has two minimal prime ideals $P = (U)$ and $Q = (V)$. Since R is a domain, any extended $\widehat{R}$-module N must have the property that N_P and N_Q have the same vector-space dimension (over $\widehat{R}_P$ and $\widehat{R}_Q$, respectively). Thus the $\widehat{R}$-module $\widehat{R}/P$ is not extended. (This behavior was the basis for the first example of failure of the Krull-Remak-Schmidt theorem for finitely generated modules over local rings. See the example due to R. G. Swan in [Ev]. The idea is developed further in [Wi01].) The module $\widehat{R}/P$ is free on the punctured spectrum and therefore, by Elkik's theorem [El], is extended from the Henselization R^{h}. With $\widehat{R}/P \cong \widehat{R} \otimes_{R^{\mathrm{h}}} V$, we see that the R^{h}-module V is not extended from R.

Next, we turn to the question of whether every finitely generated module over S is a direct summand of a finitely generated extended module. This weaker property is often useful in questions concerning ascent of finite representation type (cf. [Wi98, Lemma 2.1]). Although the next result is not explicitly stated in [Wi98], the main ideas of the proof occur there. Note that we do not require that $R/\mathfrak{m} = S/\mathfrak{n}$.

Theorem 3.4. *Let $\varphi\colon R \to S$ be a flat local homomorphism, and assume S is separable over R (that is, the diagonal map $S \otimes_R S \to S$ splits as $S \otimes_R S$-modules). Then every finitely generated S-module is a direct summand of a finitely generated extended module.*

Proof. Given a finitely generated S-module N, we apply $- \otimes_S N$ to the diagonal map, getting a split surjection of S-modules $\pi\colon S \otimes_R N \twoheadrightarrow N$, where the S-module structure on $S \otimes_R N$ comes from the S-action on S, not on N. Thus we have a split injection of S-modules $j\colon N \to S \otimes_R N$. Now write N as a direct union of finitely generated R-modules M_i. The flatness of φ implies that $S \otimes_R N$ is a direct union of the modules $S \otimes_R M_i$. The finitely generated S-module $j(N)$ must be contained in some $S \otimes_R M_i$. Since $j(N)$ is a direct summand of $S \otimes_R N$, it must be a direct summand of the smaller module $S \otimes_R M_i$. □

Corollary 3.5. *Let $R \to R^{\mathrm{h}}$ be the Henselization of the local ring R. Then every finitely generated R^{h}-module is a direct summand of a finitely generated extended module.*

Proof. Let N be a finitely generated R^{h}-module. Since $R \to R^{\mathrm{h}}$ is a direct limit of étale neighborhoods $R \to S_i$, N is extended from some S_i. Now apply Theorem 3.4 to the extension $R \to S_i$. □

The analogous result can fail for the completion:

Example 3.6. Let $(R, \mathfrak{m})$ be a countable local ring of dimension at least two. Then R has only countably many isomorphism classes of finitely generated modules. Using the Krull-Remak-Schmidt theorem over $\widehat{R}$, we see that only countably many isomorphism classes of indecomposable $\widehat{R}$-modules occur in direct-sum decompositions of finitely generated extended modules. We claim, on the other hand, that $\widehat{R}$ has uncountably many isomorphism classes of finitely generated indecomposable modules. To see this, we recall that $\widehat{R}$, being complete, has countable prime avoidance; see [SV]. By Krull's principal ideal theorem, the maximal ideal of $\widehat{R}$ is the union of the height-one prime ideals. It follows that $\widehat{R}$ must have uncountably many height-one primes P, and the $\widehat{R}$-modules $\widehat{R}/P$ are pairwise non-isomorphic and indecomposable.

If $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ is flat and satisfies (†), we know that every finite-length S-module is extended. We close with an example showing that the condition $k = l$ cannot be deleted, even for a module-finite étale extension of Artinian local rings.

Example 3.7. Let $R = \mathbb{R}[X, Y]/(X, Y)^2$ and $S = \mathbb{C} \otimes_{\mathbb{R}} R = \mathbb{C}[X, Y]/(X, Y)^2$. We claim that, for $c \in \mathbb{C}$, the module $N := S/(X + cY)$ is extended (if and) only if $c \in \mathbb{R}$. The minimal presentation of N is $S \xrightarrow{X+cY} S \to N \to 0$. If N were extended, the 1×1 matrix $X + cY$ would be equivalent to a matrix over R. In other words, we would have $X + cY = u(r + sX + tY)$ for some unit u of S and suitable elements $r, s, t \in \mathbb{R}$. Writing $u = a + bX + dY$, with $a, b, d \in \mathbb{C}$ and $a \neq 0$, we see, by comparing coefficients of $1, X$ and Y, that $c = t/s \in \mathbb{R}$.

References

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Stud. Adv. Math., vol. 39, 1993.

[CPST] L. Christensen, G. Piepmeyer, J. Striuli and R. Takahashi, *Finite Gorenstein representation type implies simple singularity*, preprint.

[DI] F. DeMeyer and E. Ingraham, *Separable Algebras over Commutative Rings*, Lecture Notes in Mathematics, vol. 181, Springer-Verlag, Berlin, 1971.

[El] R. Elkik, *Solutions d'équations à coefficients dans un anneau hensélien*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 553-603.

[Ev] E. G. Evans, Jr., *Krull-Schmidt and cancellation over local rings*, Pacific J. Math. **46** (1973), 115–121.

[FSW] A. Frankild and S. Sather-Wagstaff, *Reflexivity and ring homomorphisms of finite flat dimension*, Comm. Algebra **35** (2007), 461–500.

[G] R. Guralnick, *Lifting homomorphisms of modules*, Illinois J. Math. **29** (1985), 153–156.

[Iv] B. Iversen, *Generic Local Structure of the Morphisms in Commutative Algebra*, Lecture Notes in Mathematics, vol. 310, Springer-Verlag, Berlin, 1973.

[J] C. U. Jensen, *On the vanishing of* $\varprojlim^{(i)}$, J. Algebra **15** (1970), 151–166.

[K] E. Kunz, *Characterizations of regular local rings for characteristic* p, Amer. J. Math. **91** (1969), 772–784.

[LO] L. S. Levy and C. J. Odenthal, *Package deal theorems and splitting orders in dimension* 1, Trans. Amer. Math. Soc. **348** (1996), 3457–3503.

[RG] M. Raynaud and L. Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.

[SV] R. Y. Sharp and P. Vámos, *Baire's category theorem and prime avoidance in complete local rings*, Arch. Math. (Basel) **44** (1985), 243–248.

[Wei] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[Wes] D. Weston, *On descent in dimension two and nonsplit Gorenstein modules*, J. Algebra **118** (1988), 263–275.

[Wi98] R. Wiegand, *Local rings of finite Cohen-Macaulay type*, J. Algebra **203** (1998), 156–168.

[Wi01] ———, *Direct-sum decompositions over local rings*, J. Algebra **240** (2001), 83–97.

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