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Large indecomposable modules over local rings ☆

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Abstract

For commutative, Noetherian, local ring R of dimension one, we show that, if R is not a homomorphic image of a Dedekind-like ring, then R has indecomposable finitely generated modules that are free of arbitrary rank at each minimal prime. For Cohen–Macaulay ring R, this theorem was proved in [W. Hassler, R. Karr, L. Klingler, R. Wiegand, Indecomposable modules of large rank over Cohen–Macaulay local rings, Trans. Amer. Math. Soc., in press]; in this paper we handle the general case.

Keywords: Indecomposable module; Torsion-free rank; Dedekind-like ring

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring. If R is not a principal ideal ring, there are indecomposable finitely generated modules requiring arbitrarily many generators (cf. [19, Theorem 2] or Proposition 2.1). Moreover, if R is a domain of dimension at least 2, there are indecomposable *torsion-free* R modules of arbitrarily large rank [2, Proposition 1.2]. On the other

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hand, there are one-dimensional rings, e.g., the curve singularities of finite Cohen–Macaulay type [8], for which there is a bound on the ranks of the indecomposable torsion-free modules. The main goal of this paper is to show that in *almost all* cases one can find indecomposable modules that are not necessarily torsion-free but still have arbitrarily large torsion-free rank. In particular, the rings $\mathbb{Z}_{(2)}[\sqrt{2}\,]$ and $k[\![X,Y]\!]/(Y^2-X^3)$, which have finite Cohen–Macaulay type, have indecomposables of rank n for every $n\in\mathbb{N}$. Obviously we cannot build big indecomposables if R is a discrete valuation domain. More generally, if R is a *Dedekind-like ring* (cf. Definition 1.1 below), e.g., $\mathbb{Z}_{(2)}[2\sqrt{3}]$ or $\mathbb{R}[\![X,Y]\!]/(X^2+Y^2)$, then by [13] the torsion-free rank of every indecomposable finitely generated R-module is at most 2. It turns out that Dedekind-like rings and their homomorphic images are the *only* rings for which our construction cannot be carried out.

Our Main Theorem provides an indecomposable module which is free of specified rank at each prime P in a given finite set $\mathcal{P}\subseteq \operatorname{Spec}(R)-\{\mathfrak{m}\}$. In dimension greater than one we have to allow for the fact that if $M_P\cong R_P^{(n)}$ and Q is a prime ideal contained in P, then $M_Q\cong R_Q^{(n)}$. For $P_1,P_2\in\mathcal{P}$ we write $P_1\sim P_2$ if $P_1\cap P_2$ contains a prime ideal of R (not necessarily in $\mathcal{P}$). (Of course "$\sim$" is not necessarily transitive.)

Definition 1.1. The commutative, Noetherian local ring $(R,\mathfrak{m},k)$ is *Dedekind-like* [12, Definition 2.5] provided R is one-dimensional and reduced, the integral closure $\overline{R}$ of R in the total quotient ring of R is generated by at most 2 elements as an R-module, and $\mathfrak{m}$ is the Jacobson radical of $\overline{R}$. We call $(R,\mathfrak{m},k)$ an *exceptional Dedekind-like ring* provided, in addition, $\overline{R}/\mathfrak{m}$ is a purely inseparable field extension of k of degree 2.

We note that there is also a notion of global Dedekind-like rings [14, Definition 10.1]. In this article "Dedekind-like" always means Dedekind-like and local.

Theorem 1.2 *(Main Theorem). Let $(R,\mathfrak{m},k)$ be a commutative, Noetherian local ring.*

(1) *Suppose R is not a homomorphic image of a Dedekind-like ring. Let $\mathcal{P}$ be a finite set of non-maximal prime ideals of R, and let n_P be a non-negative integer for each $P\in\mathcal{P}$. Assume that $n_P=n_Q$ whenever $P\sim Q$. Then there exist infinitely many pairwise non-isomorphic indecomposable finitely generated R-modules X such that, for each $P\in\mathcal{P}$, the localization X_P is a free R_P-module of rank n_P.*
(2) *Conversely, assume R is not an exceptional Dedekind-like ring, but that R is a homomorphic image of some Dedekind-like ring. If X is an indecomposable finitely generated R-module and P is a non-maximal prime, then X_P is either* 0 *or is isomorphic to R_P or $R_P^{(2)}$.*

We do not know whether or not exceptional Dedekind-like rings have indecomposables of large rank, but we suspect that they do not. It is interesting to note that every ring that is not a homomorphic image of a Dedekind-like ring is finite-length wild [12, Definition 2.2], that is, its category of finite-length modules has wild representation type. On the other hand, over any non-exceptional Dedekind-like ring, there is a complete classification, up to isomorphism, of *all* finitely generated modules (cf. [13]). The situation with exceptional Dedekind-like rings still needs to be worked out, but the expectation is that they have tame representation type.

We will prove part (2) of the Main Theorem at the end of this section. In Section 2 we give a direct construction that works whenever some power of $\mathfrak{m}$ requires at least three generators. The case of a one-dimensional Cohen–Macaulay ring was treated in [9]. When R is not Cohen–Macaulay and every power of $\mathfrak{m}$ is two-generated, the construction is much more difficult, and that case is the most laborious part of the paper.

In Section 3 we establish some easy results on bimodules, and in Section 4 we apply these to the situation ${}_AE_B$, where $A = \operatorname{End}_R(M)$, $B = \operatorname{End}_R(N)$ and $E = \operatorname{Ext}^1_R(N, M)$. Here M is a suitable indecomposable module of finite length, and N is a module with positive depth. Theorem 4.2 gives a general method of building an element $\xi \in \operatorname{Ext}^1_R(N, M)$ such that the middle module X in a short exact sequence $0 \to M \to X \to N \to 0$ representing ξ is indecomposable. In Section 5 we use a method pioneered by Drozd [5] and Ringel [18] and adapted by Klingler and Levy [12], to build the requisite indecomposable finite-length modules M. Finally, in Section 6, we apply the results from Sections 3–5, to build the desired indecomposable modules.

To conclude this section, we prove part (2) (the "converse") of the Main Theorem. The assertion is vacuous if $\dim(R) = 0$. Therefore suppose $\dim(R) = 1$. Let $R = D/J$, where D is a Dedekind-like ring. If D is an exceptional Dedekind-like ring, then D is a domain. But then $R = D$, and this is the case we have excluded from consideration. Therefore D is not exceptional. Write $P = Q/J$, where Q is a non-maximal, hence minimal, prime ideal of D. Viewing M as a D-module, we see, using [14, Corollary 16.4], that M_Q is either 0 or is isomorphic to D_Q or $D_Q^{(2)}$. Since the natural map $D_Q \to R_P$ is an isomorphism, the desired conclusion follows.

2. Proof of the Main Theorem when some power of $\mathfrak{m}$ needs 3 generators

The main result of this section is Proposition 2.2, but we will warm up with a simpler construction that will not actually be needed until Section 6. This construction is far from new. See, for example, the papers of Higman [11], Heller and Reiner [10], and Warfield [19]. Similar constructions can be found in the classification, up to simultaneous equivalence, of pairs of matrices. (Cf. Dieudonné's discussion [4] of the work of Kronecker [15] and Weierstrass [20].)

Proposition 2.1 *(The Warmup). Let $(\Lambda, \mathfrak{m}, k)$ be a commutative, Artinian local ring with $\mathfrak{m}^2 = (0)$, and let m be a positive integer. Let I_m denote the $m \times m$ identity matrix and H_m the $m \times m$ Jordan block having ones on the superdiagonal and zeros elsewhere. Assume $\mathfrak{m}$ is minimally generated by $\{x, y\}$, and put $\Psi := yI_m + xH_m$. We let Ψ operate on $\Lambda^{(m)}$ by left multiplication and put $M := \operatorname{coker}(\Psi)$.*

(1) *M is an indecomposable Λ-module requiring exactly m generators.*
(2) *For every non-zero element $t \in \mathfrak{m}$, $\operatorname{socle}(M/tM) \cong k^{(m)}$.*

Proof. For (1), since the entries of Ψ are in $\mathfrak{m}$, the Λ-module M needs exactly m generators. Moreover, the associated graded module $\operatorname{gr}_{\mathfrak{m}}(M)$ is the standard form of one of the indecomposable modules in the classification of $k[X, Y]/(X^2, XY, Y^2)$-modules, found in the references above, from which it follows immediately that M is indecomposable.

For the convenience of the reader, we will give a direct proof that M is indecomposable. Suppose that $f \in \operatorname{End}_\Lambda(M)$ is idempotent but not surjective; it suffices to prove that $f = 0$. There exist matrices F and G making the following diagram commute:

$$\begin{tikzcd}
\Lambda^{(m)} \arrow[r, "\Psi"] \arrow[d, "G"'] & \Lambda^{(m)} \arrow[r] \arrow[d, "F"'] & M \arrow[d, "f"'] \\
\Lambda^{(m)} \arrow[r, "\Psi"] & \Lambda^{(m)} \arrow[r] & M.
\end{tikzcd}$$

The equation $F \cdot \Psi = \Psi \cdot G$ yields $yF + xF \cdot H_m = yG + xH_m \cdot G$. Since the images of x and y in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over k, we obtain, after reducing all entries of F, G

and H_m modulo $\mathfrak{m}$, $\overline{F} = \overline{G}$ and $\overline{F} \cdot \overline{H}_m = \overline{H}_m \cdot \overline{G}$. Therefore $\overline{F}$ and $\overline{H}_m$ commute. Since $\overline{H}_m$ is non-derogatory, $\overline{F} \in k[\overline{H}_m]$, and hence $\overline{F}$ is an upper-triangular matrix with a constant diagonal. Since F is not surjective, neither is $\overline{F}$ (Nakayama's lemma), and it follows that $\overline{F}^m = 0$. Therefore $\operatorname{im}(f) = \operatorname{im}(f^m) \subseteq \mathfrak{m}M$, whence $1 - f$ is surjective. Since f is idempotent, $f = 0$.

To prove (2), we note that $M/tM = \operatorname{coker}(\Phi)$, where $\Phi = [\Psi \; tI_m]$. Suppose first that $t = by$, where b is a unit of Λ. Elementary column operations transform Φ to the matrix $[xH_m \; yI_m]$. Therefore $M/tM \cong k^{(m-1)} \oplus \Lambda/(y)$, and (2) follows. The other possibility is that $t = ax + by$, where a is a unit. In this case we can do elementary column operations to replace the superdiagonal elements of Ψ by multiples of y. Further column operations transform the matrix to the form $[yI_m \; xI_m]$, and we have $M/tM \cong k^{(m)}$. □

Proposition 2.2. *Let $(R, \mathfrak{m}, k)$ be a commutative, Noetherian local ring for which some power $\mathfrak{m}^r$ of the maximal ideal requires at least 3 generators. Let $\mathcal{P}$ be a finite set of non-maximal prime ideals of R, and let n_P be a non-negative integer for each $P \in \mathcal{P}$. Assume that $n_P = n_Q$ whenever $P \sim Q$. Let $n_1 < \cdots < n_t$ be the distinct integers in $\{n_P \mid P \in \mathcal{P}\}$, and put $n := n_1 + \cdots + n_t$. Given any integer $q \geqslant n$, there is an indecomposable finitely generated R-module M such that*

(1) *M needs exactly $n + q$ generators, and*
(2) *$M_P \cong R_P^{(n_P)}$ for $P \in \mathcal{P}$.*

Proof. Choose $x \in \mathfrak{m}^r - (\mathfrak{m}^{r+1} \cup (\bigcup \mathcal{P}))$, $y \in \mathfrak{m}^r - ((\mathfrak{m}^{r+1} + Rx) \cup (\bigcup \mathcal{P}))$ and $z \in \mathfrak{m}^r - ((\mathfrak{m}^{r+1} + Rx + Ry) \cup (\bigcup \mathcal{P}))$. Thus x, y and z are outside the union of the primes in $\mathcal{P}$, and their images in $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ are linearly independent.

For $i = 1, \dots, t$, let $\mathcal{P}_i = \{P \in \mathcal{P} \mid n_P = n_i\}$. Put $S_i = R - \bigcup \mathcal{P}_i$, and let K_i be the kernel of the natural map $R \to S_i^{-1}R$. We claim that $0 \in S_i S_j$ if $i \neq j$. If not, there would be a prime ideal Q disjoint from the multiplicative set $S_i S_j$. But then Q would be contained in $P_i \cap P_j$ for some $P_i \in \mathcal{P}_i$ and $P_j \in \mathcal{P}_j$, contradicting $P_i \nsim P_j$. It follows that $S_i^{-1} S_j^{-1} R = 0$ if $i \neq j$, that is, $K_i S_j^{-1} R = S_j^{-1} R$ if $i \neq j$. Therefore we can choose, for each $i = 1, \dots, t$, an element

$$\xi_i \in K_i \mathfrak{m}^{r+1} - \bigcup_{j \neq i} \Bigl(\bigcup \mathcal{P}_j \Bigr).$$

The image of ξ_i in $S_j^{-1}R$ is 0 if $i = j$ and a unit if $i \neq j$.

Let I_l denote the $l \times l$ identity matrix and $0_{l \times m}$ the $l \times m$ zero matrix. Let $H = H_q$ be the nilpotent $q \times q$ Jordan block having ones on the superdiagonal and zeros elsewhere. Consider the following matrix:

$$A = \begin{bmatrix} \Xi & \Gamma \\ 0_{q \times n} & \Delta \end{bmatrix} \in \operatorname{Mat}_{n+q \times n+q}(R),$$

where

$$\Xi := \begin{bmatrix} \xi_1 I_{n_1} & 0 & \cdots & 0 \\ 0 & \xi_2 I_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \xi_t I_{n_t} \end{bmatrix} \in \operatorname{Mat}_{n \times n}(R),$$

$\Gamma := [xI_n \; 0_{n,q-n}] \in \mathrm{Mat}_{n\times q}(R)$, and $\Delta := zI_q + yH_q \in \mathrm{Mat}_{q\times q}(R)$.

We let A operate on $R^{(n)} \oplus R^{(q)}$ by left multiplication, and we put $M := M_q := \mathrm{coker}(A)$. Since the entries of A are in $\mathfrak{m}$, M_q requires exactly $n+q$ generators. To show that M is indecomposable, suppose $f \in \mathrm{End}_R(M)$ is idempotent and not surjective. We shall show that $f = 0$. We can lift f to homomorphisms F and G which render the following diagram commutative:

$$\begin{tikzcd}
R^{(n)} \oplus R^{(q)} \arrow[r, "A"] \arrow[d, "G"'] & R^{(n)} \oplus R^{(q)} \arrow[r] \arrow[d, "F"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
R^{(n)} \oplus R^{(q)} \arrow[r, "A"] & R^{(n)} \oplus R^{(q)} \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

If we write F and G as 2×2 block matrices, this diagram yields the equation

$$\begin{bmatrix} F_{11}\Xi & F_{11}\Gamma + F_{12}\Delta \\ F_{21}\Xi & F_{21}\Gamma + F_{22}\Delta \end{bmatrix} = FA = AG = \begin{bmatrix} \Xi G_{11} + \Gamma G_{21} & \Xi G_{12} + \Gamma G_{22} \\ \Delta G_{21} & \Delta G_{22} \end{bmatrix}. \tag{1}$$

Since x, y, z and the ξ_i are in $\mathfrak{m}^r$, we can consider the images, in $\mathfrak{m}^r/\mathfrak{m}^{r+1}$, of the entries of FA and AG. Using the facts that images of x, y and z are k-linearly independent in $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ and that $\xi_i \in \mathfrak{m}^{r+1}$ for all i, we can derive the following equations from (1), where bars denote reduction modulo $\mathfrak{m}$ and U denotes the top left $n \times n$ block of G_{22}:

$$\overline{F}_{12} = 0, \qquad \overline{F}_{11} = \overline{U}, \qquad \overline{F}_{21} = 0, \qquad \overline{F}_{22} = \overline{G}_{22} \quad \text{and} \quad \overline{F}_{22}\overline{H}_q = \overline{H}_q\overline{G}_{22}.$$

As in the proof of Lemma 2.1, we see that $\overline{G}_{22}$ is upper triangular with constant diagonal. The same then holds for $\overline{F}$, and we conclude as before that $f = 0$.

It remains to prove that $S_i^{-1}M \cong (S_i^{-1}R)^{(n_i)}$ for all i. Fix an index $i \leqslant t$, and consider the image A' in $\mathrm{Mat}_{(n+q)\times(n+q)}(S_i^{-1}R)$ of the matrix A. We recall that the ξ_j, $j \neq i$ become units in A', while ξ_i maps to 0. Also, x, y and z map to units. Using these facts, one can easily do elementary row and column operations over $S_i^{-1}R$ to show that A' is equivalent to the $(n+q) \times (n+q)$ matrix B with I_{n+q-n_i} in the top left corner and zeros elsewhere. Thus $S_i^{-1}M \cong \mathrm{coker}(A') \cong \mathrm{coker}(B) \cong (S_i^{-1}R)^{(n_i)}$ as desired. □

3. Bimodules

Throughout this section let R be a commutative Noetherian ring, and let A and B be module-finite R-algebras (not necessarily commutative). Let ${}_AE_B$ be an $A-B$-bimodule. We assume E is *R-symmetric*, that is, $re = er$ for $r \in R$ and $e \in E$. Furthermore we assume that E is module-finite over R. The Jacobson radical of a (not necessarily commutative) ring C is denoted by $\mathrm{J}(C)$, and the ring C is said to be *local* provided $C/\mathrm{J}(C)$ is a division ring, equivalently [6, Proposition 1.10], the set of non-units of C is closed under addition. The following lemma assembles some useful trivialities that allow us to transfer ring properties across the bimodule E.

Lemma 3.1. *Let $\alpha : {}_AA \to {}_AE$ and $\beta : B_B \to E_B$ be module homomorphisms, and assume that $\alpha(1_A) = \beta(1_B)$. Put $C := \beta^{-1}(\alpha(A))$.*

(1) *If $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $\alpha(a_i) = \beta(b_i)$, $i = 1, 2$, then $\alpha(a_1a_2) = \beta(b_1b_2)$.*

(2) *C is an R-subalgebra of B.*
(3) $\ker(\beta) \cap C$ *is an ideal of C; thus $D := \beta(C)$ has a unique ring structure making $\beta' : C \twoheadrightarrow D$ (the map induced by β) a ring homomorphism.*
(4) *Assume $\alpha(A) \subseteq \beta(B)$. Then the map $\alpha' : A \twoheadrightarrow D$ induced by α is a ring homomorphism (where D has the ring structure of* (3)).

Proof. (1) We have $\alpha(a_1a_2) = a_1\alpha(a_2) = a_1\beta(b_2) = a_1\beta(1_Bb_2) = a_1\beta(1_B)b_2 = a_1\alpha(1_A)b_2 = \alpha(a_11_A)b_2 = \alpha(a_1)b_2 = \beta(b_1)b_2 = \beta(b_1b_2)$. This proves (1), and it follows that C is a subring of B. A similar argument, using the fact that E is R-symmetric, shows that $1_Br \in C$ for each $r \in R$. Thus C is an R-subalgebra of B.

For (3), let $b_1, b_2 \in C$, with $b_2 \in \ker(\beta)$. Choosing $a_1, a_2 \in A$ as in (1), we have $\beta(b_1b_2) = \alpha(a_1a_2) = a_1\alpha(a_2) = a_1\beta(b_2) = 0$. Since $\ker(\beta) \cap C$ is clearly a right ideal of C, it is an ideal. To prove (4), let $a_1, a_2 \in A$, and choose $b_1, b_2 \in B$ as in (1). Then $\alpha(a_1a_2) = \beta(b_1b_2) = \beta(b_1)\beta(b_2) = \alpha(a_1)\alpha(a_2)$. □

Theorem 3.2. *With notation of Lemma* 3.1, *assume $\alpha(1_A) = \beta(1_B)$ and* $\ker(\beta) \subseteq \mathrm{J}(B)$. *If A is local and $\alpha(1_A) \neq 0$, then C is local.*

Proof. Suppose first that $\alpha(A) \subseteq \beta(B)$. With D as in Lemma 3.1, we have surjective ring homomorphisms

$$\begin{tikzcd} A \arrow[r, two heads, "\alpha'"] & D & C. \arrow[l, two heads, "\beta'"'] \end{tikzcd}$$

Therefore D is a (non-trivial) local ring, and to show that C is local, it will suffice to show that $\ker(\beta') \subseteq \mathrm{J}(C)$. Since $\ker(\beta) \subseteq \mathrm{J}(B)$, it is enough to show that $\mathrm{J}(B) \cap C \subseteq \mathrm{J}(C)$. As B is a module-finite R-algebra, left invertibility and right-invertibility are the same in B (thus we simply use the word “invertible”). Suppose now that $x \in \mathrm{J}(B) \cap C$. To show that $x \in \mathrm{J}(C)$ we must show that $z := 1 + yx$ is invertible in C for each $y \in C$. Since z *is* invertible in B, write $bz = 1$, with $b \in B$. Since B is module-finite over R, b is integral over R, say, $b^n + r_1b^{n-1} + \cdots + r_{n-1}b + r_n = 0$, with $r_i \in R$. Multiplying this equation by z^{n-1}, we see that $b \in C$, as desired.

For the general case, put $G := \alpha^{-1}(\beta(B))$. By (2) of Lemma 3.1 (with the roles of A and B interchanged), G is an R-subalgebra of A. To see that C is local, it will suffice to show that every non-unit of G is a non-unit of A. Since A is integral over R, the argument in the preceding paragraph does the job. □

4. Extensions

Let R be a commutative Noetherian ring, and let M and N be finitely generated R-modules. Put $A := \mathrm{End}_R(M)$ and $B := \mathrm{End}_R(N)$. Note that each of the R-modules $\mathrm{Ext}^n_R(N, M)$ has a natural $A - B$-bimodule structure. Indeed, any $f \in B$ induces an R-module homomorphism $f^* : \mathrm{Ext}^n_R(N, M) \to \mathrm{Ext}^n_R(N, M)$. For $x \in \mathrm{Ext}^n_R(N, M)$ put $x \cdot f = f^*(x)$. The left A-module structure is defined similarly, and the fact that $\mathrm{Ext}^n_R(N, M)$ is a bimodule follows from the fact that $\mathrm{Ext}^n_R(\_,\_)$ is an additive bifunctor. Note that $\mathrm{Ext}^n_R(N, M)$ is R-symmetric, since, for $r \in R$, multiplications by r on N and on M induce the same endomorphism of $\mathrm{Ext}^n_R(N, M)$.

Suppose now that $n = 1$, and put $E = \mathrm{Ext}^1_R(N, M)$, regarded as the set of equivalence classes of extensions $0 \to M \to X \to N \to 0$. Let $\alpha : {}_AA \to {}_AE$ and $\beta : B_B \to E_B$ be module homomorphisms satisfying $\alpha(1_A) = \beta(1_B) =: [\sigma]$. Then α and β are, up to signs, the connecting homomorphisms in the long exact sequences of Ext obtained by applying $\mathrm{Hom}_R(\_,M)$ and $\mathrm{Hom}_R(N,\_)$, respectively, to the short exact sequence σ. (When one computes Ext via resolutions one must adorn maps with appropriate ± signs, in order to ensure naturality of the connecting homomorphisms. In what follows, the choice of sign will not be important.)

Let $(R, \mathfrak{m})$ be a commutative, Noetherian local ring and M a finitely generated R-module. We call $\mathrm{H}^0_{\mathfrak{m}}(M) := \{x \in M \mid \mathfrak{m}^i x = 0 \text{ for some } i \geqslant 1\}$ the *finite length part* of M, and we put $M^{\natural} := M/\mathrm{H}^0_{\mathfrak{m}}(M)$. More generally (since it is no additional work), we consider a commutative Noetherian ring R (not necessarily local) and a torsion theory $(\mathcal{T}, \mathcal{F})$ and denote by $M^{\natural}$ the reduction of M modulo torsion. (See [7] for a definition and discussion of the general properties of torsion theories.)

Lemma 4.1. *Let R be a commutative Noetherian ring, let M and N be finitely generated R-modules, with M torsion and N torsion-free* (*with respect to some torsion theory*). *Let A, B and E be as above, and let $\alpha : A \to E$ and $\beta : B \to E$ be module homomorphisms with $\alpha(1_A) = \beta(1_B) = [\sigma]$, where σ is the short exact sequence*

$$0 \to M \xrightarrow{i} X \xrightarrow{\pi} N \to 0. \tag{σ}$$

Let $\rho : \mathrm{End}_R(X) \to \mathrm{End}_R(N) = B$ be the canonical homomorphism (*reduction modulo torsion*). *Then the image of ρ is exactly the ring $C := \beta^{-1}\alpha(A) \subseteq B$.*

Proof. Since $N = X^{\natural}$, the map ρ makes sense. Noting that $\mathrm{Hom}_R(M, N) = 0$, we apply various Hom functors to σ to obtain the following exact diagram:

$$\begin{tikzcd}
 & \mathrm{Hom}_R(N,X) \arrow[r] \arrow[d] & \mathrm{Hom}_R(X,X) \arrow[d, "\pi_*"] & \\
0 \arrow[r] & B \arrow[r, "\chi", "\cong"'] \arrow[d, "\beta"'] & \mathrm{Hom}_R(X,N) \arrow[r] \arrow[d, "i_*"'] & 0 \\
A \arrow[r, "\alpha"] & E \arrow[r, "\pi^*"] & \mathrm{Ext}^1_R(X,M). &
\end{tikzcd}$$

The top square commutes, and the bottom square commutes up to sign. Clearly $\rho = \chi^{-1}\pi_*$, and an easy diagram chase shows that the image of $\chi^{-1}\pi_*$ is C. □

Theorem 4.2. *Keep the notation and hypotheses of Lemma* 4.1.

(1) *Suppose C has no idempotents other than* 0 *and* 1. *If $X = U \oplus V$* (*a decomposition as R-modules*), *then either U or V is a torsion module.*
(2) *Suppose A is local and* $\ker(\beta)$ *is contained in the Jacobson radical of B. Then X is indecomposable.*

Proof. Suppose $X = U \oplus V$, with both U and V non-zero, and let $f \in \operatorname{End}_R(X)$ be the projection on U (relative to the decomposition $X = U \oplus V$). Then $\pi : X \to N$ induces an isomorphism $\bar{\pi} : U^\natural \oplus V^\natural \to N$, and $\rho(f) \in \operatorname{End}_R(N)$ is the projection on $\bar{\pi}(U^\natural)$. If $U^\natural$ and $V^\natural$ were both non-zero, $\rho(f)$ would be a non-trivial idempotent in C, contradiction. This proves (1).

To prove (2), we note that M is indecomposable. Therefore we may assume that $N \neq 0$. Then $B \neq 0$, and since $\ker(\beta) \subseteq \mathrm{J}(B)$, we have $\alpha(1_A) = \beta(1_B) \neq 0$. Now Theorem 3.2 implies that C is local, and by (1) either U or V is a torsion module. Since $M = \mathrm{H}^0_{\mathfrak{m}}(X)$, we may assume that $U \subseteq M$. Then U is a direct summand of M, whence $U = M$. But then the short exact sequence σ splits, contradicting $\alpha(1_A) \neq 0$. □

Corollary 4.3. *Let* $(R, \mathfrak{m}, k)$ *be a commutative, Noetherian local ring, let* M *be an* R*-module of finite length, and let* N *be a finitely generated* R*-module with* $\mathrm{H}^0_{\mathfrak{m}}(N) = 0$. *Put* $A := \operatorname{End}_R(M)$ *and* $B := \operatorname{End}_R(N)$. *Suppose there exists a right* B*-module homomorphism* $\beta : B_B \to E_B := \operatorname{Ext}^1_R(N, M)$ *such that* $\ker(\beta) \subseteq \mathrm{J}(B)$. *Assume* A *is local, and let* $0 \to M \to X \to N \to 0$ *represent* $\beta(1_B) \in E$. *Then* X *is indecomposable.*

Lemma 4.4. *Let* N *be a finitely generated module over a commutative, Noetherian local ring* $(R, \mathfrak{m})$, *let* Γ *be an* R*-subalgebra of* $\operatorname{End}_R(N)$, *and let* $g \in \Gamma$. *If* $g(N) \subseteq \mathfrak{m}N$, *then* $g \in \mathrm{J}(\Gamma)$.

Proof. It will suffice to show that $1 + hg$ is a unit of Γ for every $h \in \Gamma$. For each $x \in M$ we have $x = (1 + hg)(x) - q(g(x)) \in (1 + hg)(M) + \mathfrak{m}M$. By Nakayama's lemma, $1 + hg$ is surjective and therefore (as M is Noetherian) an automorphism. The inverse (in $\operatorname{End}_R(N)$) of $1 + hg$ is integral over R and therefore is in $R[1 + hg] \subseteq \Gamma$. □

5. Building a suitable finite-length module

Definition 5.1. A commutative, Artinian local ring $(\Lambda, \mathfrak{m}, k)$ is a *Drozd ring* provided its associated graded ring is the k-algebra $\mathrm{gr}_{\mathfrak{m}}(\Lambda) \cong k[X, Y]/(X^2, XY^2, Y^3)$.

The following observation [12, Lemma 4.2] will be used repeatedly in the construction.

Lemma 5.2. *Let* $(\Lambda, \mathfrak{m}, k)$ *be a Drozd ring, and let* $x, y \in \Lambda$ *be generators of* $\mathfrak{m}$ *with* $x^2 = 0$. *Then any* $c \in \mathfrak{m}$ *can be expressed in the form*

$$c = u_1 x + u_2 y + u_3 xy + u_4 y^2,$$

where each u_i *is either a unit or* 0.

The idea of the construction below originated in work of Drozd [5] and Ringel [18]. The construction was adapted by Klingler and Levy [12] to show that the category of finite-length modules over a Drozd ring has wild representation type.

We will use the terms "column space" and "image" interchangeably. We denote by H the $n \times n$ nilpotent upper-triangular Jordan block, and by I the $n \times n$ identity matrix. In the following, all blocks are $n \times n$, so we will omit the subscripts on I, H and 0.

Proposition 5.3. *Let $(\Lambda, \mathfrak{m}, k)$ be a Drozd ring, and let $x, y \in \Lambda$ with $(x, y) = \mathfrak{m}$ and $x^2 = 0$. Let $n \geqslant 1$. Then there exists an indecomposable finitely generated Λ-module M such that*

$$\frac{(0 :_M (x, y^2))}{xM} \cong k^{(n)}. \tag{2}$$

Proof. Let $\Psi : \Lambda^{(4n)} \to \Lambda^{(3n)}$ be defined by the $3n \times 4n$ matrix

$$\Psi = \begin{bmatrix} yI & xI & 0 & 0 \\ 0 & -y^2I & xI & -yI \\ 0 & 0 & -(H+I)y^2 & xI \end{bmatrix},$$

and define M by the exact sequence $\Lambda^{(4n)} \xrightarrow{\Psi} \Lambda^{(3n)} \xrightarrow{\varepsilon} M \to 0$.

To show that M is indecomposable, suppose f is an idempotent endomorphism of M. Let $\Gamma = \{g \in \operatorname{End}_\Lambda(\Lambda^{(3n)}) \mid g(\operatorname{im}\Psi) \subseteq \operatorname{im}\Psi\}$. Since ε is a projective cover, the induced map $\Gamma \to \operatorname{End}_\Lambda(M)$ is surjective, and its kernel is contained in $\mathrm{J}(\Gamma)$ by Lemma 4.4. Since Γ is left Artinian, $\mathrm{J}(\Gamma)$ is nilpotent, and thus f lifts to an idempotent $F \in \Gamma$ (cf. [1, §27]). It will suffice to show that F is either 0 or 1. Now we invoke [12, Lemma 4.8], which implies that F has the following block form:

$$F = \begin{bmatrix} F_{11} & * & * \\ \alpha & F_{22} & * \\ \beta & \gamma & F_{33} \end{bmatrix},$$

where

(1) each block is an $n \times n$ matrix,
(2) $F_{11} \equiv F_{22} \equiv F_{33} \bmod \mathfrak{m}$,
(3) $(H + I) \cdot F_{11} \equiv F_{11} \cdot (H + I) \bmod \mathfrak{m}$, and
(4) the entries of α, β and γ are in $\mathfrak{m}$.

Letting bars denote reduction modulo $\mathfrak{m}$, we have

$$\overline{F} = \begin{bmatrix} \overline{F_{11}} & * & * \\ 0 & \overline{F_{11}} & * \\ 0 & 0 & \overline{F_{11}} \end{bmatrix}.$$

Since $\overline{F_{11}}$ commutes with $\overline{H}$, $\overline{F_{11}}$ belongs to $k[\overline{H}]$, which is a local ring. Moreover, since $\overline{F}^2 = \overline{F}$, it follows that $\overline{F_{11}}^2 = \overline{F_{11}}$. Therefore $\overline{F_{11}}^2 = 0$ or 1. An easy computation then shows that $\overline{F} = 0$ or 1. By Lemma 4.4 the kernel of the map $\operatorname{End}_\Lambda(\Lambda^{(3n)}) \to \operatorname{End}_k(k^{(3n)})$ is contained in the Jacobson radical of $\operatorname{End}_\Lambda(\Lambda^{(3n)})$. It follows that $F = 0$ or 1, and therefore that $f = 0$ or 1. This shows that M is indecomposable.

We claim that $(0 :_M (x, y^2))$ is generated by the images, under ε, of the columns of the matrix

$$\varphi := \begin{bmatrix} xI & 0 & 0 & 0 & 0 & I \\ 0 & xI & 0 & yI & 0 & 0 \\ 0 & 0 & xI & 0 & y^2I & -yI \end{bmatrix}$$

(where each block is $n \times n$). An easy calculation shows that both x and y^2 knock the column space of $\boldsymbol{\varphi}$ into the column space of Ψ, so the purported generators are, at least, in $(0 :_M (x, y^2))$.

To prove the claim, suppose $\boldsymbol{\alpha} \in \Lambda^{(3n)}$ and $x\boldsymbol{\alpha}$ and $y^2\boldsymbol{\alpha}$ are both in the image of Ψ. We will show that $\boldsymbol{\alpha} \in \operatorname{im}(\boldsymbol{\varphi})$.

We can write

$$x\boldsymbol{\alpha} = \Psi \cdot \boldsymbol{\beta} \quad \text{and} \quad y^2\boldsymbol{\alpha} = \Psi \cdot \boldsymbol{\gamma} \tag{3}$$

with $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \Lambda^{(4n)}$. Write

$$\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \boldsymbol{\alpha}_3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\beta} = \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \boldsymbol{\beta}_3 \\ \boldsymbol{\beta}_4 \end{bmatrix},$$

where the $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_j$ are in $\Lambda^{(n)}$. The first equation in (3) yields

$$\begin{bmatrix} x\boldsymbol{\alpha}_1 \\ x\boldsymbol{\alpha}_2 \\ x\boldsymbol{\alpha}_3 \end{bmatrix} = \begin{bmatrix} y\boldsymbol{\beta}_1 + x\boldsymbol{\beta}_2 \\ -y^2\boldsymbol{\beta}_2 + x\boldsymbol{\beta}_3 - y\boldsymbol{\beta}_4 \\ -y^2(H+I)\cdot \boldsymbol{\beta}_3 + x\boldsymbol{\beta}_4 \end{bmatrix}.$$

By Lemma 5.2 we can write the $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ in the form

$$\boldsymbol{\alpha}_i = \boldsymbol{u}_{i,0} + \boldsymbol{u}_{i,1}x + \boldsymbol{u}_{i,2}y + \boldsymbol{u}_{i,3}xy + \boldsymbol{u}_{i,4}y^2,$$
$$\boldsymbol{\beta}_i = \boldsymbol{v}_{i,0} + \boldsymbol{v}_{i,1}x + \boldsymbol{v}_{i,2}y + \boldsymbol{v}_{i,3}xy + \boldsymbol{v}_{i,4}y^2,$$

where the entries of $\boldsymbol{u}_{i,j}$ and $\boldsymbol{v}_{i,j}$ are either units or 0. Since the images of x and y in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over k, the equation $x\boldsymbol{\alpha}_1 = y\boldsymbol{\beta}_1 + x\boldsymbol{\beta}_2$ yields $\boldsymbol{v}_{1,0} = \mathbf{0}$ and $\bar{\boldsymbol{u}}_{1,0} = \bar{\boldsymbol{v}}_{2,0}$, where bars denote reduction modulo $\mathfrak{m}$. From $x\boldsymbol{\alpha}_2 = -y^2\boldsymbol{\beta}_2 + x\boldsymbol{\beta}_3 - y\boldsymbol{\beta}_4$, it follows that $\bar{\boldsymbol{u}}_{2,0} = \bar{\boldsymbol{v}}_{3,0}$ and $\boldsymbol{v}_{4,0} = \mathbf{0}$ and, since the socle elements xy and y^2 are linearly independent over k, that $\bar{\boldsymbol{v}}_{2,0} = -\bar{\boldsymbol{v}}_{4,2}$. From $x\boldsymbol{\alpha}_3 = -y^2(H+I)\cdot\boldsymbol{\beta}_3 + x\boldsymbol{\beta}_4$, it follows that $\bar{\boldsymbol{u}}_{3,0} = \bar{\boldsymbol{v}}_{4,0}$ and hence that $\boldsymbol{u}_{3,0} = \mathbf{0}$.

Using the equation $x\boldsymbol{\alpha}_3 = -y^2(H+I)\cdot\boldsymbol{\beta}_3 + x\boldsymbol{\beta}_4$ again, we see that $\bar{\boldsymbol{u}}_{3,2} = \bar{\boldsymbol{v}}_{4,2}$. Further, since $H+I$ is invertible, it follows that $\boldsymbol{v}_{3,0} = \mathbf{0}$ and hence that $\boldsymbol{u}_{2,0} = \mathbf{0}$.

To summarize, we have $\bar{\boldsymbol{u}}_{3,2} = \bar{\boldsymbol{v}}_{4,2} = -\bar{\boldsymbol{v}}_{2,0} = -\bar{\boldsymbol{u}}_{1,0}$, and $\boldsymbol{u}_{2,0} = \boldsymbol{u}_{3,0} = \mathbf{0}$. Putting $\boldsymbol{w} := \boldsymbol{u}_{1,0}$, we have $\boldsymbol{u}_{3,2} = -\boldsymbol{w} + x\boldsymbol{\mu} + y\boldsymbol{\nu}$ for suitable $\boldsymbol{\mu}, \boldsymbol{\nu} \in \Lambda^{(n)}$. Then

$$\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{w} & +x\boldsymbol{u}_{1,1} & +y\boldsymbol{u}_{1,2} & +xy\boldsymbol{u}_{1,3} & +y^2\boldsymbol{u}_{1,4} \\ \mathbf{0} & +x\boldsymbol{u}_{2,1} & +y\boldsymbol{u}_{2,2} & +xy\boldsymbol{u}_{2,3} & +y^2\boldsymbol{u}_{2,4} \\ -y\boldsymbol{w} & +x\boldsymbol{u}_{3,1} & +\mathbf{0} & +xy(\boldsymbol{u}_{3,3}+\boldsymbol{\mu}) & +y^2(\boldsymbol{u}_{3,4}+\boldsymbol{\nu}) \end{bmatrix}. \tag{4}$$

From (4) it follows that $\boldsymbol{\alpha} \in \operatorname{im}(\boldsymbol{\varphi})$, as desired. This completes the proof of our claim.

It is easy to see, using the invertibility of $H+I$, that the image of the left-most $3n \times 5n$ submatrix of $\boldsymbol{\varphi}$ is contained in $x\Lambda^{(3n)} + \operatorname{im}(\Psi)$. Letting $\gamma_1, \dots, \gamma_n$ be the last n columns of $\boldsymbol{\varphi}$, we see that $(0 :_M (x, y^2))/xM$ is generated by $\zeta_1 := \varepsilon(\gamma_1) + xM, \dots, \zeta_n := \varepsilon(\gamma_n) + xM$. Since $x\gamma_i,\ y\gamma_i \in x\Lambda^{(3n)} + \operatorname{im}(\Psi)$ for each i, we see that $(0 :_M (x, y^2))/xM$ is a k-vector space of dimension at most n.

To show that $\frac{(0:_M(x,y^2))}{xM} \cong k^{(n)}$, we need only show that $\zeta_1, \dots, \zeta_n$ are linearly independent. Given a relation $\sum_{i=1}^n \lambda_i \zeta_i = 0$, with $\lambda_i \in \Lambda$, we have $\sum_{i=1}^n \lambda_i \gamma_i \in \operatorname{im}(\Psi) + x\Lambda^{(3n)} \subseteq \mathfrak{m}\Lambda^{(3n)}$. This relation obviously forces $\lambda_i \in \mathfrak{m}$ for all i, as desired. □

6. Proof of the Main Theorem when every power of $\mathfrak{m}$ is 2-generated

This section is devoted to the proof of (1) of Theorem 1.2 in the remaining case—when every power of $\mathfrak{m}$ is generated by at most two elements. If $\dim R \geqslant 2$, then $\mathfrak{m}$ needs at least 3 generators unless R is a two-dimensional regular local ring. But in that case $\mathfrak{m}^2$ needs 3 generators. Thus Proposition 2.2 applies if $\dim(R) \geqslant 2$. If $\dim(R) = 0$, then $\mathcal{P} = \emptyset$, and R is not a principal ideal ring because, by the Cohen Structure Theorem (e.g., [17, Theorem 3.1]), an Artinian local principal ideal ring is a homomorphic image of a discrete valuation domain and hence of a Dedekind-like ring. Therefore, if $\dim(R) = 0$, then $\mathfrak{m}$ needs exactly two generators, and Proposition 2.1 (applied to $R/\mathfrak{m}^2$) provides an infinite list of pairwise non-isomorphic indecomposable R-modules.

Thus we may assume that R is one-dimensional. If, in addition, R is Cohen–Macaulay, we quote [9, Theorem 1.2] to obtain the desired infinite family of indecomposable modules with prescribed ranks at the minimal primes.

The following theorem is a special case of the "ring-theoretic dichotomy" theorem of Klingler and Levy [14, Theorem 14.3]:

Theorem 6.1. *Let $(\Lambda, \mathfrak{m}, k)$ be a one-dimensional local ring whose maximal ideal $\mathfrak{m}$ is generated by at most two elements. Then exactly one of the following possibilities occurs*:

(1) *Λ is a homomorphic image of a Dedekind-like ring.*
(2) *Λ has a Drozd ring as a homomorphic image.*

Since our ring R is, by assumption, not a homomorphic image of a Dedekind-like ring, it must have a Drozd ring as a homomorphic image. (Actually, the dichotomy theorem in [14] applies to indecomposable Noetherian rings of arbitrary dimension, and in case (1) there is a second possibility—that Λ is a *Klein ring* [12, Definition 2.8]. However, since Klein rings are Artinian, they do not appear in Theorem 6.1. Also, in case (2) there is a second possibility—that R have an *Artinian triad* [12, Definition 2.4] as a homomorphic image. But since the maximal ideal of an Artinian triad needs three generators, this possibility does not occur in our context.)

Lemma 6.2. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local ring. Assume that $\mathfrak{m}$ and $\mathfrak{m}^2$ are two-generated and R/L is a Drozd ring for some ideal L. Write $\mathfrak{m} = Rx + Ry$, with $x^2 \in L$. Then $L = \mathfrak{m}^3$, and $\mathfrak{m}^r = y^{r-1}\mathfrak{m} = Rxy^{r-1} + Ry^r$ for each $r \geqslant 1$. If, further, R is not Cohen–Macaulay, then the following also hold*:

(1) *$\mathfrak{m}^r = Ry^r$ for all $r \gg 0$.*
(2) *R has exactly one minimal prime ideal P. Moreover, R_P is a field and R/P is a discrete valuation ring.*
(3) *P is a principal ideal, and $P \nsubseteq \mathfrak{m}^2$.*

Proof. Since R/L is a Drozd ring, $\mathfrak{m}^3 \subseteq L$. Moreover, $L \subseteq \mathfrak{m}^2$, because $\mathfrak{m}/L$ is not principal. Therefore $L \subseteq \mathfrak{m}^3$, else $\mathfrak{m}^2/L$ would be principal. Since $\mathfrak{m}^2 = (x^2, xy, y^2)$ and $x^2 \in \mathfrak{m}^3$, we have $\mathfrak{m}^2 = y\mathfrak{m}$, and it follows that $\mathfrak{m}^r = y^{r-1}\mathfrak{m}$ for all $r \geqslant 1$.

Suppose now that R is not Cohen–Macaulay, and let z be a non-zero element such that $z\mathfrak{m} = 0$. Let $z \in \mathfrak{m}^s - \mathfrak{m}^{s+1}$, say $z = axy^{s-1} + by^s$, with either a or b a unit. If a is a unit, the equation $axy^s + by^{s+1} = 0$ implies that $xy^r \in Ry^{r+1}$ for all $r \geqslant s$. Therefore $\mathfrak{m}^r = Ry^r$ for all $r > s$. If b is a unit, the equation $ax^2y^{s-1} + bxy^s = 0$ and the fact that $x^2 \in \mathfrak{m}^3$ imply that $xy^s \in \mathfrak{m}^{s+2}$. Then $xy^r \in \mathfrak{m}^{r+2}$ for all $r \geqslant s$. By Nakayama's lemma, $\mathfrak{m}^r = Ry^r$ for $r > s$. We have now proved (1) in either case.

By (1), the multiplicity $\mathrm{e}(R)$ is 1. (For a one-dimensional local ring, the multiplicity is the number of generators needed for sufficiently large powers of $\mathfrak{m}$. Cf. [16, §14].) Item (2) now follows immediately from the "associativity formula": $\mathrm{e}(R) = \sum_P \mathrm{e}(R/P)\ell(R_P)$, where ℓ denotes length (as an R_P-module) and the sum runs over the prime ideals with $\dim(R/P) = \dim(R)$—in our situation, the minimal primes. (Cf. [16, Theorem 14.7] or [3, Corollary 4.7.8].)

To prove (3), we note that $\mathfrak{m}/P$ is principal by (2), and it follows that $P \nsubseteq \mathfrak{m}^2$. Select $t \in P - \mathfrak{m}^2$, and note that $R/(t)$ is a one-dimensional local ring with principal maximal ideal, i.e., a discrete valuation domain. Therefore $Rt = P$, and the proof is complete. □

Having handled every other case in the proof of (1) of Theorem 1.2, we may now assume that $(R, \mathfrak{m}, k)$ satisfies all of the assumptions of Lemma 6.2. We let $P = Rt$ be the minimal prime ideal of R. We are given a non-negative integer n, and we seek an infinite family of pairwise non-isomorphic indecomposable modules X satisfying $X_P \cong R_P^{(n)}$. We now isolate the technical condition that will produce the modules X (at least in pivotal cases):

Proposition 6.3. *With the notation and assumptions above, suppose there is an indecomposable finite-length R-module M such that $\dim_k(\mathrm{socle}_R(\mathrm{Ext}^1_R(R/P, M))) \geqslant n$. Then there is an indecomposable finitely generated R-module X such that*

(1) $\mathrm{H}^0_{\mathfrak{m}}(X) \cong M$,
(2) $X/\mathrm{H}^0_{\mathfrak{m}}(X) \cong (R/P)^{(n)}$, *and*
(3) $X_P \cong R_P^{(n)}$.

Proof. Put $E_1 = \mathrm{Ext}^1_R(R/P, M)$. We return to the set-up of Section 4, taking $N := (R/P)^{(n)}$. Recall that $A := \mathrm{End}_R(M)$, $B := \mathrm{End}_R(N) = \mathrm{Mat}_{n\times n}(R/P)$ and $E := \mathrm{Ext}^1_R(N, M) = E_1^n$. If we write elements of E as row vectors $(1 \times n)$ with entries in E_1, the right B-module structure is given by matrix multiplication. Since M has finite length, A is local [6, Lemmas 2.20 and 2.21].

Let $e_1, \dots, e_n$ be linearly independent elements of $\mathrm{socle}_R(E_1)$, and put $\boldsymbol{e} := [e_1, \dots, e_n] \in E$. We define a right B-module homomorphism $\beta : B_B \to E_B$ by $1 \mapsto \boldsymbol{e}$. We claim that $\ker(\beta) \subseteq \mathrm{J}(B)$. For, suppose $\varphi \in \ker(\beta)$, and write $\varphi = [a_{ij}]$, with $a_{ij} \in R/P$. Then $\boldsymbol{e}\varphi = 0$, that is, $e_1a_{1j} + \dots + e_na_{nj} = 0$ for each $j = 1, \dots, n$. Linear independence of the e_i now implies that $a_{ij} \in \mathfrak{m}/P$ for each i, j. Then $\varphi \in \mathrm{J}(B)$, and the claim is proved.

Now Corollary 4.3 provides a short exact sequence

$$0 \to M \to X \to N \to 0,$$

in which X is indecomposable. Since $M_P = 0$ and (by Lemma 6.2) $N_P \cong R_P^{(n)}$, assertions (1), (2) and (3) are clear. □

Now we can complete the proof of the Main Theorem in the easier case, when $(0:_R t) \subseteq \mathfrak{m}^2$. (Recall that t generates the unique minimal prime P.) Given an arbitrary integer $m \geqslant n$, apply the "warmup" construction to $\Lambda := R/\mathfrak{m}^2$, getting an indecomposable finite-length R-module M such that $M/tM \cong k^{(m)}$. Applying $\operatorname{Hom}_R(\_, M)$ to the short exact sequence

$$0 \to Rt \to R \to R/(t) \to 0,$$

we obtain an exact sequence

$$\operatorname{Hom}_R(R, M) \to \operatorname{Hom}_R(Rt, M) \to E_1 \to 0, \tag{5}$$

where $E_1 = \operatorname{Ext}^1_R(R/P, M)$ as before. Now $Rt \cong R/(0:_R t)$, and since $(0:_R t)M = (0)$, the map $f \mapsto f(t)$ provides an isomorphism $\operatorname{Hom}_R(Rt, M) \cong M$. Combining this isomorphism with the usual isomorphism $\operatorname{Hom}_R(R, M) \cong M$ $(g \mapsto g(1))$, we transform (5) to the exact sequence $M \xrightarrow{t} M \to E_1 \to 0$. Thus $E_1 \cong M/tM \cong k^{(m)}$. By Proposition 6.3, we get an indecomposable module X such that $X_P \cong R_P^{(n)}$ and $\mathrm{H}^0_{\mathfrak{m}}(X) \cong M$. By varying m, we obtain infinitely many non-isomorphic modules M (since M needs m generators) and therefore infinitely many non-isomorphic modules X.

For the rest of this section we assume that $(0:_R t) \nsubseteq \mathfrak{m}^2$. Since $t \notin \mathfrak{m}^2$ by Lemma 6.2, we have $\mathfrak{m} = Rt + Ru$ for some u. We claim that

$$t^2 \in \mathfrak{m}^3.$$

To prove this, choose $z \in (0:_R t) - \mathfrak{m}^2$, and write $z = at + bu$, where either a or b is a unit. Suppose first that b is a unit. Then $\mathfrak{m} = Rt + Rz$. By Lemma 6.2, there is an element $x \notin \mathfrak{m}^2$ with $x^2 \in \mathfrak{m}^3$. Write $x = ct + dz$, where either c or d is a unit. Then $c^2t^2 + d^2z^2 = x^2 \in \mathfrak{m}^3$. It follows that $\mathfrak{m}^2/\mathfrak{m}^3$ is principal (generated by either t^2 or z^2). But this contradicts the fact that R maps onto a Drozd ring. Therefore b is not a unit, and now the equation $at^2 + but = 0$ shows that $t^2 \in \mathfrak{m}^3$ as desired.

At this point, now that we have shown that $t^3 \in \mathfrak{m}^2$, it makes sense to refresh notation, writing $P = Rx$ and $\mathfrak{m} = Rx + Ry$. To summarize, we have

$$P = Rx, \qquad \mathfrak{m} = Rx + Ry, \quad \text{and} \quad x^2 \in \mathfrak{m}^3. \tag{6}$$

We now complete the proof under the additional assumption

$$x^2 = xy^2 = 0. \tag{7}$$

Choose any integer $m \geqslant n$, and apply Proposition 5.3 to the Drozd ring $\Lambda := R/\mathfrak{m}^3$. We get an indecomposable R-module M satisfying (2) and requiring exactly m generators. We claim that $(0:_R x) = (x, y^2)$. The inclusion "$\supseteq$" is clear from (7). For the reverse, let $z \in (0:_R x)$, and write $z = ax + by$. Then $bxy = 0$. If b were a unit, we would have $\mathfrak{m}^2 = Ry^2$, contradicting the fact that $R/\mathfrak{m}^3$ is a Drozd ring. Thus $b \in \mathfrak{m}$, and the claim follows.

As before, we obtain an exact sequence

$$\operatorname{Hom}_R(R, M) \to \operatorname{Hom}_R(Rx, M) \to E_1 \to 0.$$

Since $Rx \cong R/(x, y^2)$, we have an isomorphism $\mathrm{Hom}_R(Rx, M) \cong (0 :_M (x, y^2))$ (taking f to $f(x)$). It follows easily that $E_1 \cong \frac{(0:_M(x,y^2))}{xM}$. Now (2) shows that $E_1 \cong k^{(m)}$, and, as before, we can use Proposition 6.3 to produce and infinite family of pairwise non-isomorphic indecomposable modules X such that $X_P \cong R_P^{(n)}$.

Finally, we complete the proof when (7) is not necessarily satisfied. Since $x^2 \in \mathfrak{m}^3$ by (6), $S := R/(x^2, xy^2)$ maps onto the Drozd ring $R/\mathfrak{m}^3$. Therefore, by Theorem 6.1, S is not a homomorphic image of a Dedekind-like ring. Moreover, S is not Cohen–Macaulay, since $xy \notin (x^2, xy^2)$ (else $\mathfrak{m}^2$ would be principal) but $\mathfrak{m}xy \subseteq (x^2, xy^2)$. By case (7), we obtain infinitely many pairwise non-isomorphic S-modules X such that $X_Q \cong S_Q^{(n)}$, where $Q = P/(x^2, xy^2)$. Now view these modules as R-modules and note that the natural map $R_P \to S_Q$ is an isomorphism. This completes the proof of Theorem 1.2.

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