Direct Sum Cancellation for Modules over One-dimensional Rings

Wolfgang Hassler*
Institut für Mathematik, Karl-Franzens Universität Graz
Heinrichstr. 36, A-8010 Graz, Austria
email: wolfgang.hassler@uni-graz.at

Roger Wiegand†
Department of Mathematics, University of Nebraska
Lincoln, NE 68588-0323, USA
email: rwiegand@math.unl.edu

January 31, 2004

1 Introduction

Let R be a one-dimensional Noetherian domain with finite normalization $\overline{R}$. In the eighties the second-named author and S. Wiegand developed a mechanism for studying the cancellation problem for finitely generated torsion-free R-modules. The key idea, described in [Wie84] and [WW87], is to represent a given torsion-free module M as a pullback:

$$\begin{array}{ccc} M & \longrightarrow & \overline{R}M \\ \downarrow & & \downarrow \\ M/\mathfrak{f}M & \longrightarrow & \overline{R}M/\mathfrak{f}M \end{array}$$

*This research was initiated while the author was a visitor at the University of Nebraska, supported by the Fonds zur Förderung der wissenschaftlichen Forschung.

†Partially supported by grants from the National Science Foundation, the National Security Agency and the Mathematical Sciences Research Institute, where much of this research was done.

Here $\mathfrak{f} := \{r \in R \mid r\overline{R} \subseteq R\}$ is the conductor, and $\overline{R}M$ denotes the module $\overline{R} \otimes_R M$ modulo torsion. The power of this approach comes from the fact that one is working mostly with the finite-length modules $M/\mathfrak{f}M$ and $\overline{R}M/\mathfrak{f}M$. This approach fails for modules that are not torsion-free. The analogous diagram is not necessarily a pullback if M has torsion, and, even more significantly, one cannot tell which parts of the finite-length modules $M/\mathfrak{f}M$ and $\overline{R}M/\mathfrak{f}M$ come from the torsion part of M.

Over the past two decades several authors, notably Guralnick [Gur86], [Gur87], and Klingler and Levy [KL1]—[KL4] have obtained remarkable results on cancellation and related issues for modules with torsion. A major unanswered question, however, has been whether every ring R as above with *torsion-free cancellation* (that is, $M \oplus L \cong N \oplus L \implies M \cong N$ for all finitely generated torsion-free modules M, N, L) actually has *cancellation* ($M \oplus L \cong N \oplus L \implies M \cong N$ for all finitely generated modules M, N, L). The research described in this paper began as an attempt to answer this question.

In this paper we use two approaches to extend the results in [Wie84] and [WW87]. One is to kill a sufficiently high power of the conductor in order to obtain a pullback. The disadvantage of this approach is that no fixed power will work for all modules. The other approach is to pass to the "singular semilocalization" $S^{-1}R$, where S is the complement of the union of the finitely many singular maximal ideals of R. Our main results are stated in terms of the singular semilocalization, although the pullbacks obtained by killing a power of $\mathfrak{f}$ will play a crucial role in the development of the machinery.

In Sections 3 and 4 we will develop analogues of the basic results of [Wie84] and [WW87], valid for all finitely generated modules, rather than just torsion-free modules. Section 5 gives an application (Theorem 5.3) to coordinate rings of affine curves and also studies the related question of power cancellation.

In Section 6 we begin with an application (Theorem 6.1) to curves over an infinite perfect field and then concentrate on a very special class of rings (see Notation 6.2). Within this class are the Dedekind-like rings, whose finitely generated modules have been completely classified by Klingler and Levy [KL1] – [KL4]. We give a necessary and sufficient condition (Theorem 6.10) for a Dedekind-like ring to have cancellation and use it to show that there are Dedekind-like rings (Example 6.12) having torsion-free cancellation but not cancellation. This answers the question that originally motivated this research, as well as Open Problem 20.5 in [KL4]. We show, on the other hand, that torsion-free cancellation implies cancellation for Dedekind-like orders in an algebraic number field. Finally, in Example 6.20, we find a quadratic order having torsion-free cancellation but not cancellation. This last example depends on the construction of indecomposable (mixed) modules of torsion-free rank two over certain non-Dedekind-like rings, e.g., the cusp $k[[t^2, t^3]]$ and orders such as $\mathbb{Z}[2\sqrt{-1}]$.

We thank the referee for a careful reading of the paper.

2 Preliminaries

All rings in this paper (except for certain endomorphism rings, e.g., in the proof of Proposition 4.8) are commutative with identity. Our focus will be on one-dimensional reduced Noetherian rings, and we will encounter some minor hurdles in dealing with zero-divisors. We denote the total quotient ring of a ring R by $\mathcal{Q}(R)$ and the integral closure of R in $\mathcal{Q}(R)$ by $\overline{R}$. For rings $R \subseteq A$ the conductor ideal $\{x \in R \mid xA \subseteq R\}$ is denoted by $(R : A)$. Given an R-module M, we denote by M_{tors} the *torsion submodule* of M, that is, the kernel of the map $M \longrightarrow \mathcal{Q}(R) \otimes_R M$. Thus an element $x \in M$ is in M_{tors} if and only if x is killed by a non-zerodivisor of R.

For lack of a suitable reference we record three simple lemmas.

Lemma 2.1 *Let R be a reduced ring and $S \subseteq R$ a multiplicatively closed set. Then the natural map $S^{-1}\overline{R} \longrightarrow \overline{S^{-1}R}$ is an isomorphism.*

Proof. It is enough to show that the canonical injection $S^{-1}\mathcal{Q}(R) \longrightarrow \mathcal{Q}(S^{-1}R)$ is an isomorphism, for then the proof is essentially the same as for a domain. But even the natural map $\mathcal{Q}(R) \longrightarrow \mathcal{Q}(S^{-1}R)$ is surjective. To see this, we note that for any reduced Noetherian ring A, $\mathcal{Q}(A) = \prod_{P \in \text{Minspec}(A)} \mathcal{Q}(A/P)$. Thus the map $\mathcal{Q}(R) \longrightarrow \mathcal{Q}(S^{-1}R)$ is the projection of $\prod_{P \in \text{Minspec}(A)} \mathcal{Q}(A/P)$ onto $\prod_{P \in X} \mathcal{Q}(A/P)$, where X is the set of minimal primes disjoint from S. ■

The hypothesis that R be reduced cannot be omitted. For example, let k be a field and put $R := k[[x, y, z]]/(y^2 - x^3)(x, y, z)$. Since the maximal ideal annihilates the non-zero element $y^2 - x^3$, $R = \mathcal{Q}(R) = \overline{R}$. On the other hand, $R[\frac{1}{z}] \cong K[[x, y]]/(y^2 - x^3)$, where K is the quotient field of $k[[z]]$, and thus $\overline{R}[\frac{1}{z}] \neq \overline{R[\frac{1}{z}]}$.

Lemma 2.2 *Let R be a reduced ring such that $\overline{R}$ is a finitely generated R-module. Let $\mathfrak{f} = (R : \overline{R})$ denote the conductor. Then $\mathfrak{f}R_{\mathfrak{p}} = (R_{\mathfrak{p}} : \overline{R_{\mathfrak{p}}})$ for all prime ideals $\mathfrak{p}$ of R. In particular, $R_{\mathfrak{p}}$ is integrally closed if and only if $\mathfrak{p}$ does not contain $\mathfrak{f}$.*

Proof. Use Lemma 2.1 together with the fact that colons localize for finitely generated modules. ■

Lemma 2.3 *Let R be a ring and $I \subseteq R$ an ideal. Let $\mathfrak{M}$ be a set of maximal ideals of R, and assume that every maximal ideal of R that contains I is in $\mathfrak{M}$. Put $S = R - \bigcup \mathfrak{M}$. Then the natural map $\varphi : R/I \longrightarrow S^{-1}(R/I)$ is an isomorphism.*

Proof. The localization $\varphi_{\mathfrak{m}}$ of φ is an isomorphism for every maximal ideal $\mathfrak{m}$ of R. ■

Notation 2.4 For the remainder of this section, let $D = D_1 \times \ldots \times D_s$, where each D_i is either a field or a Dedekind domain. (In our applications, D will be the integral closure of the one-dimensional reduced Noetherian ring R. We are allowing for the possibility that R might have a field as a direct factor.) Let M be a finitely generated D-module. Then $M = M_1 \oplus \ldots \oplus M_s$, where each M_i is a finitely generated D_i-module. Put $V_i = M_i/(M_i)_{\text{tors}}$, a projective D_i-module of rank, say, r_i. By definition, $\det M$ is the class in $\text{Pic}(D)$ of the rank-one projective D-module $L := \bigwedge^{r_1} V_1 \oplus \ldots \oplus \bigwedge^{r_s} V_s$. If, now, ϕ is a D-endomorphism of M, the endomorphism of L induced by ϕ is multiplication by a unique element of D, and we denote this element by $\det \phi$. By convention, if some $r_i = 0$ we put $\bigwedge^{r_i} V_i = D_i$. Also, in this case, the i^{th} component of $\det(\phi)$ is taken to be 1. Note that $\det(\phi) = \det(1_K \otimes \phi)$, where $K = \mathcal{Q}(D)$.

Lemma 2.5 *Let D be as in Notation 2.4, and let I be an ideal of D. Let M be a finitely generated D-module and ϕ a D-endomorphism inducing an automorphism of M/IM. If $\det \phi \equiv 1 \pmod{I}$, then there exists a $\psi \in \text{Aut}_D(M)$ such that ϕ and ψ induce the same automorphism of M/IM.*

Proof. We may assume that D is a Dedekind domain and $I \subseteq D$ is a non-zero ideal. Write $M = F \oplus T$, where F is projective and $T = M_{\text{tors}}$. Let $\mathfrak{M}$ be the set of maximal ideals of D that contain I.

Decomposing T into its primary components, we get a decomposition $T = T_1 \oplus T_2$ such that $\mathrm{Supp}(T_1) \subseteq \mathfrak{M}$ and $\mathrm{Supp}(T_2) \cap \mathfrak{M} = \emptyset$. We can write ϕ (relative to the decomposition $M = F \oplus T_1 \oplus T_2$) as a 3×3 matrix

$$\phi = \begin{bmatrix} \beta & 0 & 0 \\ f & \alpha_1 & 0 \\ g & 0 & \alpha_2 \end{bmatrix}$$

with $\beta : F \longrightarrow F$, $\alpha_i : T_i \longrightarrow T_i$, $f : F \longrightarrow T_1$ and $g : F \longrightarrow T_2$.

We claim that α_1 is an isomorphism. It is enough to check this locally at each maximal ideal $\mathfrak{m}$. If $\mathfrak{m} \notin \mathfrak{M}$ then $(T_1)_\mathfrak{m} = 0$, so we may assume that $\mathfrak{m} \in \mathfrak{M}$. Since I is contained in the Jacobson radical of $D_\mathfrak{m}$ and ϕ is an automorphism modulo I, we see that $\phi_\mathfrak{m}$ is surjective and therefore an automorphism. It follows easily from the triangular form of $\phi_\mathfrak{m}$ that $(\alpha_1)_\mathfrak{m}$ is an isomorphism, and the claim is proved.

Recalling Notation 2.4, we see that $\det \phi = \det \beta$. Since $\det \beta \equiv 1 \pmod{I}$, there exists by [Wie84, Theorem 1.1] an element $\chi \in \mathrm{Aut}_R(F)$ such that χ and β induce the same homomorphism on F/IF. Define $\psi : M \longrightarrow M$ (relative to the decomposition $M = F \oplus T_1 \oplus T_2$) by the matrix

$$\psi = \begin{bmatrix} \chi & 0 & 0 \\ f & \alpha_1 & 0 \\ g & 0 & 1_{T_2} \end{bmatrix}.$$

It is now easy to see that ψ is surjective (and hence bijective) and that ψ and ϕ induce the same automorphism of M/IM. ■

Proposition 2.6 *Let D be as in Notation 2.4, M a finitely generated D-module and $I \subseteq D$ an ideal. Let S be the complement of the union of all maximal ideals containing I, and let $\phi \in \mathrm{Aut}_{S^{-1}D}(S^{-1}M)$ with $\det \phi = 1$. Then there exists an automorphism $\psi \in \mathrm{Aut}_D(M)$ such that ϕ and ψ induce the same automorphism of M/IM.*

PROOF. Since $\mathrm{End}_{S^{-1}D}(S^{-1}M) = S^{-1}\mathrm{End}_D(M)$, there exist a D-endomorphism $\theta : M \longrightarrow M$ and an element $t \in S$ such that $\phi = t^{-1}\theta$. Choose $u \in S$ with $tu \equiv 1 \mod I$ and put $\chi = u\theta$. Then χ and ϕ induce the same D/I-automorphism of M/IM. Since $\det \chi \equiv 1 \pmod{I}$, Lemma 2.5 yields the desired automorphism ψ. ■

3 An action of the group $(S^{-1}\overline{R})^\times$ on finitely generated R-modules

In this section, R always denotes a one-dimensional reduced ring whose integral closure $\overline{R}$ is a finitely generated R-module. We denote the conductor $(R : \overline{R})$ by $\mathfrak{f}$, and we put $S = R - \bigcup \mathfrak{M}$, where $\mathfrak{M}$ is the set of maximal ideals of R that contain $\mathfrak{f}$. The ring $S^{-1}R$ is called the *singular semilocalization* of R. The group of units of a ring A is denoted by $A^\times$.

To avoid trivial special cases, we assume throughout this section that $R \neq \overline{R}$, equivalently, $S^{-1}R$ is a non-zero ring. Note that $S^{-1}R$ is a semilocal ring whose localizations at maximal ideals are neither fields nor discrete valuation rings. By Lemma 2.1, the integral closure of $S^{-1}R$ (in its total quotient ring) is $S^{-1}\overline{R}$.

Lemma 3.1 *Keep the notation and assumptions above, and let M be an R-module.*

i.) The natural square

$$\begin{array}{ccc} M & \longrightarrow & \overline{R} \otimes_R M \\ \downarrow & & \downarrow \phi \\ S^{-1}M & \xrightarrow{\psi} & S^{-1}\overline{R} \otimes_R M \end{array} \tag{1}$$

is a pullback.

ii.) Suppose that M is finitely generated. Then, for every sufficiently large integer n, the following diagram with the natural maps is a pullback:

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & \overline{R} \otimes_R M \\ \beta_n \downarrow & & \downarrow \phi_n \\ M/\mathfrak{f}^n M & \xrightarrow{\psi_n} & \frac{\overline{R} \otimes_R M}{\mathfrak{f}^n(\overline{R} \otimes_R M)} \end{array} \tag{2}$$

PROOF. i.) We prove the assertion locally. Suppose first that the maximal ideal $\mathfrak{m}$ contains the conductor $\mathfrak{f}$. When we localize (1) at $\mathfrak{m}$, the vertical maps become isomorphisms, and we certainly obtain a pullback. Now suppose that $\mathfrak{m}$ does not contain the conductor. If we localize at $\mathfrak{m}$ we get the diagram

$$\begin{array}{ccc} M_{\mathfrak{m}} & \xrightarrow{=} & M_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ T \otimes_R M & \xrightarrow{=} & T \otimes_R M \end{array} ,$$

where $T = R_{\mathfrak{m}} \otimes_R S^{-1}R$. This square is again a pullback.

ii.) By Lemma 2.2 we may assume that R is local and $\overline{R} \neq R$. We have to show that the complex

$$0 \longrightarrow M \xrightarrow{\left[\begin{smallmatrix}\alpha \\ \beta_n\end{smallmatrix}\right]} (\overline{R} \otimes_R M) \oplus M/\mathfrak{f}^n M \xrightarrow{[\,\phi_n \; -\psi_n\,]} \frac{\overline{R} \otimes_R M}{\mathfrak{f}^n(\overline{R} \otimes_R M)}$$

is exact. We first show that $\mathrm{Ker}([\,\phi_n \; -\psi_n\,]) \subseteq \mathrm{Image}(\left[\begin{smallmatrix}\alpha \\ \beta_n\end{smallmatrix}\right])$ for every $n \geq 1$. Let $\xi \in \overline{R} \otimes_R M$ and $\eta = y + \mathfrak{f}^n M \in M/\mathfrak{f}^n M$, with $\phi_n(\xi) = \psi_n(\eta)$. Then $1 \otimes y - \xi \in \mathfrak{f}^n(\overline{R} \otimes_R M)$. It is easy to see that every element of $\mathfrak{f}^n(\overline{R} \otimes_R M)$ is of the form $1 \otimes z$ with $z \in \mathfrak{f}^n M$. Thus we get $\xi = 1 \otimes v$, where $v = y - z$ with $z \in \mathfrak{f}^n M$. Then $\left[\begin{smallmatrix}\xi \\ \eta\end{smallmatrix}\right] = \left[\begin{smallmatrix}\alpha \\ \beta_n\end{smallmatrix}\right](v)$, as desired.

Choose a non-zerodivisor $\lambda \in R$ with $\lambda M_{\mathrm{tors}} = 0$, and choose $n \geq 1$ large enough so that $\mathfrak{f}^n \subseteq \lambda R$. With this choice of n, we show that $\left[\begin{smallmatrix}\alpha \\ \beta_n\end{smallmatrix}\right]$ is injective, i.e., $\ker(\alpha) \cap \ker(\beta_n) = 0$. Note that $\ker(\alpha) \subseteq M_{\mathrm{tors}} = \mathrm{Ker}(M \longrightarrow \overline{R} \otimes_R M \longrightarrow \mathcal{Q}(R) \otimes_R M)$. Suppose that $x \in M_{\mathrm{tors}} \cap \mathfrak{f}^n M$. Write $x = \lambda y$ with $y \in M$. Since λ is a non-zerodivisor, $y \in M_{\mathrm{tors}}$, and hence $x = 0$, as desired. ■

Definition 3.2 We call (1) the standard pullback of M, and we call (2) the standard pullback of M with respect to the n^{th} power of $\mathfrak{f}$.

Perhaps it is worth pointing out that the $\mathfrak{f}$-adic completion of (1) is the inverse limit of the diagrams (2), as $n \to \infty$.

The next lemma characterizes the diagrams that arise as standard pullbacks of finitely generated R-modules, and, more importantly, shows how to construct modules via pullbacks.

Lemma 3.3 *Let U be a finitely generated $S^{-1}R$-module and V a finitely generated $\overline{R}$-module. Let $\psi' : U \longrightarrow S^{-1}V$ be an $S^{-1}R$-homomorphism such that the induced $S^{-1}\overline{R}$-homomorphism $\widetilde{\psi} : S^{-1}\overline{R} \otimes_{S^{-1}R} U \longrightarrow S^{-1}V$ is an isomorphism.*

i.) Let M be the module defined by the pullback diagram

$$\begin{array}{ccc} M & \longrightarrow & V \\ \downarrow & & \downarrow \phi' \\ U & \stackrel{\psi'}{\longrightarrow} & V \otimes_{\overline{R}} S^{-1}\overline{R} \end{array} \qquad (3)$$

Then M is a finitely generated R-module and (3) *is naturally isomorphic to the standard pullback of M.*

ii.) For $n \geq 1$ let $\psi'_n : U/\mathfrak{f}^n U \longrightarrow V/\mathfrak{f}^n V$ $(= S^{-1}V/\mathfrak{f}^n(S^{-1}V))$ denote the $R/\mathfrak{f}^n$-homomorphism induced by ψ'. Consider the pullback

$$\begin{array}{ccc} M_n & \longrightarrow & V \\ \downarrow & & \downarrow \phi'_n \\ U/\mathfrak{f}^n U & \stackrel{\psi'_n}{\longrightarrow} & V/\mathfrak{f}^n V \end{array} \qquad (4)$$

and let M be as in (3)*. Then the natural maps $M \longrightarrow U \longrightarrow U/\mathfrak{f}^n U$ and $M \longrightarrow V$ induce a homomorphism $\xi_n : M \longrightarrow M_n$. Moreover, if n is sufficiently large, then ξ_n is an isomorphism and (4) is naturally isomorphic to diagram (2), the standard pullback of M with respect to the n^{th} power of $\mathfrak{f}$.*

Proof. We first show that (3) is naturally isomorphic to the standard pullback of M. Consider the diagram

$$\begin{array}{ccccccc} M & & & \longrightarrow & & & \overline{R} \otimes_R M \\ & =\searrow & & & & \swarrow \gamma & \\ & & M & \longrightarrow & V & & \\ \chi \downarrow & & \downarrow & & \downarrow \phi' & & \downarrow \phi \\ & & U & \stackrel{\psi'}{\longrightarrow} & S^{-1}V & & \\ & \alpha \nearrow & & & & \nwarrow \beta & \\ S^{-1}M & & & \stackrel{\psi}{\longrightarrow} & & & S^{-1}\overline{R} \otimes_R M \end{array} \qquad (5)$$

where χ, ψ and ϕ are the natural maps, and α, β and γ are the induced maps making the left, right and top trapezoids commute. Of course the inner and outer squares commute. To see that the bottom trapezoid commutes, note that $\chi(M)$ generates $S^{-1}M$ as an $S^{-1}R$-module; therefore we can back up to M and chase around the other five commutative faces of the "cube". If we localize (3) with respect to S, we see that α is actually an isomorphism. Since ψ' induces an isomorphism $\widetilde{\psi} : U \otimes_{S^{-1}R} S^{-1}\overline{R} \longrightarrow V \otimes_{\overline{R}} S^{-1}\overline{R}$, we see that β is an isomorphism as well. In order to prove that γ is an isomorphism, it is therefore enough to show that γ becomes an isomorphism when we localize it at each non-singular maximal ideal $\mathfrak{m}$ of R. Thus, let $\mathfrak{m}$ be a maximal ideal not containing $\mathfrak{f}$ and localize everything with respect to the multiplicative set $R - \mathfrak{m}$. Now γ agrees with the top arrow of the inner square of (5), so it will suffice to show that this arrow is an isomorphism. Now the bottom arrow of the inner cube, namely ψ', coincides with the isomorphism $\tilde{\psi}$ (after localization at $\mathfrak{m}$) and hence is an isomorphism. Since the inner square is a pullback, its top arrow must be an isomorphism as well. This shows that γ is an isomorphism. Thus (5) provides a natural isomorphism between (3) and the standard pullback for M.

Next we show that M is a finitely generated R-module. Since $M \longrightarrow \mathcal{Q}(R) \otimes_R M$ factors through $\overline{R} \otimes_R M$, the natural map

$$M/M_{\text{tors}} \longrightarrow (\overline{R} \otimes_R M)/(\overline{R} \otimes_R M)_{\text{tors}}$$

is injective. Since $\overline{R} \otimes_R M \cong V$ and V is finitely generated as an R-module, we see that M/M_{tors} is finitely generated. Therefore it is enough to show that M_{tors} is finitely generated. We have an injection $M \longrightarrow U \oplus V$, and the torsion submodule of V is finitely generated as an R-module. Therefore it will suffice to show that the R-torsion submodule T of U is finitely generated. But T coincides with the $S^{-1}R$-torsion submodule of U and therefore has finite length as an $S^{-1}R$-module. Since every simple $S^{-1}R$-module is simple as an R-module, we're done.

ii.) Fix any integer $n \geq 1$, and consider the following diagram:

$$\begin{array}{ccccccc}
M & & & \longrightarrow & & & \overline{R} \otimes_R M \\
& \xi_n \searrow & & & & \cong\swarrow \gamma & \\
& & M_n & \longrightarrow & V & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \phi_n \\
& & U/\mathfrak{f}^n U & \xrightarrow{\psi'_n} & V/\mathfrak{f}^n V & & \\
& \alpha_n \nearrow\cong & & & & \cong\nwarrow \beta_n & \\
M/\mathfrak{f}^n M & & & \xrightarrow{\psi_n} & & & \frac{\overline{R} \otimes_R M}{\mathfrak{f}^n(\overline{R} \otimes_R M)}
\end{array}$$

Here the inner square is the pullback diagram (4). The isomorphism γ is the same as in (5), and α_n, β_n and ψ_n are obtained by reducing the isomorphisms α, β and ψ of (5) modulo $\mathfrak{f}^n$. At this point (without the map ξ_n), the diagram (consisting of two squares and two trapezoids) is commutative, and since the inner square is a pullback, there is a unique map ξ_n making the "cube" commute. Now we use Lemma 3.1 to choose n so large that the outer square is a pullback. Then ξ_n is an isomorphism, and we have the desired natural isomorphism between the two pullbacks. ■

Our next aim is to define an action of the group $(S^{-1}\overline{R})^\times$ on isomorphism classes of finitely generated R-modules. The orbits and the stabilizers of this action will play a crucial role in our further investigations. For logical precision, we choose, once and for all, a *representative set* $\mathrm{FinGen}(R)$ of finitely generated R-modules. That is, $\mathrm{FinGen}(R)$ is a set of finitely generated R-modules with the property that every finitely generated R-module M is isomorphic to a unique element $[M] \in \mathrm{FinGen}(R)$.

Notation 3.4 Since $S^{-1}\overline{R}$ is a finite product $D_1 \times \ldots \times D_s$ of semilocal principal ideal domains D_i, we have an internal decomposition of the group $(S^{-1}\overline{R})^\times$. Suppose that F is a finitely generated $S^{-1}\overline{R}$-module. Then F has a decomposition $F_1 \times \ldots \times F_s$, where each F_i is a finitely generated D_i-module. Let J be the set of indices j $(1 \le j \le s)$ such that the D_j-module F_j has nonzero torsion-free rank. Given an element $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_s) \in (S^{-1}\overline{R})^\times$, define a new element $\varepsilon|F \in (S^{-1}\overline{R})^\times$ by letting the j^{th} coordinate of $\varepsilon|F$ be ε_j if $j \in J$ and 1 if $j \notin J$. If M is a finitely generated R-module, respectively $S^{-1}R$-module, we put $\varepsilon|M := \varepsilon|S^{-1}\overline{R} \otimes_R M$, respectively, $\varepsilon|M := \varepsilon|S^{-1}\overline{R} \otimes_{S^{-1}R} M$.

For convenience we put $\Omega(R) := (S^{-1}\overline{R})^\times$ and $\Omega^*(R) := \mathrm{Coker}(\overline{R}^\times \longrightarrow (S^{-1}\overline{R})^\times)$, since these groups will occur repeatedly in what follows. Notice that since the decomposition of $S^{-1}\overline{R}$ actually comes from a decomposition of $\overline{R}$ (though if R is not connected some of the components of $\overline{R}$ may disappear when we pass to $S^{-1}\overline{R}$), we have an induced decomposition of $\Omega^*(R)$. Thus, if $\varepsilon \in \Omega(R)$ and $\varepsilon \mapsto \varepsilon^* \in \Omega^*(R)$, we can define $\varepsilon^*|M$ unambiguously to be the image of $\varepsilon|M$ in $\Omega^*(R)$. If G is a subgroup of one of the groups $\Omega(R)$ or $\Omega^*(R)$, and if V is a finitely generated module over R, $S^{-1}R$ or $S^{-1}\overline{R}$, we let $G|V$ denote the group $\{(\varepsilon|V) \mid \varepsilon \in G\}$. Note that $G|V$ is not necessarily a subgroup of G; rather, it is the projection of G on $\Omega(R)$ or $\Omega^*(R)$.

The construction. Let M be a finitely generated R-module, and let $\varepsilon \in \Omega(R)$. Choose an *arbitrary* automorphism $\theta \in \mathrm{Aut}_{S^{-1}\overline{R}}(S^{-1}\overline{R} \otimes_R M)$ with $\det\theta = \varepsilon|M$. (To see that such automorphisms exist, let J and the D_j be as in Notation 3.4, for the module $F := S^{-1}\overline{R} \otimes_R M$. For each $j \in J$, F_j has a direct summand isomorphic to D_j. Thus we can write $F = G \oplus H$, where $G \cong \prod_{j \in J} D_j$. Now take θ to be multiplication by $\varepsilon|M$ on G and the identity on H.) Define an R-module M^θ (eventually to be denoted by M^ε) by the following pullback diagram:

$$\begin{array}{ccc} M^\theta & \longrightarrow & \overline{R} \otimes_R M \\ \downarrow & & \downarrow \phi \\ S^{-1}M & \xrightarrow{\theta\psi} & S^{-1}\overline{R} \otimes_R M \end{array} \tag{6}$$

(Here ϕ and ψ are the maps in (1), the standard pullback for M.) We first show that the isomorphism class of M^θ does not depend on the choice of θ. Thus suppose that θ' is another automorphism with $\det\theta' = \varepsilon|M$. By Lemma 3.3 we can also obtain M^θ and $M^{\theta'}$ as pullbacks whose bottom lines are quotients modulo $\mathfrak{f}^n$ (for some sufficiently large power n):

$$\begin{array}{ccccccc} M^{\theta'} & & & \longrightarrow & & & \overline{R} \otimes_R M \\ & \cong\searrow \xi & & & & \cong\swarrow \chi & \\ & & M^\theta & \longrightarrow & \overline{R} \otimes_R M & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & M/\mathfrak{f}^n M & \xrightarrow{\theta_n\psi_n} & \frac{\overline{R}\otimes_R M}{\mathfrak{f}^n(\overline{R}\otimes_R M)} & & \\ & =\nearrow & & & & \cong\nwarrow \chi_n & \\ M/\mathfrak{f}^n M & & & \xrightarrow{\theta'_n\psi_n} & & & \frac{\overline{R}\otimes_R M}{\mathfrak{f}^n(\overline{R}\otimes_R M)} \end{array}$$

Here $\chi_n = \theta_n(\theta'_n)^{-1}$, and $\chi \in \mathrm{Aut}_{\overline{R}}(\overline{R} \otimes_R M)$ is any lifting of χ_n. (Such an automorphism χ exists by Proposition 2.6.) The resulting diagram (without the map $\xi : M^{\theta'} \to M^\theta$) commutes, and now the induced map ξ is the desired isomorphism.

Therefore we can define an operation of $\Omega(R)$ on $\mathrm{FinGen}(R)$ by setting $[M]^\varepsilon = [M^\theta]$, where $\det\theta = \varepsilon|M$. Next we show that the operation we have just defined is a group action, that is, $([M]^\varepsilon)^\delta = [M]^{\varepsilon\delta}$ for all finitely generated R-modules M and for all $\varepsilon, \delta \in \Omega(R)$. Let θ, η be $S^{-1}\overline{R}$-automorphisms of $S^{-1}\overline{R} \otimes_R M$ with $\det\theta = \varepsilon|M$ and $\det\eta = \delta|M$. By Lemma 3.3 and diagram (5) there is an isomorphism $(1_{M^\theta}, \alpha, \beta, \gamma)$ from the standard pullback for M^θ to the pullback (6). Putting $\eta' = \beta^{-1}\eta\beta \in \mathrm{Aut}_{S^{-1}\overline{R}}(S^{-1}\overline{R} \otimes_R M^\theta)$, we see that the same 4-tuple of maps gives an isomorphism from the pullback defining $(M^\theta)^{\eta'}$ to the pullback defining $M^{\eta\theta}$. Since, for $\zeta \in \mathrm{Aut}_{S^{-1}\overline{R}}(S^{-1}\overline{R} \otimes_R M)$, the isomorphism class of M^ζ depends only on $\det\zeta$, we see that $(M^\theta)^\eta \cong M^{\theta\eta}$, as desired. Thus we indeed have an action of $\Omega(R)$ on $\mathrm{FinGen}(R)$.

Next we show that the image of the natural map $\overline{R}^\times \longrightarrow (S^{-1}\overline{R})^\times$ acts trivially on $\mathrm{FinGen}(R)$. Let M be an arbitrary finitely generated module, and let $\varepsilon \in \mathrm{Image}(\overline{R}^\times \longrightarrow (S^{-1}\overline{R})^\times)$. Since a finitely generated torsion-free module over a Dedekind domain is a direct sum of rank-one projectives, we can pick $\chi \in \mathrm{Aut}_{\overline{R}}(\overline{R}\otimes_R M)$ such that the determinant of the map $\theta \in \mathrm{Aut}_{S^{-1}\overline{R}}(S^{-1}\overline{R}\otimes_R M)$ induced by χ is equal to $\varepsilon|M$. Then χ, θ and the identity map on $S^{-1}M$ yield an isomorphism from the standard pullback (1) of M to the pullback (6). Thus $M \cong M^\theta \cong M^\varepsilon$.

Let us summarize what we have proved:

Proposition 3.5 *The correspondence* $([M],\varepsilon) \mapsto [M]^\varepsilon$ *described above is an action of* $\Omega(R)$ *on* $\mathrm{FinGen}(R)$*. The subgroup* $H := \mathrm{Image}(\overline{R}^\times \longrightarrow (S^{-1}\overline{R})^\times)$ *of* $\Omega(R)$ *acts trivially. Thus, if we denote by* ε^* *the coset of* ε *modulo* H*, the induced correspondence* $([M],\varepsilon^*) \mapsto [M]^{\varepsilon^*} := [M]^\varepsilon$ *is an action of* $\Omega^*(R)$ *on* $\mathrm{FinGen}(R)$.

Harmless imprecision. From now on, if M is a finitely generated R-module and ε belongs to either $\Omega(R)$ or $\Omega^*(R)$, we will denote by M^ε any finitely generated R-module N for which $[N] = [M]^\varepsilon$, keeping in mind that M^ε is defined only up to (non-canonical) isomorphism. Thus we will forget all about FinGen, and we will make no notational distinction between the actions of $\Omega(R)$ and $\Omega^*(R)$ on finitely generated R-modules.

Proposition 3.6 *Let* ε *be an element of* $\Omega(R)$ *(or of* $\Omega^*(R)$*). The following hold for all finitely generated* R*-modules* M *and* N*:*

i.) $\overline{R}\otimes_R M^\varepsilon \cong \overline{R}\otimes_R M$.

ii.) $(M^\varepsilon)_\mathfrak{m} \cong M_\mathfrak{m}$ *for all maximal ideals* $\mathfrak{m}$ *of* R.

iii.) $M^\varepsilon \oplus N \cong (M\oplus N)^{\varepsilon|M}$.

iv.) $M^\varepsilon \cong M^{\varepsilon|M}$.

Proof. i.) and ii.) follow from the fact that (6) is isomorphic to the standard pullback of M^θ; iii.) and iv.) are clear from the construction. ∎

The next proposition characterizes the orbits of the group action (but see Corollary 4.3 below for a characterization that shows the connection with the cancellation problem).

Proposition 3.7 *Let M and N be finitely generated R-modules. Then the following conditions are equivalent:*

i.) $M^{\varepsilon} \cong N$ *for some* ε *(in either* $\Omega(R)$ *or* $\Omega^*(R)$*).*

ii.) $\overline{R} \otimes_R M \cong \overline{R} \otimes_R N$ *and* $M_{\mathfrak{m}} \cong N_{\mathfrak{m}}$ *for every maximal ideal* $\mathfrak{m}$ *of* R*.*

iii.) $\overline{R} \otimes_R M \cong \overline{R} \otimes_R N$ *and* $S^{-1}M \cong S^{-1}N$*.*

Proof. i.) implies ii.) by Proposition 3.6, and clearly ii.) implies iii.). To prove that iii.) implies i.), choose isomorphisms $\alpha : \overline{R} \otimes_R M \longrightarrow \overline{R} \otimes_R N$ and $\beta : S^{-1}M \longrightarrow S^{-1}N$. These maps induce isomorphisms $\gamma := S^{-1}\alpha$ and $\delta := 1_{S^{-1}\overline{R}} \otimes \beta$ from $S^{-1}\overline{R} \otimes_R M$ onto $S^{-1}\overline{R} \otimes_R N$. If we put $\theta = \gamma^{-1}\delta$ we see that α, β and γ provide an isomorphism from the pullback (6) defining M^{θ} to standard pullback of N. ∎

4 Delta groups and the cancellation problem for finitely generated modules

Standing assumptions and notation. For the rest of the paper, R denotes a one-dimensional Noetherian reduced ring with finite normalization $\overline{R}$. As before we let S be the complement of the union of the singular maximal ideals of R (i.e., those maximal ideals containing the conductor $\mathfrak{f} = (R : \overline{R})$).

In the following we define the "delta groups" associated to a finitely generated R-module M. These invariants determine the stabilizers of the group actions defined in §3 and tell exactly when cancellation holds. Since we consider both the action of $\Omega(R)$ and the induced action of $\Omega^*(R)$, we will define a delta group for each action. Recall Notation 2.4 concerning determinants.

Definition 4.1 Let M be a finitely generated R-module.

i.) Put $\Delta(M) = \Delta_R(M) := \{\det(1_{S^{-1}\overline{R}} \otimes_{S^{-1}R} \phi) \mid \phi \in \mathrm{Aut}_{S^{-1}R}(S^{-1}M)\} \subseteq \Omega(R)$.

ii.) Let $\Delta^*(M) = \Delta^*_R(M)$ denote the image of $\Delta_R(M)$ in $\Omega^*(R)$.

Notice, for example, that $\Delta_R(R) = (S^{-1}R)^{\times}$ and $\Delta_R(\overline{R}) = \Omega(R)$. In the rest of this section we will develop the basic rules for working with delta groups. We will give most results both in terms of the action of the group $\Omega(R)$ and in terms of the induced action of $\Omega^*(R)$.

Proposition 4.2 *Let M be a finitely generated R-module.*

i.) For $\varepsilon \in \Omega(R) = (S^{-1}\overline{R})^\times$, we have $M^\varepsilon \cong M$ if and only if $\varepsilon|M \in \Delta(M) \cdot \mathrm{Image}(\overline{R}^\times \longrightarrow (S^{-1}\overline{R})^\times)$.

ii.) For $\varepsilon \in \Omega^*(R)$, we have $M^\varepsilon \cong M$ if and only if $\varepsilon|M \in \Delta^*(M)$.

Proof. Clearly i.) and ii.) say the same thing, so we will prove i.).

Suppose $M^\varepsilon \cong M$. Following the recipe in the construction of M^ε, choose $\theta \in \mathrm{Aut}_{S^{-1}\overline{R}}(S^{-1}\overline{R} \otimes_R M)$ with $\det\theta = \varepsilon|M$. Then $M^\varepsilon \cong M^\theta$, and we can choose an isomorphism $\alpha : M^\varepsilon \longrightarrow M$. Consider the diagram below, in which the inner square is the standard pullback for M and the outer square is (6), which, by Lemma (3.3) is isomorphic to the standard pullback for M^θ. Now α induces isomorphisms β and γ making the left and top trapezoids commute. Then γ induces an isomorphism δ making the right trapezoid commute. We *define* $\eta := \delta\theta$, so that the triangle commutes.

$$\begin{array}{ccccccc}
M^\varepsilon & & & \longrightarrow & & & \overline{R} \otimes_R M \\
 & \alpha \searrow \cong & & & & \cong \swarrow \gamma & \\
 & & M & \longrightarrow & \overline{R} \otimes_R M & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & S^{-1}M & \xrightarrow{\psi} & S^{-1}\overline{R} \otimes_R M & & \\
 & \beta \nearrow \cong & & & \eta \uparrow\cong & \cong \nwarrow \delta & \\
S^{-1}M & & \xrightarrow{\psi} & & S^{-1}\overline{R} \otimes_R M & \xrightarrow{\theta} & S^{-1}\overline{R} \otimes_R M
\end{array} \qquad (7)$$

The argument we used with (5) shows that the bottom face of the "cube" commutes. Then $\eta\psi = \psi\beta$, whence $\det\eta \in \Delta_R(M)$. Also, since δ is induced by γ, we see that $\det\delta \in \mathrm{Image}(\overline{R}^\times \longrightarrow (S^{-1}\overline{R})^\times)$. Thus $\varepsilon|M = \det\theta = (\det\eta) \cdot (\det\delta)^{-1} \in \Delta(M) \cdot \mathrm{Image}(\overline{R}^\times \longrightarrow (S^{-1}\overline{R})^\times)$.

Conversely, suppose that $\varepsilon|M = \delta\lambda$, with $\delta \in \Delta(M)$ and with $\lambda \in \mathrm{Image}(\overline{R}^\times \longrightarrow (S^{-1}\overline{R})^\times)$. By Proposition 3.5, we have $M^\varepsilon \cong M^\delta$. Since $\delta \in \Delta_R(M)$ we can choose $\beta \in \mathrm{Aut}_{S^{-1}R}(S^{-1}M)$ so that $\det(1_{S^{-1}\overline{R}} \otimes \beta) = \delta$.

Put $\theta = 1_{S^{-1}\overline{R}} \otimes \beta$. We obtain an isomorphism of pullbacks:

$$\begin{array}{ccccccc}
M^\theta & & & \longrightarrow & & & \overline{R} \otimes_R M \\
& \alpha \searrow \cong & & & & \swarrow = & \\
& & M & \longrightarrow & \overline{R} \otimes_R M & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & S^{-1}M & \xrightarrow{\psi} & S^{-1}\overline{R} \otimes_R M & & \\
& \beta \nearrow \cong & & & & \nwarrow = & \\
S^{-1}M & & & \xrightarrow{\theta\psi} & & & S^{-1}\overline{R} \otimes_R M
\end{array}$$

Therefore $M^\varepsilon \cong M^\delta \cong M^\theta \cong M$. ∎

Corollary 4.3 *Let M and N be finitely generated R-modules. The following conditions are equivalent:*

i.) $N \cong M^\varepsilon$ *for some* $\varepsilon \in \Omega(R)$ *(equivalently, for some* $\varepsilon \in \Omega^*(R)$*).*

ii.) $M \oplus \overline{R} \cong N \oplus \overline{R}$.

iii.) $M \oplus L \cong N \oplus L$ *for some finitely generated R-module L.*

Proof. Assume i.), and put $\delta := \varepsilon | M$. By Proposition 3.6 we have $M^\varepsilon \oplus \overline{R} \cong (M \oplus \overline{R})^\delta \cong M \oplus \overline{R}^\delta$ (since $\delta | \overline{R} = \delta$). But $\delta \in \Omega(R) = \Delta_R(\overline{R})$, so $\overline{R}^\delta \cong \overline{R}$ by 4.2. Thus we have ii.).

Obviously ii.) implies iii.). Now assume iii.). Since cancellation holds over local rings (cf. [Eva73]) and over Dedekind domains, we have $M_\mathfrak{m} \cong N_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R and $\overline{R} \otimes_R M \cong \overline{R} \otimes_R N$. Now Proposition 3.7 gives i.), and the proof is complete. ∎

By combining Corollary 4.3 with Proposition 4.2, we obtain the following criterion for cancellation.

Corollary 4.4 *The following conditions are equivalent, for a finitely generated R-module M:*

i.) $M \oplus L \cong N \oplus L \implies M \cong N$ *for all finitely generated R-modules N, L.*

ii.) $\Omega(R) | M \subseteq \Delta_R(M) \cdot \mathrm{Image}(\overline{R}^\times \longrightarrow (S^{-1}\overline{R})^\times)$.

iii.) $\Omega^*(R)|M \subseteq \Delta^*_R(M)$.

Even though cancellation is by no means a local property (since it *always* holds locally), the delta groups $\Delta_R(M)$ can be computed locally. To make this precise, and to set the stage for our constructions in §6, we define the delta groups of a finitely generated $S^{-1}R$-module F in the obvious way:

$$\begin{gathered}\Delta(F) = \Delta_{S^{-1}R}(F) := \{\det(1_{S^{-1}\overline{R}} \otimes_{S^{-1}R} \phi) \mid \phi \in \mathrm{Aut}_{S^{-1}R}(S^{-1}F)\} \subseteq \Omega(R).\\ \Delta^*(F) = \Delta^*_{S^{-1}R}(F) := \text{ image of } \Delta_R(F) \text{ in } \Omega^*(R).\end{gathered} \tag{8}$$

Most of our results could be stated in terms of these "semilocal delta groups", in view of the following, whose proof is self-evident:

Proposition 4.5 *Let M and N be finitely generated R-modules, and put $F := S^{-1}M$.*

i.) $\Delta_R(M) = \Delta_{S^{-1}R}(F)$ *(and similarly for the groups Δ^*).*

ii.) $\Omega(R)|M = \Omega(R)|F$ *(and similarly for Ω^*).*

iii.) If $M_\mathfrak{m} \cong N_\mathfrak{m}$ over $R_\mathfrak{m}$ for every singular maximal ideal $\mathfrak{m}$ of R, then $\Delta(M) = \Delta(N)$ and $\Delta^*(M) = \Delta^*(N)$.

We have essentially reduced the cancellation problem over R to the problem of computing the delta groups of modules over the semilocal ring $S^{-1}R$. The only quantity that comes from the "global" ring R is the group $\overline{R}^\times$ of units of the integral closure of R. In fact, as we shall see in Corollary 4.7, the delta groups themselves can be computed locally (not just semilocally).

Proposition 4.6 *Let M be a finitely generated R-module. There is a positive integer n such that $(1 + S^{-1}\mathfrak{f}^n)|M \subseteq \Delta_R(M)$.*

Proof. After replacing R by $S^{-1}R$ we may assume, by Proposition 4.5, that R is semilocal and that $\mathfrak{f}$ is contained in the Jacobson radical of R. Choose n so large that (2) is a pullback diagram. Let $\varepsilon \in 1 + \mathfrak{f}^n \subseteq \overline{R}^\times$. We seek an automorphism ϕ of M such that $\det(1_{\overline{R}} \otimes \phi) = \varepsilon|M$. With the notation of (3.4), and with $F := \overline{R} \otimes_R M$, we can split off a free direct summand of F isomorphic to D_j for each $j \in J$ (the set of coordinates where F is not torsion). Thus we can write $\overline{R} \otimes_R M = L \oplus N$, where $L \cong \prod_{j \in J} D_j$. Let $\psi \in \mathrm{Aut}_{\overline{R}}(\overline{R} \otimes_R M)$ be given by multiplication by ε on

L and the identity on N. Then $\det(\psi) = \varepsilon|M$, and it will suffice to find an automorphism φ of M such that $\psi = 1_{\overline{R}} \otimes \varphi$. Since ψ induces the identity map ι on $(\overline{R} \otimes_R M)/\mathfrak{f}^n(\overline{R} \otimes_R M)$, the automorphisms ψ, ι and $1_{M/\mathfrak{f}^n M}$ induce an automorphism of the pullback diagram (2). Now we let φ be the induced automorphism of M. ■

The following corollary of (4.6) will be used in §6 when we study Dedekind-like rings:

Corollary 4.7 *Let M be a finitely generated R-module, let $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be the singular maximal ideals of R, and let $\nu_i : S^{-1}\overline{R} \to \overline{R}_{\mathfrak{m}_i}$ be the natural map, for $i = 1, \ldots, t$. Then $\Delta_R(M) = \{u \in (S^{-1}\overline{R})^\times \mid \nu_i(u) \in \Delta_{R_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}) \ \forall\ i = 1, \ldots, t\}$.*

Proof. We may assume that $R = S^{-1}R$. The conductor $\mathfrak{f} = (R : \overline{R})$ is then contained in the Jacobson radical J of R. If $u \in \Delta_R(M)$, then clearly $\nu_i(u) \in \Delta_{R_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i})$ for each i. For the reverse inclusion, we must allow for the fact that some of the components of M may be torsion modules. Given any singular maximal ideal $\mathfrak{m}$ of R, we note that the following diagram commutes:

$$\begin{array}{ccc} \overline{R}^\times & \xrightarrow{\nu} & (\overline{R}_{\mathfrak{m}})^\times \\ |M \downarrow & & \downarrow |M_{\mathfrak{m}} \\ \overline{R}^\times & \xrightarrow{\nu} & (\overline{R}_{\mathfrak{m}})^\times \end{array} \qquad (9)$$

Here ν is the natural map and the vertical maps are the "restriction" maps $\varepsilon \mapsto \varepsilon|M$ (and $\varepsilon \mapsto \varepsilon|M_{\mathfrak{m}}$) of Notation 3.4.

Now let $u \in \overline{R}^\times$, and suppose that $\nu_i(u) \in \Delta_{R_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i})$ for each i. Then $\nu_i(u) = \nu_i(u)|M_{\mathfrak{m}_i}$ for each i, and from (9) we see that $u = u|M$. For each $i, 1 \le i \le t$, let ϕ_i be an $R_{\mathfrak{m}_i}$-automorphism of $M_{\mathfrak{m}_i}$ with $\det(1_{\overline{R}_{\mathfrak{m}_i}} \otimes \phi_i) = \nu_i(u)$. Write $\phi_i = \frac{\psi_i}{s_i}$ with $\psi \in \mathrm{End}_R(M)$ and $s_i \in R - \mathfrak{m}_i$.

Choose n as in Proposition 4.6, and choose a positive integer p such that $J^p \subseteq \mathfrak{f}^n$. For $i = 1, \ldots, t$, select $a_i \in (\mathfrak{m}_1 \cdot \ldots \cdot \widehat{\mathfrak{m}}_i \cdot \ldots \cdot \mathfrak{m}_t)^p$ such that $a_i s_i \equiv 1 \pmod{\mathfrak{m}_i^p}$. Put $\psi := a_1\psi_1 + \ldots + a_t\psi_t \in \mathrm{End}_R(M)$, and let $\theta \in \mathrm{End}_{R/J^p}(M/J^pM)$ be the map induced by ψ. Over each component $R/\mathfrak{m}_i^p$ of the ring R/J^p, θ agrees with the automorphism induced by ϕ_i. Therefore θ is an automorphism, and by Nakayama's lemma $\psi \in \mathrm{Aut}_R(M)$. Hence $\det(1_{\overline{R}} \otimes \psi) \in \Delta_R(M)$. Since $\det(1_{\overline{R}} \otimes \psi) \equiv u \pmod{\mathfrak{f}^n}$ and $(1 + \mathfrak{f}^n)|M \subseteq \Delta_R(M)$ by Proposition 4.6, we have $u \in \Delta_R(M)$ as desired. ■

In the next section, in order to get more subtle results on cancellation, we will need the following result (cf. [WW87, Lemma 1.7]) on the "additivity" of delta groups:

Proposition 4.8 *Let T denote either R or $S^{-1}R$, and let F and G be finitely generated T-modules. Then $\Delta_T(F \oplus G) = \Delta_T(F) \cdot \Delta_T(G)$ and $\Delta^*_T(F \oplus G) = \Delta^*_T(F) \cdot \Delta^*_T(G)$*

PROOF. It will suffice to take $T = S^{-1}R$ and show that $\Delta_T(F \oplus G) = \Delta_T(F) \cdot \Delta_T(G)$. We obviously have $\Delta_T(F \oplus G) \supseteq \Delta_T(F) \cdot \Delta_T(G)$. In order to prove the opposite inclusion we write every endomorphism φ of $F \oplus G$ in the form $\varphi = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$, where $\alpha \in \mathrm{End}_R(F)$, $\delta \in \mathrm{End}_R(G)$, $\beta \in \mathrm{Hom}_R(G, F)$ and $\gamma \in \mathrm{Hom}_R(F, G)$. Let $\varphi = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in \mathrm{Aut}_R(F \oplus G)$ and let $\varphi^{-1} = \left[\begin{smallmatrix} \widetilde{\alpha} & \widetilde{\beta} \\ \widetilde{\gamma} & \widetilde{\delta} \end{smallmatrix}\right]$. Then $\alpha\widetilde{\alpha} + \beta\widetilde{\gamma} = \mathrm{id}_F$. Since $\mathrm{End}_R(F)$ is semilocal (in the non-commutative sense — see [Fac98, p. 7]), there exists by [Bas68, Ch. 5, Proposition 3.4] an element $\mu \in \mathrm{End}_R(F)$ such that $\alpha + \beta\widetilde{\gamma}\mu$ is a unit of $\mathrm{End}_R(F)$. Since the automorphism $\sigma := \left[\begin{smallmatrix} 1 & 0 \\ \widetilde{\gamma}\mu & 1 \end{smallmatrix}\right]$ has determinant 1, we may replace φ by $\varphi\sigma$ without changing its determinant. The $(1, 1)$ entry of $\varphi\sigma$ is $\alpha + \beta\widetilde{\gamma}\mu$. Hence we may assume without restriction that the $(1, 1)$ entry of φ, still called α, is an isomorphism. Put $\tau = \left[\begin{smallmatrix} 1 & -\alpha^{-1}\beta \\ 0 & 1 \end{smallmatrix}\right]$. Then $\varphi\tau$ is triangular, and $\det \varphi = \det(\varphi\tau) \in \Delta^*(F)\Delta^*(G)$ as desired. ■

5 Stable isomorphism and power cancellation

Keep the notation and assumptions in the opening paragraph of Section 4. We say that two R-modules M and N are stably isomorphic if $M \oplus R^n \cong N \oplus R^n$ for some $n \geq 1$. The following theorem generalizes [WW87, Theorem 1.8].

Theorem 5.1 *Let M, N and L be finitely generated R-modules. Then the following conditions are equivalent:*

i.) $M \oplus L^n \cong N \oplus L^n$ *for some integer* $n \geq 1$.

ii.) $M \oplus L \cong N \oplus L$.

PROOF. The theorem follows directly from Proposition 4.8 and Proposition 5.2 below. ■

Proposition 5.2 *Let M, N and L be finitely generated R-modules. Then $M \oplus L \cong N \oplus L$ if and only if there is an element $\varepsilon \in \Delta^*(L)$ such that $\varepsilon|M = \varepsilon$ and $N \cong M^\varepsilon$. In particular, if M is a torsion module, then $M \oplus L \cong N \oplus L \implies M \cong N$.*

Proof. Suppose $\varepsilon \in \Delta^*(L)$ with $\varepsilon|M = \varepsilon$ and $M^\varepsilon \cong N$. Then $N \oplus L \cong M^\varepsilon \oplus L \cong (M \oplus L)^\varepsilon$ by Proposition 3.6. Since $\varepsilon \in \Delta^*(L)$, we have $\varepsilon|L = \varepsilon$. Therefore $(M \oplus L)^\varepsilon \cong M \oplus L^\varepsilon \cong M \oplus L$.

To prove the converse suppose that $M \oplus L \cong N \oplus L$. Then $N \cong M^\eta$ for some $\eta \in \Omega^*(R)$. By part iv.) of 3.6 we may assume that $\eta|M = \eta$. Then we must also have $\eta|(M \oplus L) = \eta$. Now $(M \oplus L)^\eta \cong (M \oplus L)$, and therefore $\eta \in \Delta^*(M)\Delta^*(L)$ by Proposition 4.2 and Proposition 4.8. Write $\eta = \zeta\varepsilon$ with $\zeta \in \Delta^*(M)$ and $\varepsilon \in \Delta^*(L)$. Clearly $\zeta = \zeta|M$, and since $\eta = \eta|M$ we must have $\varepsilon = \varepsilon|M$. Since $M^\varepsilon \cong M^\eta \cong N$, the proof is complete.

For the last statement, we observe that if M is torsion then $\varepsilon|M = 1$. ∎

The next result, which generalizes [WW87, Corollary 2.5] to modules with torsion, shows that stable isomorphism implies isomorphism if, for example, R is the coordinate ring of an irreducible affine curve over the complex numbers. Recall that the finitely generated R-module M has *constant rank* r provided $\mathcal{Q}(R) \otimes_R M$ is a free $\mathcal{Q}(R)$-module of rank r.

Theorem 5.3 *Let R be a one-dimensional reduced ring which is a finitely generated algebra over an algebraically closed field k. Let M and N be finitely generated R-modules of constant rank r. If M and N are stably isomorphic and $\operatorname{char}(k) \nmid r$, then $M \cong N$.*

Proof. By Theorem 5.1 we have $M \oplus R \cong N \oplus R$. If $r = 0$, then $M \cong N$ by Proposition 5.2. Thus we may assume that $r > 0$. By Proposition 5.2 and Proposition 3.5 we have $N \cong M^\varepsilon$ for some $\varepsilon \in \Delta(R) = (S^{-1}R)^\times$. Choose an integer n as in Proposition 4.6, and let u be the image of ε in $S^{-1}R/S^{-1}\mathfrak{f}^n = R/\mathfrak{f}^n$. By [WW87, (2.3)] there is an element $v \in S^{-1}R/S^{-1}\mathfrak{f}^n$ with $v^r = u$. Letting $w \in (S^{-1}R)^\times$ be any preimage of v, we have $\frac{\varepsilon}{w^r} \in 1 + S^{-1}\mathfrak{f}^n \subseteq \Delta_R(M)$. (Note that $G|M = G$ for any subgroup $G \subseteq \Omega(R)$ since M has non-zero constant rank.) But clearly $w^r \in \Delta_R(M)$ (take scalar multiplication by w for the map ϕ in Definition 4.1). Thus $\varepsilon \in \Delta_R(M)$, and $N \cong M^\varepsilon \cong M$. ∎

We remark that the hypothesis of constant rank cannot be omitted, even if the field k has characteristic zero and the modules M and N are torsion-free.

The ring in [WW87, Example 3.3] is a one-dimensional reduced ring, finitely generated as a $\mathbb{C}$-algebra. This ring has two finitely generated torsion-free modules M and N such that $M \oplus R \cong N \oplus R$, yet there is no positive integer q for which $M^q \cong N^q$. The ring has two irreducible components, and M and N have rank 1 on one component of $\mathcal{Q}(R)$ and rank 2 on the other.

We now treat the problem of *power cancellation*, which has been investigated in [Goo76], [Gur86], [Gur87], [LW85] and [Lev85]. We say that power cancellation holds for R if for all finitely generated R-modules M, N and L with $M \oplus L \cong N \oplus L$ there exists an integer $n \geq 1$ such that $M^n \cong N^n$. If some n works for all M, N and L, we say that R has power cancellation with bounded exponent. Recall that the *exponent* of a group G is the least common multiple of the orders of the elements of G.

Lemma 5.4 *Let M be a finitely generated R-module.*

1. *For a positive integer n, the following conditions are equivalent:*
   - (a) *$M \oplus L \cong N \oplus L \implies M^n \cong N^n$, for all finitely generated R-modules L, N.*
   - (b) *The group $\frac{\Omega^*(R)|M}{\Delta^*(M)}$ has exponent dividing n.*
2. *The following conditions are equivalent:*
   - (a) *If $M \oplus L \cong N \oplus L$ with L and N finitely generated R-modules, then $M^n \cong N^n$ for some positive integer n.*
   - (b) *$\frac{\Omega^*(R)|M}{\Delta^*(M)}$ is a torsion group.*

Proof. We prove that a.) implies b.) in parts i.) and ii.) simultaneously. Let ε be an arbitrary element of $\Omega^*(R)|M$. Put $N = M^\varepsilon$. By Corollary 4.3 we have $M \oplus \overline{R} \cong N \oplus \overline{R}$.

Therefore, by a.) we have $M^n \cong N^n$ (for *some* n in part ii.)). Since $\varepsilon|M = \varepsilon$, part iii.) of Proposition 3.6 implies that $(M^n)^{\varepsilon^n} \cong (M^\varepsilon)^n$. Thus $(M^n)^{\varepsilon^n} \cong M^n$, and by Proposition 4.2 we have $\varepsilon^n \in \Delta^*(M^n)$, which equals $\Delta^*(M)$ by Proposition 4.8. Thus we have b.).

For the converse, suppose $M \oplus L \cong N \oplus L$, with N and L finitely generated R-modules. By Corollary 4.3 we have $M^\varepsilon \cong N$ for some $\varepsilon \in \Omega^*(R)$. By Proposition 3.6 we can assume that $\varepsilon \in \Omega^*(R)|M$. Assuming b.), we have $\varepsilon^n \in \Delta^*(M)$ (for *some* positive integer n in part ii.)). Now $\Delta^*(M) = \Delta^*(M^n)$, by Proposition 4.8. Thus $M^n \cong (M^n)^{\varepsilon^n} \cong (M^\varepsilon)^n \cong N^n$. ■

Lemma 5.4 does not immediately give a useful criterion for power cancellation, since it appears that one would need to know the delta groups of all finitely generated modules in order to decide whether or not the ring has power cancellation. Surprisingly, there is a criterion that depends only on the kernel $\mathrm{D}(R)$ of the map $\mathrm{Pic}(R) \to \mathrm{Pic}\,\overline{R})$. Before stating the result (Theorem 5.7 below) we observe the following:

Lemma 5.5 $\mathrm{D}(R)$ *is isomorphic to* $\Omega^*(R)/\Delta^*(R)$.

Proof. A finitely generated R-module V represents an element of $\mathrm{D}(R)$ if and only if $V_{\mathfrak{m}} \cong R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R and $\overline{R} \otimes_R V \cong \overline{R}$. Thus, by Proposition 3.7, $\mathrm{D}(R) = \{[R^\varepsilon] \mid \varepsilon \in \Omega^*(R)\}$. The desired result now follows from (2) of Proposition 4.2. ■

Notation 5.6 For the next theorem we need to clarify the notion of "rank" when R is not necessarily a domain. The notion we define is unnatural but well-suited to the proof. (We will use this notation only in the present section of the paper.) Referring to Notation 3.4, write $S^{-1}\overline{R} = D_1 \times \ldots \times \mathrm{D}_s$, where each D_i is a semilocal principal ideal domain. Given a finitely generated $S^{-1}R$-module F, put $G := S^{-1}\overline{R} \otimes_{S^{-1}R} F$. Write $G = G_1 \times \ldots \times G_s$, where each G_i is a finitely generated D_i-module. Let $r_i = r_i(F)$ be the (torsion-free) rank of G_i, and let $J = J(F) := \{i \mid 1 \le i \le s \text{ and } r_i > 0\}$. If $J \neq \emptyset$ (that is, F is not a torsion module), we define $\mathrm{RANK}(F)$ to be the least common multiple of $\{r_i \mid i \in J\}$; if $J = \emptyset$, we put $\mathrm{RANK}(F) = 1$.

Theorem 5.7 *Let R be, as always, a one-dimensional reduced Noetherian ring with finite normalization.*

- *i.) Power cancellation holds for R if and only if* $\mathrm{D}(R)$ *is a torsion group.*
- *ii.) If R has power cancellation with bounded exponent, then* $\mathrm{D}(R)$ *is a bounded group.*
- *iii.) If* $\mathrm{D}(R)$ *is bounded and there is a bound on the ranks of the indecomposable $S^{-1}R$-modules, then R has power cancellation with bounded exponent (possibly larger than the exponent of* $\mathrm{D}(R)$*).*

Proof. To prove ii.) and the "only if" direction of i.), we put $M = R$ in Lemma 5.4, and then apply Lemma 5.5.

Next we prove iii.) and the "if" direction of i.). Suppose that $\mathrm{D}(R)$ is a torsion group, and let M be an arbitrary finitely generated R-module. Put $F = S^{-1}M$, and write $F = F_1 \oplus \ldots \oplus F_t$, where each F_j is an indecomposable $S^{-1}R$-module. Let m be the least common multiple of the numbers $\mathrm{RANK}(F_j), 1 \le j \le t$. Let ε be an arbitrary element of $\Omega^*(R)|F$ (see Notation 3.4). We will show that $\varepsilon^{lm} \in \Delta^*(F)$, where l is the order of some element $\tilde{\varepsilon} \in \Omega^*(R)/\Delta^*(R)$. Of course $\Omega^*(R)|F = \Omega^*(R)|M$ and $\Delta^*(F) = \Delta^*(M)$ (see Notation 3.4). Since $\Omega^*(R)/\Delta^*(R) \cong \mathrm{D}(R)$, this will prove both iii.) and the "if" direction of i.) and will complete the proof of the theorem.

Suppose first that $t = 1$, whence $m = \mathrm{RANK}(F)$. Let $D_1, \ldots, D_s$, $r_i = r_i(F)$ and $J = J(F)$ be as in Notation 5.6. Let $\pi \in S^{-1}\overline{R}$ represent the coset $\varepsilon \in \Omega^*(R)|M$, and write $\pi = (\pi_1, \ldots, \pi_s)$, with $\pi_i \in D_i$. For $1 \le i \le s$, we put $a_i = \frac{m}{r_i}$ if $i \in J$, and $a_i = 0$ if $i \notin J$. Define $\rho := (\pi_1^{a_1}, \ldots, \pi_s^{a_s}) \in S^{-1}\overline{R}$, and let $\tilde{\varepsilon} \in \Omega^*(R)/\Delta^*(R)$ be the image of ρ under the natural maps $S^{-1}\overline{R} \twoheadrightarrow \Omega^*(R) \twoheadrightarrow \Omega^*(R)/\Delta^*(R)$. Letting l be the order of $\tilde{\varepsilon}$, we can write ρ^l as a product $\rho^l = \delta\zeta$, where $\delta \in \Delta_R(R) = (S^{-1}R)^\times$ and $\zeta \in \mathrm{Image}(\overline{R}^\times \to (S^{-1}\overline{R})^\times)$.

Viewing δ as an element of $(S^{-1}\overline{R})^\times$, write $\delta = (\delta_1, \ldots, \delta_s)$, with $\delta_i \in (D_i)^\times$. The $S^{-1}R$-automorphism of F given by multiplication by δ induces an automorphism of $(S^{-1}\overline{R}) \otimes_{S^{-1}R} F$ with determinant $d := (\delta_1^{r_1}, \ \ldots, \delta_s^{r_s})$ (recall Notation 2.4). Thus $d \in \Delta_R(M)$. Since the elements d and π^{lm} of $S^{-1}\overline{R}$ have the same image in $\Omega^*(R)|M$, it follows that $\varepsilon^{lm} \in \Delta^*(M)$, as desired.

Suppose now that $t > 1$, and put $L := F_2 \oplus \ldots \oplus F_t$. We can write $\varepsilon = \alpha\beta$, with $\alpha \in \Omega^*(R)|F_1$ and $\beta \in \Omega^*(R)|L$. Let $p = \mathrm{RANK}(M_1)$ and let q be the least common multiple of $\{\mathrm{RANK}(F_j) : 2 \le j \le t\}$. By induction on t, there are elements $\tilde{\alpha}$ and $\tilde{\beta} \in \Omega^*(R)/\Delta^*(R)$ of orders, say, u and v respectively, such that $\alpha^{pu} \in \Delta^*(F_1)$ and $\beta^{qv} \in \Delta^*(L)$. Choose any element $\tilde{\varepsilon} \in \Omega^*(R)/\Delta^*_R(R)$ whose order l is a common multiple of u and v. Since m is a common multiple (in fact the least common multiple) of p and q, we have $\varepsilon^{lm} = \alpha^{lm}\beta^{lm} \in \Delta^*(F_1) \cdot \Delta^*(L) = \Delta^*(F)$, by Proposition 4.8. ■

As an immediate corollary of this theorem we obtain a new proof of [KL4, Theorem 19.12] on power cancellation for Dedekind-like rings (see §6 for relevant definitions). At this point all we need to know is that every Dedekind-like ring R *does* satisfy the standing assumptions from the beginning of §4 and that if M is an indecomposable finitely generated module over $S^{-1}R$ (or even over R, for that matter), then $\mathrm{RANK}(M) \leq 2$, [KL4, Corollary 6.16].

Corollary 5.8 *Let R be a Dedekind like ring with finite normalization. Then power cancellation with bounded exponent holds for R if and only if $\mathrm{D}(R)$ has finite exponent.*

Our next aim is to establish a connection between stable isomorphism and power cancellation.

Lemma 5.9 *Let M be a finitely generated R-module of constant rank r, and let $\varepsilon \in \Delta^*(R)$. Then $\varepsilon^r \in \Delta^*(M)$.*

PROOF. We may assume $r > 0$. Lift ε to an element $u \in \Delta(R) = (S^{-1}R)^\times$. Multiplication by u is then an automorphism of $S^{-1}M$, and the induced automorphism of $S^{-1}\overline{R} \otimes_{S^{-1}R} S^{-1}M$ has determinant u^r. Therefore $\varepsilon^r \in \Delta^*(M)$. ■

The following Proposition generalizes [WW87, Proposition 2.9].

Proposition 5.10 *Let M and N be stably isomorphic R-modules of constant rank r. Then $M^r \cong N^r$ if $r > 0$. If $r = 0$ then M and N are isomorphic.*

PROOF. By Proposition 5.2 there is an element $\varepsilon \in \Delta^*(R)$ with $M^\varepsilon \cong N$ and $\varepsilon|M = \varepsilon$. If $r = 0$ then $\varepsilon = 1$ and we are done. Assuming $r > 0$, we have, by Lemma 5.9 and Proposition 4.8, $\varepsilon^r \in \Delta^*(M) = \Delta^*(M^r)$. Now $N^r \cong (M^\varepsilon)^r \cong (M^r)^{\varepsilon^r} \cong M^r$. ■

We conclude this section by showing that "cancellation within a genus" [GL91, Corollary 5.10] is an immediate consequence of our machinery.

Proposition 5.11 *Let M, N and L be finitely generated R-modules, with $M \oplus L \cong N \oplus L$. If $L_\mathfrak{m} \cong M_\mathfrak{m}$ (as $R_\mathfrak{m}$-modules) for every singular maximal ideal $\mathfrak{m}$, then $M \cong N$.*

PROOF. By Proposition 5.2 there is an element $\varepsilon \in \Delta^*(L)$ such that $N \cong M^\varepsilon$. Since $\varepsilon \in \Delta^*(M)$ by Proposition 4.5, we have $M^\varepsilon \cong M$. ■

6 Cancellation vs. torsion-free cancellation

We keep the notation and conventions established at the beginning of §4. We say that R satisfies *cancellation*, respectively *torsion-free cancellation*, provided $M \oplus L \cong N \oplus L \implies M \cong N$ for all finitely generated R-modules M, N, L, respectively all finitely generated torsion-free R-modules M, N, L. In this section we will answer the question that motivated the research presented in this paper: Does torsion-free cancellation imply cancellation?

We begin with a positive result, though it is somewhat unsatisfying due to the paucity of examples (other than Dedekind domains) satisfying the equivalent conditions in the theorem.

Theorem 6.1 *Let k be an infinite perfect field, and let R be a one-dimensional affine domain, finitely generated as a k-algebra.*

1. *Suppose* $\mathrm{D}(R) = 0$ *and M, N and L are finitely generated R-modules with $M \oplus L \cong N \oplus L$. If* $\mathrm{char}(k) \nmid r := \mathrm{rank}(M)$*, then $M \cong N$.*
2. *Assume* $\mathrm{char}(k) = 0$*. The following conditions are equivalent:*
   - *(a)* $\mathrm{D}(R) = 0$.
   - *(b)* $\mathrm{D}(R)$ *is finitely generated (as an abelian group).*
   - *(c) R has torsion-free cancellation.*
   - *(d) R has cancellation.*

PROOF. By [Wie89′, (0.1)], (a), (b) and (c) of Part 2 are equivalent; and of course (d) implies (c). Thus Part 2 is a consequence of Part 1 (and Proposition (5.2) if $r = 0$). To prove Part 1 we may assume that R is not a Dedekind domain. Then, by [Wie89′, (1.6)], R has exactly one singular maximal ideal $\mathfrak{m}$, and $\mathfrak{m}$ is the conductor $(R : \overline{R})$; moreover, $\overline{R} = F \oplus \mathfrak{m}$, where the field F is the integral closure of k in $\overline{R}$.

Let M be an arbitrary finitely generated R-module, and let $\varepsilon \in (S^{-1}\overline{R})^\times = (\overline{R}_\mathfrak{m})^\times$. It will suffice, by Corollary 4.3, to show that $M^\varepsilon \cong M$. Since $\overline{R} = F \oplus \mathfrak{m}$, we can write $\varepsilon = c(1 + x)$, where $c \in F^\times$ and $x \in \mathfrak{m}\overline{R}_\mathfrak{m} = \mathfrak{m}R_\mathfrak{m}$.

Using Proposition 4.6, we choose n so large that $1 + \mathfrak{m}^n \subseteq \Delta_R(M)$. By [WW87, (2.3)] (or the zero-dimensional case of Hensel's lemma), the image of $1 + x$ in $(R_\mathfrak{m}/\mathfrak{m}^n R_\mathfrak{m})^\times$ is an r^{th} power. As in the proof of Theorem 5.3, we see that $1 + x \in \Delta_R(M)$. Since c is a unit of $\overline{R}$, $M^\varepsilon \cong M$. ∎

For an example of a singular affine domain with torsion-free cancellation, let $R = k + XF[X]$, where F/k is a finite algebraic extension. For this ring, $\overline{R} = F[X]$ and $\overline{R}/\mathfrak{f} = F$. Torsion-free cancellation follows from the fact that every unit of $\overline{R}/\mathfrak{f}$ lifts to a unit of $\overline{R}$, [Wie84, Corollary 2.4].

From now on we will concentrate on rings of multiplicity 2, since torsion-free cancellation is reasonably well understood in that case. We begin with the very special class of *Dedekind-like* rings, and, at least in the local case, we will use notation consistent with that of [KL1] and [KL2].

Notation 6.2 Throughout this section, $(\Lambda, \mathfrak{M}, k)$ is a one-dimensional reduced, local Noetherian ring (with maximal ideal $\mathfrak{M}$ and residue field k), and Γ is the integral closure of Λ in its total quotient ring. Moreover, we assume

i.) $\mathfrak{M}$ is the conductor $(\Lambda : \Gamma)$, and

ii.) Γ is generated by two elements as a Λ-module.

Remark/Definition 6.3 Since $\Gamma/\mathfrak{M}$ is a k-algebra of dimension at most 2, we have the following possibilities:

i.) $\Gamma/\mathfrak{M} = k$, in which case Λ is a discrete valuation ring.

ii.) $\Gamma/\mathfrak{M}$ is a field F, separable of degree 2 over k, in which case we say that Λ is *unsplit Dedekind-like.*

iii.) $\Gamma/\mathfrak{M} \cong k \times k$, in which case we say that Λ is *split Dedekind-like.*

iv.) $\Gamma/\mathfrak{M}$ is a purely inseparable field extension of degree 2 over k.

v.) $\Gamma/\mathfrak{M} = k[\epsilon]$ with $\epsilon \neq 0$ and $\epsilon^2 = 0$.

Definition 6.4 The local ring Λ as in (6.2) is said to be *Dedekind-like* provided i.), ii.) or iii.) of (6.3) holds. The ring R (one-dimensional, Noetherian, reduced, with finite integral closure) is said to be *Dedekind-like* provided $R_\mathfrak{m}$ is Dedekind-like for every maximal ideal $\mathfrak{m}$ of R. The Dedekind-like ring R is said to be *split*, respectively *unsplit*, provided $R_\mathfrak{m}$ is split, respectively unsplit for every singular maximal ideal $\mathfrak{m}$.

The rings satisfying iv.) of (6.3) are considered to be Dedekind-like in [KL1] but not in [KL2]. Since our work here depends heavily on the classification in [KL2] of the indecomposable modules, we will not consider these rings to be Dedekind-like. We should mention also that the global Dedekind-like rings considered in [KL3] and [KL4] are not required to have finite integral closure and may therefore have infinitely many singular maximal ideals.

Over a Dedekind-like ring every finitely generated torsion-free module is isomorphic to a direct sum of ideals, by [Bas63, §7]. Moreover, if the local ring Λ is split Dedekind-like, then every indecomposable finitely generated module has rank at most one (at each minimal prime ideal), [KL4, (6.8), (6.11)]. For unsplit Dedkekind-like local rings, however, there are indecomposable modules of rank 2, and our aim is to compute the delta groups of these modules. Actually, we shall define an explicit subgroup $\mathcal{N}$ of $\Gamma^\times$ (corresponding to the image of the norm map $F^\times \to k^\times$) and show that $\Delta_\Lambda(M) \supseteq \mathcal{N}$ for *every* indecomposable torsion-free Λ-module of rank 2; moreover, we shall construct a *particular* module whose delta group is *exactly* $\mathcal{N}$. These results are stated, in very different language, in [KL4, Theorem 13.5]. We believe that it will be helpful, however, to review some of the construction here, as a warmup for the ramified case.

Indecomposable modules of rank two. Let $(\Lambda, \mathfrak{M}, k)$ be an unsplit local Dedekind-like ring with normalization Γ. Then Γ is a discrete valuation domain with maximal ideal $\mathfrak{M}$ and residue field F. We choose, once and for all, an element $\pi \in \Gamma$ such that $\Gamma\pi = \mathfrak{M}$. (The construction we will outline depends on the choice of this uniformizing parameter.) Recalling that F/k is Galois of degree 2, we let τ be the nontrivial k-automorphism of F. The indecomposable finitely generated Λ-modules are based on various diagrams to be found in §2 of [KL2], but the only diagrams yielding modules of rank greater than one are the diagrams $\mathcal{D}_{\mathrm{Nrd}}$ in [KL2, (2.4)]. A consequence of the classification theorem in [KL2] is that the indecomposable torsion-free Λ-modules have rank 0, 1 or 2. Since the modules of rank 2 are our main concern, it is fortunate that their description is the simplest of the three. Still, the construction is somewhat intricate, and the reader might prefer to jump ahead to the special case where we compute the delta group explicitly.

The indecomposable modules of rank 2 arise as follows:

i.) Choose an integer $d \geq 2$.

ii.) Choose a sequence $(i_1, j_1, \ldots, i_d, j_d)$ of $2d$ symbols, with $i_1 = j_d =$

∞ and with $j_1, i_2, j_2, \ldots, j_{d-1}, i_d$ integers strictly greater than 1. The sequence must *not* be left-right symmetric. (Otherwise the resulting module would decompose.)

The module M is a subquotient of the Γ-module

$$H := \Gamma/\Gamma\pi^{i_1} \oplus \Gamma/\Gamma\pi^{j_1} \oplus \ldots \oplus \Gamma/\Gamma\pi^{i_d} \oplus \Gamma/\Gamma\pi^{j_d}. \tag{10}$$

The first and last summands above are copies of the free module Γ, and the interior summands all have finite length at least 2.

We now describe a Λ-submodule V of H, obtained by "top-gluing" the summand indexed by i_k to that indexed by j_k. (This submodule is denoted by S in [KL2], but we wish to avoid a conflict with our notation for the complement of the union of the singular maximal ideals when we discuss the global situation later.) Given any $q \in \{i_1, \ldots, j_d\}$, let $\nu : \Gamma/\Gamma\pi^q \to F = \Gamma/\Gamma\pi$ be the canonical surjection, and put

$$V := \{(x_1, y_1, \ldots, x_d, y_d) \in H \mid \tau\nu(x_k) = \nu(y_k), k = 1, \ldots, d\}. \tag{11}$$

The rank-two modules are quotients of these modules V, obtained by "bottom-gluing" the socles of the ${j_k}^{\text{th}}$ and ${i_{k+1}}^{\text{st}}$ summands of H, $k = 1, \ldots, d-1$. Note that $\operatorname{Soc}(H) \subseteq V$, since the integers $j_1, i_2, \ldots, j_{d-1}, i_d$ are all greater than 1. For $k = 1, \ldots, d-1$, define a Λ-isomorphism

$$\sigma = \sigma_k : \Gamma\pi^{j_k - 1}/\Gamma\pi^{j_k} \to \Gamma\pi^{i_{k+1}-1}/\Gamma\pi^{i_{k+1}}$$

by

$$r\pi^{j_k-1} + \Gamma\pi^{j_k} \mapsto \overline{r}\pi^{i_{k+1}-1} + \Gamma\pi^{i_{k+1}},$$

where $\overline{r} \in \Gamma$ is chosen in such a way that $\overline{r} + \Gamma\pi = \tau(r + \Gamma\pi)$. (The element $\overline{r}$ is not well-defined, but the map σ is. Essentially, we are identifying the socle $\Gamma\pi^{j_k-1}/\Gamma\pi^{j_k}$ of $\Gamma/\Gamma\pi^{j_k}$ with F via the map $\pi^{j_k-1} + \Gamma\pi^{j_k} \mapsto 1$, making a similar identification of the socle of $\Gamma/\Gamma\pi^{i_{k+1}}$ with F, and then identifying σ with the non-trivial k-automorphism τ of F.)

We put $M := V/K$, where

$$K := \{(x_1, y_1, \ldots, x_d, y_d) \in \operatorname{Soc}(H) \mid \sigma(y_{j_k}) = x_{i_{k+1}}, k = 1, \ldots, d-1\}. \tag{12}$$

A very small part of the classification theorem [KL2, (2.7),(2.8)] states that every module M obtained in this way is indecomposable of rank 2, that every indecomposable module of rank 2 arises in this fashion, and that two modules so obtained are isomorphic if and only if their sequences $(i_1, j_1, \ldots, i_2, j_d)$ either are equal or are mirror images of each other.

Now let N_k^F be the norm map taking $\alpha \in F^\times$ to $\alpha \cdot \tau(\alpha) \in k^\times$. Let $\mathcal{N}_\Lambda = \nu^{-1}(\mathrm{Image}(\mathrm{N}_k^F))$, where $\nu : \Gamma \to F = \Gamma/\mathfrak{M}$ is the natural map. Thus $\mathcal{N}_\Lambda$ is the subgroup of $\Lambda^\times$ consisting of elements of the form $\alpha\overline{\alpha}$, where $\alpha \in \Gamma$ and $\overline{\alpha}$ is any element such that $\nu(\overline{\alpha}) = \tau\nu(\alpha)$.

Given such elements $\alpha, \overline{\alpha}$, and given the module $M = V/K$ constructed above, we let $\phi \in \mathrm{Aut}_\Gamma(H)$ be given by the $2d \times 2d$ diagonal matrix with α and $\overline{\alpha}$ alternating along the diagonal. It is easy to see that $\phi(V) \subseteq V$ and $\phi(K) \subseteq K$. Therefore ϕ induces an automorphism ψ of M. When we pass to Γ and kill the torsion, the middle $2d-2$ coordinates disappear, and we see that $\det(1_\Gamma \otimes_\Lambda \psi) = \alpha\overline{\alpha}$. We have proved the following result:

Proposition 6.5 *Let Λ be a local unsplit Dedekind-like ring, and let M be an indecomposable finitely generated Λ-module of rank 2. Then $\Delta_\Lambda(M) \supseteq \mathcal{N}_\Lambda$.*

We take a brief intermission and relax our assumptions on Λ, in order to record the following general result on local Dedekind-like rings:

Corollary 6.6 *Let Λ be a local Dedekind-like ring, and let M be an arbitrary finitely generated Λ-module.*

i.) If Λ is split, then $\Delta_\Lambda(M) \supseteq \Lambda^\times|M$.

ii.) If Λ is unsplit, then $\Delta_\Lambda(M) \supseteq \mathcal{N}_\Lambda|M$.

Proof. Note that the integer s in Notation 3.4 is either 1 or 2. It follows that, for finitely generated Λ-modules M_1 and M_2, we have $\Lambda^\times|(M_1 \oplus M_2) \subseteq (\Lambda^\times|M_1) \cdot (\Lambda^\times|M_2)$ and $\mathcal{N}_\Lambda|(M_1 \oplus M_2) \subseteq (\mathcal{N}_\Lambda|M_1) \cdot (\mathcal{N}_\Lambda|M_2)$. Therefore, by Proposition 4.8, we may assume that M is indecomposable. Moreover, we may assume that M has positive rank, since otherwise $\Lambda^\times|M = \{1\}$. Now we appeal to [KL4, (6.8) and (6.11)], which tells us the possible ranks of M. In case i.), if Λ is a domain, then M has rank 1. If Λ is not a domain, there are two minimal primes $\mathfrak{p}$ and $\mathfrak{q}$, and $(\dim_{R_\mathfrak{p}}(M_\mathfrak{p}), \dim_{R_\mathfrak{q}}(M_\mathfrak{q}))$ is $(1,1), (1,0)$ or $(0,1)$. In any case, multiplication by an arbitrary element $\varepsilon \in \Lambda^\times|M$ gives a Λ-automorphism of M with determinant ε. In case ii.), Λ is a domain, and the rank of M is either 1 or 2. If M has rank 1, multiplication by an arbitrary element $\varepsilon \in \mathcal{N}_\Lambda|M$ gives a Λ-automorphism of M with determinant ε. Finally, if M has rank 2, we appeal to Proposition 6.5. ∎

Returning to our assumption that Λ is local and unsplit Dedekind-like, we now construct a *specific* Λ-module M such that $\Delta_\Lambda(M) = \mathcal{N}_\Lambda$. This construction, together with Proposition 6.6 will give us necessary and sufficient conditions for cancellation over any Dedekind-like ring. Keeping the notation of (10)–(12), we take $d = 2$, $j_1 = 3$ and $i_2 = 2$. Thus

$$H = \Gamma \oplus \Gamma/\Gamma\pi^3 \oplus \Gamma/\Gamma\pi^2 \oplus \Gamma,$$
$$V = \{(x, y, u, v) \in H \mid \tau\nu(x) = \nu(y) \text{ and } \tau\nu(u) = \nu(v)\},$$
$$K = \{(x, y, u, v) \in H \mid x = v = 0, y \in \Gamma\pi^2/\Gamma\pi^3, u \in \Gamma\pi/\Gamma\pi^2, \sigma(y) = u\}, \text{ and}$$
$$M = V/K.$$

In order to compute $\Delta_\Lambda(M)$ we will first describe $\mathrm{End}_\Lambda(M)$. Put $E := \{\phi \in \mathrm{End}_\Gamma(H) \mid \phi(V) \subseteq V \text{ and } \phi(K) \subseteq K\}$. The following result from [KL2] is crucial here:

Theorem 6.7 *[KL2, (2.11)] The natural homomorphism $E \to \mathrm{End}_\Lambda(M)$ is surjective.*

Thus we want compute E. We begin by noting that

$$\mathrm{End}_\Gamma(H) = \begin{bmatrix} \Gamma & 0 & 0 & \Gamma \\ \Gamma/\Gamma\pi^3 & \Gamma/\Gamma\pi^3 & \Gamma\pi/\Gamma\pi^3 & \Gamma/\Gamma\pi^3 \\ \Gamma/\Gamma\pi^2 & \Gamma/\Gamma\pi^2 & \Gamma/\Gamma\pi^2 & \Gamma/\Gamma\pi^2 \\ \Gamma & 0 & 0 & \Gamma \end{bmatrix} \tag{13}$$

acting on the left. (Thus we should think of elements of H as columns.) Notice in particular that the $2,3$-entry of each element of $\mathrm{End}_\Gamma(H)$ represents a map from $\Gamma/\Gamma\pi^2$ to $\Gamma/\Gamma\pi^3$ and therefore is in $\Gamma\pi/\Gamma\pi^3$.

Let $F = k[\theta]$, and choose $\gamma, \overline{\gamma} \in \Gamma$ such that $\nu(\gamma) = \theta$ and $\nu(\overline{\gamma}) = \tau(\gamma)$. (Recall that τ is the non-trivial k-automorphism of F and that ν denotes the canonical surjection $\Gamma/\Gamma\pi^p \to F$, for $2 \le p \le \infty$.) Note that $\Gamma = \Lambda + \Lambda\gamma$ and that $\mathfrak{M} = \Gamma\pi = \Lambda\pi + \Lambda\gamma\pi$.

Given $p, q \in \{2, 3, \infty\}$ and elements $a \in \Gamma/\Gamma\pi^p$ and $b \in \Gamma/\Gamma\pi^q$, we write $a \equiv b$ provided $\nu(a) = \nu(b)$, and we write $\overline{a} \equiv b$ provided $\tau\nu(a) = \nu(b)$. Thus, for $\alpha = \begin{bmatrix} x & y & u & v \end{bmatrix}^{\mathrm{tr}} \in H$, we have $\alpha \in V \iff \overline{x} \equiv y$ and $\overline{u} \equiv v$. In particular, V contains the following four elements:

$$\xi_1 := \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_2 := \begin{bmatrix} \overline{\gamma} \\ \gamma \\ 0 \\ 0 \end{bmatrix}, \quad \xi_3 := \begin{bmatrix} 0 \\ 0 \\ \gamma \\ \overline{\gamma} \end{bmatrix}, \quad \xi_4 := \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

(Of course the second and third entries of each ξ_ℓ are really cosets modulo $\Gamma\pi^3$ and $\Gamma\pi^2$ respectively.) Now let $\phi = [a_{ij}] \in E$. Let us write down the congruences that are forced by the relations $\phi(\xi_\ell) \in V$, keeping in mind that the a_{ij} are as in (13):

$$\phi\xi_1 \in V \implies \overline{a_{11}} \equiv a_{21} + a_{22} \text{ and } \overline{a_{41}} \equiv a_{31} + a_{32}$$
$$\phi\xi_2 \in V \implies \overline{a_{11}}\gamma \equiv a_{21}\overline{\gamma} + a_{22}\gamma \text{ and } \overline{a_{41}}\gamma \equiv a_{31}\overline{\gamma} + a_{32}\gamma.$$

Since $\gamma \not\equiv \overline{\gamma}$, one easily deduces that

$$\overline{a_{11}} \equiv a_{22}, \quad a_{21} \equiv 0, \quad \overline{a_{41}} \equiv a_{32}, \quad a_{31} \equiv 0.$$

Similarly, using the elements ξ_5 and ξ_6, we get the congruences

$$\overline{a_{44}} \equiv a_{33}, \quad a_{34} \equiv 0, \quad \overline{a_{14}} \equiv a_{23}, \quad a_{24} \equiv 0.$$

Now the relation $a_{23} \equiv 0$ (from (13)) implies that

$$\phi \equiv \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a' & 0 & 0 \\ 0 & c' & b & 0 \\ c & 0 & 0 & b' \end{bmatrix}, \tag{14}$$

with entries in Γ and with $\overline{a} \equiv a'$, $\overline{b} \equiv b'$ and $\overline{c} \equiv c'$.

Next we determine the additional conditions on ϕ imposed by the relation $\phi(K) \subseteq K$. Note that K contains $\alpha := \begin{bmatrix} 0 & \pi^2 & \pi & 0 \end{bmatrix}^{\mathrm{tr}}$ and $\beta := \begin{bmatrix} 0 & \gamma\pi^2 & \overline{\gamma}\pi & 0 \end{bmatrix}^{\mathrm{tr}}$. Since $a_{23} \equiv 0$, we write $a_{23} = h\pi$ with $h \in \Gamma$. Then

$$\phi(\alpha) = \begin{bmatrix} 0 \\ (a_{22} + h)\pi^2 \\ a_{33}\pi \\ 0 \end{bmatrix} \quad \text{and} \quad \phi(\beta) = \begin{bmatrix} 0 \\ (a_{22}\gamma + h\overline{\gamma})\pi^2 \\ a_{33}\overline{\gamma}\pi \\ 0 \end{bmatrix}.$$

Since these elements must be in K, we have $\overline{a_{33}} \equiv a_{22} + h$ and $\overline{a_{33}}\gamma \equiv a_{22}\gamma + h\overline{\gamma}$. Since $\gamma \not\equiv \gamma$, it follows that $h \equiv 0$ and $\overline{a_{33}} \equiv a_{22}$. Therefore we have $a \equiv b$ in (14). As before, when we pass to Γ and kill torsion, the middle two coordinates disappear, and we see that $\det(1_\Gamma \otimes \phi) \equiv a\overline{a}$. These computations show that $\Delta_\Lambda(M) \subseteq \mathcal{N}_\Lambda$. Recalling the reverse inclusion from Proposition 6.5, we have, in summary:

Proposition 6.8 *Let Λ be a local unsplit Dedekind-like ring. The Λ-module M constructed above has $\Delta_\Lambda(M) = \mathcal{N}_\Lambda$.*

Notation 6.9 Now let R be an arbitrary Dedekind-like ring, with singular semilocalization $S^{-1}R$. Assume $R \neq \overline{R}$, that is, $S^{-1}R$ is a non-zero ring. Let $\mathfrak{m}$ be a singular maximal ideal of R. If $R_\mathfrak{m}$ is unsplit, let $\mathcal{H}_\mathfrak{m}$ be the subgroup (denoted by $\mathcal{N}_{R_\mathfrak{m}}$ above) of $(R_\mathfrak{m})^\times$ consisting of elements whose residues modulo $\mathfrak{m}R_\mathfrak{m}$ are in the image of the norm map $(\overline{R}_\mathfrak{m}/\mathfrak{m}\overline{R}_\mathfrak{m})^\times \to (R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m})^\times$. If $R_\mathfrak{m}$ is split, let $\mathcal{H}_\mathfrak{m} = R_\mathfrak{m}^\times$. Finally, let $\mathcal{H}_R$ be the inverse image of $\prod_\mathfrak{m} \mathcal{H}_\mathfrak{m}$ under the natural map $(S^{-1}R)^\times \to \prod_\mathfrak{m}(R_\mathfrak{m})^\times$ ($\mathfrak{m}$ ranging over the singular maximal ideals). If there are no singular maximal ideals, that is, $R = \overline{R}$, we put $\mathcal{H}_R = \Omega(R)$.

Now we can state our main results on cancellation over Dedekind-like rings:

Theorem 6.10 *Let R be a Dedekind-like ring. Then R has cancellation if and only if $\mathcal{H}_R \cdot (\mathrm{Image}(\overline{R}^\times \to (S^{-1}\overline{R})^\times)) = \Omega(R)$.*

Proof. If $R = \overline{R}$, then R has cancellation and $\mathcal{N}_R = \Omega(R)$. Assume from now on that $R \neq \overline{R}$.

Let M be an arbitrary finitely generated R-module, and let $\delta \in \mathcal{H}_R$. For each singular maximal ideal $\mathfrak{m}$, commutativity of (9) shows that the map $(S^{-1}\overline{R})^\times \longrightarrow (\overline{R}_\mathfrak{m})^\times$ carries $\delta|M$ to an element of $\mathcal{H}_\mathfrak{m}|M_\mathfrak{m}$. Since, by Corollary 6.6, $\mathcal{H}_\mathfrak{m}|M_\mathfrak{m} \subseteq \Delta_{R_\mathfrak{m}}(M_\mathfrak{m})$, Corollary 4.7 implies that $\delta|M \in \Delta_R(M)$. Thus $\mathcal{H}_R|M \subseteq \Delta_R(M)$.

Suppose, now, that $\mathcal{H}_R \cdot (\mathrm{Image}(\overline{R}^\times \to (S^{-1}\overline{R})^\times)) = \Omega(R)$. Given any $\varepsilon \in \Omega(R)$, write $\varepsilon = \delta\gamma$, with $\delta \in \mathcal{H}_R$ and $\gamma \in \mathrm{Image}(\overline{R}^\times \to (S^{-1}\overline{R})^\times)$. Then $M^\varepsilon \cong M^\delta = M^{\delta|M} \cong M$. By Corollary 4.3, R has cancellation.

Conversely, suppose R satisfies cancellation. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be the singular maximal ideals. For each $i = 1, \ldots, t$, put $M_i = R_{\mathfrak{m}_i} \oplus R_{\mathfrak{m}_i}$ if $R_{\mathfrak{m}_i}$ is split Dedekind-like, and let M_i be the $R_{\mathfrak{m}_i}$-module of Proposition 6.8 if $R_{\mathfrak{m}_i}$ is unsplit Dedekind-like. Since all of these modules have constant rank 2, they can be glued [Wie89, (1.11)] to obtain an $S^{-1}R$-module whose localization at $\mathfrak{m}_i$ is isomorphic to M_i for each i. Therefore there is a finitely generated R-module M such that $M_{\mathfrak{m}_i} \cong M_i$ for each $i = 1, \ldots t$. Referring to the notation of (6.9) and noting that $\Delta_{R_{\mathfrak{m}_i}}(M_i) = \mathcal{H}_{\mathfrak{m}_i}$ for each i, we see from Corollary 4.7 that $\Delta_R(M) = \mathcal{H}_R$. Since M is faithful, $\Omega(R)|M = \Omega(R)$, and now ii.) of Corollary 4.4 gives us the equality we need. ∎

Corollary 6.11 *Let R be a Dedekind-like ring with torsion-free cancellation. Assume $R/\mathfrak{m}$ is a finite field for every maximal ideal $\mathfrak{m}$ for which $R_\mathfrak{m}$ is unsplit. Then R has cancellation. In particular, for a Dedekind-like order in an algebraic number field, torsion-free cancellation implies cancellation.*

PROOF. Let $\varepsilon \in \Omega(R) = (S^{-1}\overline{R})^\times$. Then $R^\varepsilon \oplus \overline{R} \cong R \oplus \overline{R}$, by Corollary 4.3. Since R has torsion-free cancellation, $R \cong R^\varepsilon$. By Proposition 4.2, $\varepsilon \in \Delta_R(R) \cdot \mathrm{Image}(\overline{R}^\times \to (S^{-1}\overline{R})^\times) = (S^{-1}R)^\times \cdot \mathrm{Image}(\overline{R}^\times \to (S^{-1}\overline{R})^\times)$. Since the norm map is surjective for finite fields, we have $\mathcal{H}_R = (S^{-1}R)^\times$, and an appeal to Theorem 6.10 completes the proof. ■

Imaginary quadratic orders. Theorem 4.5 of [Wie84] gives a complete list of the negative quadratic orders having torsion-free cancellation. Aside from the Dedekind domains, these rings are $\mathbb{Z}[2\sqrt{-1}]$, $\mathbb{Z}[\frac{3}{2}(1+\sqrt{-3})]$, $\mathbb{Z}[\sqrt{-3}]$ and $\mathbb{Z}[\sqrt{d}]$, where d is a square-free negative integer with $d \equiv 1 \pmod 8$. The first two are not Dedekind-like, but $\mathbb{Z}[\sqrt{-3}]$ is unsplit Dedekind-like, and the rings $\mathbb{Z}[\sqrt{d}]$ are split Dedekind-like. Thus we have an infinite family of non-maximal orders having cancellation.

For more general Dedekind-like rings, torsion-free cancellation may not imply cancellation, and here we are finally able to answer the question that motivated this investigation.

Example 6.12 *There exists a Dedekind-like domain having torsion-free cancellation but not cancellation.*

PROOF. Let τ and X be algebraically independent over the field $\mathbb{Q}$ of rationals. The inclusion $\mathbb{Z}[\tau^4] \to \mathbb{Z}[\tau]$ and the specialization $X \mapsto \tau$ give a surjective ring homomorphism $\phi : \mathbb{Z}[\tau^4, X] \to \mathbb{Z}[\tau]$. Let $\mathcal{P}$ be the set of prime elements of $\mathbb{Z}[\tau]$, and let $\mathcal{S} = \mathcal{P} - (\mathcal{P} \cap \mathbb{Z}[\tau^2])$.

Clearly every prime element of $\mathbb{Z}[\tau]$ is the image, via ϕ, of some prime element of $\mathbb{Z}[\tau^4, X]$. Let T_0 be a set of prime elements of $\mathbb{Z}[\tau^4, X]$ such that $\phi(T_0) = \mathcal{S}$, and let T be the multiplicative subset of $\mathbb{Z}[\tau^4, X]$ generated by T_0 and $\mathbb{Z}[\tau^4] - \{0\}$. Put $\overline{R} := T^{-1}\mathbb{Z}[\tau^4, X] = T_0^{-1}\mathbb{Q}(\tau^4)[X]$, and let $\psi : \overline{R} \to \mathbb{Q}(\tau)$ be the homomorphism induced by ϕ. Clearly ψ is surjective, and we let R be the Dedekind-like ring defined by the following pullback diagram:

$$\begin{array}{ccc} R & \longrightarrow & \overline{R} \\ \downarrow & & \downarrow \psi \\ \mathbb{Q}(\tau^2) & \hookrightarrow & \mathbb{Q}(\tau) \end{array}$$

Then R has a unique singular maximal ideal $\mathfrak{m} := \mathrm{Ker}(\psi)$, and $R_{\mathfrak{m}}$ is a local unsplit Dedekind-like ring. We will show that R has torsion-free cancellation but not cancellation.

An easy computation shows that $\overline{R}^{\times} = \mathbb{Q}(\tau^4)^{\times} \cdot \langle T_0 \rangle$. (We denote by $\langle E \rangle$ the subgroup generated by a subset E.) Therefore $\mathrm{Image}(\overline{R}^{\times} \to \mathbb{Q}(\tau)^{\times}) = \mathbb{Q}(\tau^4)^{\times} \cdot \langle \mathcal{S} \rangle$. The subgroup $\mathbb{Q}(\tau^2)^{\times}$ of $\mathbb{Q}(\tau)^{\times}$ contains $\mathcal{P} - \mathcal{S}$, so clearly $\mathbb{Q}(\tau^2)^{\times} \cdot \mathrm{Image}(\overline{R}^{\times} \to \mathbb{Q}(\tau)^{\times}) = \mathbb{Q}(\tau)^{\times}$. This means that $\mathrm{D}(R) := \mathrm{Ker}(\mathrm{Pic}(R) \to \mathrm{Pic}(\overline{R}))$ is trivial (see, e.g., the discussion preceding Corollary 2.4 in [Wie84]). By [Wie84, Theorem 2.7] R has torsion-free cancellation.

To show that cancellation fails for R, we invoke Theorem 6.10. If the criterion of 6.10 were true, then, after going modulo $\mathfrak{m}$, we would have $N \cdot \mathrm{Image}(\overline{R}^{\times} \to \mathbb{Q}(\tau)^{\times}) = \mathbb{Q}(\tau)^{\times}$, where N is the image of the norm map $\mathrm{N} = \mathrm{N}^{\mathbb{Q}(\tau)}_{\mathbb{Q}(\tau^2)} : \mathbb{Q}(\tau)^{\times} \to \mathbb{Q}(\tau^2)^{\times}$. From the formula for $\mathrm{Image}(\overline{R}^{\times} \to \mathbb{Q}(\tau)^{\times})$ in the preceding paragraph, we see that it will suffice to prove that $N \cdot \mathbb{Q}(\tau^4)^{\times} \cdot \langle \mathcal{S} \rangle$ is a proper subgroup of $\mathbb{Q}(\tau)^{\times}$.

Consider the prime elements $\pi_1 = 2 + \tau^2$ and $\pi_2 = 2 - \tau^2$ of $\mathbb{Z}[\tau]$, and let v_i be the π_i-adic valuation of $\mathbb{Q}(\tau)$, $i = 1, 2$. We claim that $v_1(z) + v_2(z)$ is even, for every $z \in N \cdot \mathbb{Q}(\tau^4)^{\times} \cdot \langle \mathcal{S} \rangle$. Now N is generated by the elements $\{\mathrm{N}(\pi) \mid \pi \in \mathcal{P}\}$. If $\pi \in \mathcal{P}$ then $\mathrm{N}(\pi) = \pi\overline{\pi}$, where $\overline{\pi}$ denotes the conjugate of π under the non-trivial $\mathbb{Q}(\tau^2)$-automorphism $\tau \mapsto -\tau$ of $\mathbb{Q}(\tau)$. Since $\overline{\pi_i} = \pi_i$ for $i = 1, 2$ we see that both $v_1(\mathrm{N}(\pi))$ and $v_2(\mathrm{N}(\pi))$ are even (either 0 or 2). Thus the claim holds for $z \in N$. Note that, as elements of $\mathbb{Q}(\tau^2)$, π_1 and π_2 are conjugates over $\mathbb{Q}(\tau^4)$. It follows that $v_1(z) = v_2(z)$ for each $z \in \mathbb{Q}(\tau^4)^{\times}$. Finally, if $z \in \mathcal{S}$, then $v_1(z) = v_2(z) = 0$, and the claim follows.

To complete the proof, we note that $v_1(\pi_1) + v_2(\pi_1) = 1$. Thus $\pi_1 \in \mathbb{Q}(\tau)^{\times} - N \cdot \mathbb{Q}(\tau^4)^{\times} \cdot \langle \mathcal{S} \rangle$. ■

Ramified rings. We now suppose $(\Lambda, \mathfrak{M}, k)$ satisfies v.) of Remark 6.3. Thus Λ is *not* Dedekind-like. One can get examples of such rings by localizing, e.g., $\mathbb{Z}[2\sqrt{-1}]$ or $k[T^2, T^3]$ at the unique singular maximal ideal. We shall construct a Λ-module reminiscent of that of Proposition 6.8 and use this module to show that the order $\mathbb{Z}[17\frac{1+\sqrt{17}}{2}]$ does not have cancellation, even though it does have torsion-free cancellation. (Recall (6.11) that this cannot happen for Dedekind-like orders.)

Notation 6.13 We keep all of the notation of (6.2), and in addition we put $A := \Gamma/\mathfrak{M}$. We assume that $A = k[\epsilon]$, with $\epsilon \neq 0$ but $\epsilon^2 = 0$. Let $\rho : \Gamma \to A$ be the natural surjection. (Note that ρ restricts to the natural map from Λ to its residue field $k = \Lambda/\mathfrak{M}$.) Choose a generator π of the maximal ideal of Γ such that $\rho(\pi) = \epsilon$. Then $\mathfrak{M} = \Gamma\pi^2$. Given any $q \geq 2$, denote the map $\Gamma/\Gamma\pi^q \twoheadrightarrow A$ induced by ρ by the same symbol ρ. Let τ be the k-automorphism of A that interchanges 1 and ϵ, that is,

$$\tau(a + b\epsilon) = b + a\epsilon, \text{ for } a, b \in k. \tag{15}$$

The map τ is Λ-linear but not Γ-linear. If p and q are extended integers, $2 \leq p, q \leq \infty$, and $(x, y) \in \Gamma/\Gamma\pi^p \oplus \Gamma/\Gamma\pi^q$ we write $x \equiv_2 y$ to mean that $\rho(x) = \rho(y)$. Also, we write $\bar{x} \equiv_2 y$ provided $\tau\rho(x) = \rho(y)$.

We now proceed to define H, V and K much as before, but we allow a little more room to work. We put

$$H := \Gamma \oplus \Gamma/\Gamma\pi^6 \oplus \Gamma/\Gamma\pi^7 \oplus \Gamma \tag{16}$$

and define V to be the following Λ-submodule of H:

$$V := \{\begin{bmatrix} x & y & u & v \end{bmatrix}^{\mathrm{tr}} \in H \mid \bar{x} \equiv_2 y \text{ and } \bar{u} \equiv_2 v\}. \tag{17}$$

We have "top-glued" the first two components of H and top-glued the last two components of H.

For the bottom-gluing, assume $2 \leq q < \infty$. We define the "bottom" of $\Gamma/\Gamma\pi^q$ to be the bottom *two* layers:

$$\mathrm{Bot}(\Gamma/(\pi^q)) := (\pi^{q-2})/(\pi^q).$$

We identify the bottom of $\Gamma/\Gamma\pi^q$ with A by sending π^{q-2} to 1 and then let $\sigma : \mathrm{Bot}(\Gamma/(\pi^6)) \to \mathrm{Bot}(\Gamma/(\pi^7))$ be the isomorphism corresponding to the automorphism τ in (15). Thus we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Bot}(\Gamma/\Gamma\pi^6) & \xrightarrow{\sigma} & \mathrm{Bot}(\Gamma/\Gamma\pi^7) \\ \downarrow\cong & & \downarrow\cong \\ A & \xrightarrow{\tau} & A \end{array} \tag{18}$$

in which the vertical arrows are defined by $\pi^4 \mapsto 1$ and $\pi^5 \mapsto 1$.

We let K be the set of columns in (17) with $x = v = 0$, with y and u in the bottoms, and with $\sigma(y) = u$. A typical element $\alpha \in K$ then looks like this:

$$\alpha = \begin{bmatrix} 0 & x\pi^4 + \Gamma\pi^6 & y\pi^5 + \Gamma\pi^7 & 0 \end{bmatrix}^{\mathrm{tr}}, \text{ with } x, y \in \Gamma \text{ and } \bar{x} \equiv_2 y. \tag{19}$$

We put $M := V/K$. We want to describe the endomorphism ring of M. To do this, we must prove a result analogous to Theorem 6.7, saying that every Λ-endomorphism of M actually comes from a Γ-endomorphism of H. The first step is to verify that $V \twoheadrightarrow M$ is a *separated cover* of M (cf. [KL2, §4]). This means (1) V is a *separated* Λ-module, that is, V is a Λ-submodule of some finitely generated Γ-module, and (2) if $V \twoheadrightarrow M$ factors as $V \twoheadrightarrow V' \twoheadrightarrow M$, then $V \twoheadrightarrow V'$ is an isomorphism.

Lemma 6.14 *The natural homomorphism $V \twoheadrightarrow M$ is a separated cover of M.*

PROOF. We will use [KL2, Lemma 4.9]. Certainly V is a separated module, since it is a Λ-submodule of some Γ-module, namely H. Next we check that the kernel K contains no non-zero Γ-submodules of H. That is, if $0 \neq \alpha \in K$, we want to show that $\Gamma\alpha \not\subseteq K$. Write α as in (19), and write $\rho(x) = a + b\epsilon$, with $a, b \in k$. Then $\rho(y) = b + a\epsilon$. On the other hand, $\pi\alpha = \begin{bmatrix}0 & \pi x\pi^4 + \Gamma\pi^6 & \pi y\pi^5 + \Gamma\pi^7 & 0\end{bmatrix}^{\mathrm{tr}}$. We have $\rho(\pi x) = a\epsilon$ and $\rho(\pi y) = b\epsilon$. In order that $\pi\alpha \in K$, we must have $a = b = 0$. But then $\alpha = 0$, as desired. We have verified condition (ii)(a) of [KL2, (4.9)].

We still have to verify condition (ii)(b) of [KL2, (4.9)], that is, $K \subseteq \mathfrak{M}V$. But this is trivial, since, for a typical element $\alpha \in K$ (see (19)) we can write $\alpha = \pi^2 \begin{bmatrix}0 & x\pi^2 + \Gamma\pi^6 & y\pi^3 + \Gamma\pi^7 & 0\end{bmatrix}^{\mathrm{tr}} \in \mathfrak{M}V$. ■

From now on, for notational simplicity, we just list coset representatives, e.g., $x\pi^2$, rather than cosets, e.g., $x\pi^2 + \Gamma\pi^6$, in the middle two coordinates of H.

Lemma 6.15 $\Gamma V = H$.

PROOF. Note that V contains the elements

$$\beta := \begin{bmatrix}1 & \pi & 0 & 0\end{bmatrix}^{\mathrm{tr}} \text{ and } \gamma := \begin{bmatrix}\pi & 1 & 0 & 0\end{bmatrix}^{\mathrm{tr}}.$$

Therefore

$$\begin{aligned}\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^{\mathrm{tr}} &= (1 + \pi^2 + \pi^4 + \pi^6)\beta - (\pi + \pi^3 + \pi^5)\gamma \in \Gamma S \quad \text{and} \\ \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}^{\mathrm{tr}} &= (1 + \pi^2 + \pi^4)\gamma - (\pi + \pi^3 + \pi^5)\beta \in \Gamma S.\end{aligned}$$

Similar machinations show that the other two obvious generators of H are in ΓV. ■

Lemma 6.16 *The natural surjection $\Gamma \otimes_\Lambda V \twoheadrightarrow H$ is an isomorphism.*

Proof. Clearly $H/\mathfrak{m}H$ is a free A-module of rank 4. Therefore, by [KL2, (5.2)], it is enough to show that $V/\mathfrak{m}V$ is 4-dimensional. We focus on the first two components, letting $H_0, V_0, \beta_0, \gamma_0$ be the projections of V, H and the elements β, γ in the proof of Lemma 6.15 on $\Gamma \oplus \Gamma/(\pi^6)$. We will show that V_0 is generated by β_0 and γ_0. Given $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in V_0$, write $\rho(x) = a1 + b\epsilon$, with $a, b \in \Lambda$. (Here $1 \in k$.) Then $\rho(y) = b1 + a\epsilon$, whence $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] - a\beta_0 - b\gamma_0 \in \pi^2 H$. Therefore it will suffice to show that $\left[\begin{smallmatrix} \pi^2 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ \pi^2 \end{smallmatrix}\right]$ are Λ-linear combinations of β_0 and γ_0. Here are the expressions we need:

$$\left[\begin{smallmatrix} \pi^2 \\ 0 \end{smallmatrix}\right] = (\pi^2 + \pi^4 + \pi^6)\beta_0 - (\pi^3 + \pi^5)\gamma_0$$
$$\left[\begin{smallmatrix} 0 \\ \pi^2 \end{smallmatrix}\right] = (\pi^2 + \pi^4)\gamma_0 - (\pi^3 + \pi^5)\beta_0$$

A similar computation with the third and fourth components shows that V can be generated by 4 elements. Obviously no fewer will suffice, by Lemma 6.15. By Nakayama's lemma, $V/\mathfrak{m}V$ has dimension 4. ∎

Combining Lemmas 6.14 – 6.16 with [KL2, (4.5), (4.12)], we get the following:

Theorem 6.17 *Put $E := \{\phi \in \mathrm{End}_\Gamma(H) \mid \phi(V) \subseteq V$ and $\phi(K) \subseteq K\}$. The natural homomorphism $E \to \mathrm{End}_\Lambda(M)$ is surjective.*

Now let $\phi \in E$. Our aim is to determine certain congruences that are forced upon the entries of ϕ. We will leave out some of the gory details, as they are rather straightforward. We note that

$$\mathrm{End}(H) = \begin{bmatrix} \Gamma & 0 & 0 & \Gamma \\ \Gamma/\Gamma\pi^6 & \Gamma/\Gamma\pi^6 & \Gamma/\Gamma\pi^6 & \Gamma/\Gamma\pi^6 \\ \Gamma/\Gamma\pi^7 & \Gamma\pi/\Gamma\pi^7 & \Gamma/\Gamma\pi^7 & \Gamma/\Gamma\pi^7 \\ \Gamma & 0 & 0 & \Gamma \end{bmatrix}. \qquad (20)$$

Since everything we will do depends only on congruence classes modulo $\Gamma\pi^2$, we choose elements $a_{ij}, b_{ij} \in \Lambda$ such that $\phi \equiv_2 [a_{ij} + b_{ij}\pi]$, carefully choosing $a_{ij} = b_{ij} = 0$ for $(i,j) \in \{(1,2),(1,3),(4,2),(4,3)\}$. Moreover, we have (by necessity) $a_{32} = 0$. We now apply ϕ to the elements β and γ in the proof of Lemma 6.15, as well as to their mirror images. The requirement that the resulting four elements must be in V gives sixteen congruences (modulo

$\Gamma\pi^2$). A straightforward analysis of these congruences shows that ϕ must have the following form:

$$\phi \equiv_2 \begin{bmatrix} a+b\pi & 0 & 0 & d+e\pi \\ b & a-b\pi & d-e\pi & e \\ c & -c\pi & f+g\pi & -g \\ c\pi & 0 & 0 & f-g\pi \end{bmatrix}, \text{ with } a,b,c,d,e,f,g \in \Lambda. \quad (21)$$

Next we examine what additional conditions are imposed by the requirement that $\phi(K) \subseteq K$. To do this, we apply ϕ to the elements $\xi := \begin{bmatrix} 0 & \pi^4 & \pi^6 & 0 \end{bmatrix}^{\mathrm{tr}}, \eta := \begin{bmatrix} 0 & \pi^5 & \pi^5 & 0 \end{bmatrix}^{\mathrm{tr}} \in K$. We compute $\phi(\xi)$ and $\phi(\eta)$, using the form of ϕ given by (21). (Note that if $c \in \Gamma$ and $x \in \mathrm{Bot}(\Gamma/(\pi^q))$ for $2 \le q < \infty$, then cx depends only on the congruence class of c modulo $\Gamma\pi^2$, since $\mathrm{Bot}(\Gamma/\Gamma\pi^q$ is an A-module.) We have

$$\phi(\xi) = \begin{bmatrix} 0 \\ (a-b\pi)\pi^4 \\ (-c+f\pi)\pi^5 \\ 0 \end{bmatrix} \text{ and } \phi(\eta) = \begin{bmatrix} 0 \\ ((a+d)\pi)\pi^4 \\ (f+(g-c)\pi)\pi^5 \\ 0 \end{bmatrix}.$$

The requirement that these two elements be in K amounts to the following conditions:

$$a \equiv_2 f, \quad b \equiv_2 c, \quad a+d \equiv_2 f, \quad g-c \equiv_2 0. \quad (22)$$

Combining these observations with Theorem 6.17, we obtain the following:

Theorem 6.18 *Let Λ be as in Notation 6.13, and let M be the Λ-module constructed above. Let $\theta \in \mathrm{End}(M)$.*

i.) There is a Γ-endomorphism ϕ of H such that $\phi(V) \subseteq V$, $\phi(K) \subseteq K$, and θ is the endomorphism of $M := V/K$ induced by ϕ.

ii.) There are elements $a, b, e \in \Lambda$ such that

$$\phi \equiv_2 \begin{bmatrix} a+b\pi & 0 & 0 & e\pi \\ b & a-b\pi & -e\pi & e \\ b & -b\pi & a+b\pi & -b \\ b\pi & 0 & 0 & a-b\pi \end{bmatrix}.$$

Finally, we can identify the delta subgroup of our module M.

Corollary 6.19 *Let Λ and M be as above, and let $\mathcal{D}$ denote the subgroup of $\Lambda^\times$ consisting of elements whose residues modulo $\mathfrak{M}$ are squares in $k^\times$. Then $\Delta_\Lambda(M) = \mathcal{D}$.*

Proof. Recall that $\mathfrak{M} = \Gamma\pi^2$. Let L be the quotient field of Λ. Given $d \in \mathcal{D}$, write $d = a^2 + z$, where $a \in \Lambda^\times$ and $z \in \mathfrak{M}$. Put $b := a + a^{-1}z$, and let ϕ be the automorphism of H given by the diagonal matrix with diagonal (a, b, a, b). Since $a \equiv_2 b$, one can check, using (17) and (19), that $\phi(V) \subseteq V$ and $\phi(K) \subseteq K$. Therefore ϕ induces a Λ-automorphism θ of M. Clearly $\det(1_L \otimes \theta) = ab = d$, and it follows that $d \in \Delta_\Lambda(M)$.

Conversely, let $d \in \Delta_\Lambda(M)$, and choose $\theta \in \mathrm{Aut}_\Lambda(M)$ with $\det(1_L \otimes \theta) = d$. Choose ϕ as in i.) of Theorem 6.18. Write $\phi = [c_{ij}]$ as in (20).

The images of the elements $\mu := \begin{bmatrix}1 & \pi & 0 & 0\end{bmatrix}^{\mathrm{tr}}$ and $\nu := \begin{bmatrix}0\\0\\\pi\\1\end{bmatrix}$ form an L-basis for $L \otimes_\Lambda M$. The matrix for the automorphism $1_L \otimes \theta$ relative to the basis $\{\mu, \eta\}$ is easily seen to be $\mu := \left[\begin{smallmatrix}c_{11} & c_{14}\\ c_{41} & c_{44}\end{smallmatrix}\right]$. By ii.) of Theorem 6.18, μ is congruent, modulo (π^2), to a matrix of the form $\left[\begin{smallmatrix}a+b\pi & e\pi\\ b\pi & a-b\pi\end{smallmatrix}\right]$, with $a, b, e \in \Lambda$. Therefore $d = \det(\mu) \equiv_2 a^2$, whence $d \in \Lambda$. Since θ is an automorphism, so is $1_\Gamma \otimes \theta$, and it follows that $d \in \Lambda^\times$. Therefore $d \in \mathcal{D}$, as desired. ∎

One can show, using Theorem 6.18, that $E/\operatorname{rad}(E) \cong k$, where E is the ring of Theorem 6.17. It follows from Theorem 6.17 that the module M constructed above has a local (in the non-commutative sense) endomorphism ring and hence is indecomposable. The authors have shown recently, using very different methods, that, in contrast to Dedekind-like rings, the ring Λ (as in Notation 6.13) has, for every positive integer n, an indecomposable module of rank n.

We conclude the paper by giving an example of a quadratic order with torsion-free cancellation but not cancellation.

Example 6.20 *The quadratic order $R := \mathbb{Z}[17\frac{1+\sqrt{17}}{2}]$ has torsion-free cancellation but not cancellation.*

Proof. Put $\omega = \frac{1+\sqrt{17}}{2}$, so that $\overline{R} = \mathbb{Z}[\omega]$. The fundamental unit of $\overline{R}$ is $3 + 2\omega$, and (ii) of [Wie84, Theorem 4.9] implies that R has torsion-free cancellation. The conductor $(R : \overline{R})$ is $\mathfrak{m} := 17\overline{R}$. Put $\Lambda = R_\mathfrak{m}$ and $\Gamma = \overline{R}_\mathfrak{m}$.

We have $k := R/\mathfrak{m} = \mathbb{F}_{17}$ and $\overline{R}/\mathfrak{m} = \mathbb{F}_{17}[\epsilon]$, where ϵ is the image of $\sqrt{17}$. Let N be a finitely generated R-module such that $N_{\mathfrak{m}}$ is isomorphic to the Λ-module M of Corollary 6.19. Choose any $\varepsilon \in \Lambda^{\times}$ such that the image c of ε in k is not a square. To show failure of cancellation, it will suffice, by Proposition 4.2 and Corollary 4.3, to show that $\varepsilon \notin \Delta_{\Lambda}(M) \cdot \text{Image}(\overline{R}^{\times} \to \Gamma^{\times})$. By Corollary 6.19, it is enough to show that c cannot be expressed in the form a^2b, where $a \in k^{\times}$ and $b \in \mathbb{F}_{17}[\epsilon]$ lifts to a unit of $\overline{R}$. The fundamental unit $3 + 2\omega$ maps to $d := 4(1 - 4\epsilon)$. If, now, $c = a^2b$ as above, write $b = \pm d^r$ for some integer r. Replacing a by $a\sqrt{-1}$ if necessary, we can assume that $b = d^r$. Now we have $c = a^2 \cdot 4^r \cdot (1 - 4r\epsilon) = (2^r \cdot a)^2 \cdot (1 - 4r\epsilon)$. Since $c \in k$, $r \equiv 0 \pmod{17}$, whence c is a square, a contradiction. ■

References

[Bas63] H. Bass. On the ubiquity of Gorenstein rings. *Math. Z.*, 82:8–28, 1963.

[Bas68] H. Bass. *Algebraic K-Theory.* Benjamin, New York, 1968.

[Eva73] E. G. Evans, Jr. Krull-Schmidt and cancellation over local rings. *Pacific J. Math.*, 46:115–121, 1973.

[Fac98] A. Facchini. *Module theory. Endomorphism rings and direct sum decompositions in some classes of modules.* Progress in Math. 167, Birkhäuser Verlag, 1998.

[Goo76] K. R. Goodearl. Power cancellation of groups and modules. *Pacific J. Math.*, 64:387–411, 1976.

[Gur86] R. M. Guralnick. Power cancellation of modules. *Pacific J. Math.*, 124:131–144, 1986.

[Gur87] R. M. Guralnick. The genus of a module II. Roiter's theorem, power cancellation and extension of scalars. *J. Number Theory*, 26:149–165, 1987.

[GL91] R. M. Guralnick and L. S. Levy. Cancellation and direct summands in dimension 1. *J. Algebra*, 142:310–347, 1991.

[KL1] L. Klingler and L. S. Levy. Representation type of commutative Noetherian rings I: Local wildness. *Pacific J. Math.*, 200:345–386, 2001.

[KL2] L. Klingler and L. S. Levy. Representation type of commutative Noetherian rings II: Local tameness. *Pacific J. Math.*, 200:387–483, 2001.

[KL3] L. Klingler and L. S. Levy. Representation type of commutative Noetherian rings III: Global wildness. Preprint, 2003.

[KL4] L. Klingler and L. S. Levy. Representation type of commutative Noetherian rings IV: Global tameness. Preprint, 2003.

[Lev85] L. S. Levy. Modules over Dedekind-like rings. *J. Algebra*, 93:1–116, 1985.

[LW85] L. S. Levy and R. Wiegand. Dedekind-like behavior of rings with 2-generated ideals. *J. Pure Appl. Algebra*, 37:41–58, 1985.

[Wie84] R. Wiegand. Cancellation over commutative rings of dimension one and two. *J. Algebra*, 88:438–459, 1984.

[Wie89] R. Wiegand. Noetherian rings of bounded representation type. *Commutative Algebra, Proceedings of a Microprogram held June 15 – July 2, 1987.* Springer, New York, 1989.

[Wie89′] R. Wiegand. Picard groups of singular affine curves over a perfect field. *Math. Z.*, 200:301–311, 1989.

[WW87] R. Wiegand and S. Wiegand. Stable isomorphism of modules over one-dimensional rings. *J. Algebra*, 107:425–435, 1987.