

# A Krull-Schmidt Theorem for One-dimensional Rings of Finite Cohen-Macaulay Type

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February 8, 2006

## Abstract

This paper determines when the Krull-Schmidt property holds for all finitely generated modules and for maximal Cohen-Macaulay modules over one-dimensional local rings with finite Cohen-Macaulay type. We classify all maximal Cohen-Macaulay modules over these rings, beginning with the complete rings where the Krull-Schmidt property is known to hold. We are then able to determine when the Krull-Schmidt property holds over the non-complete local rings and when we have the weaker property that any two representations of a maximal Cohen-Macaulay module as a direct sum of indecomposables have the same number of indecomposable summands.

**Keywords:** Krull-Schmidt, maximal Cohen-Macaulay, finite Cohen-Macaulay type

## 1 Introduction

It is well known (*cf.* [2] and [20]) that the Krull-Schmidt property holds over any complete local ring  $R$ . That is, whenever  $M_1 \oplus \cdots \oplus M_s \cong N_1 \oplus \cdots \oplus N_t$  where the  $M_i$  and  $N_j$  are indecomposable finitely generated  $R$ -modules, then  $s = t$  and after a suitable reordering of the summands,  $M_i \cong N_i$  for all  $i$ . Swan [20] and later Evans [9] give examples exhibiting the failure of Krull-Schmidt for non-complete local rings. More examples can be found in works of R. Wiegand [25] and R. and S. Wiegand [26]. It is natural to ask when the Krull-Schmidt property holds for

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\*This work consists of research done as part of the author's Ph.D. thesis, conducted at the University of Nebraska-Lincoln under the direction of Roger Wiegand.

non-complete local rings. Further, when the property fails, can we measure how badly it fails?

One approach to answering this question involves the monoid of isomorphism classes of modules over a ring. In [25] R. Wiegand considered the monoid  $+(M)$  consisting of isomorphism classes of modules that are direct summands of direct sums of finitely many copies of a given finitely generated module  $M$ . He showed that  $+(M)$  is isomorphic to an expanded submonoid  $\Lambda$  of some  $\mathbb{N}^t$ , that is, there is an  $s \times t$  matrix  $\mathcal{A}$  such that  $\Lambda = \ker(\mathcal{A}) \cap \mathbb{N}^t$ . Conversely, every expanded submonoid of  $\mathbb{N}^t$  can be realized as  $+(M)$  for an appropriate local ring  $R$  and torsion-free  $R$ -module  $M$ . In this paper we investigate which monoids arise when one considers local rings of finite representation type — rings having only finitely many isomorphism classes of indecomposable finitely generated torsion-free modules.

We will always take  $(R, \mathfrak{m}, k)$  to be a one-dimensional Cohen-Macaulay local ring (where  $\mathfrak{m}$  is the unique maximal ideal and  $k$  is the residue class field  $R/\mathfrak{m}$ ). We assume throughout that  $R$  is equicharacteristic, equivalently,  $R$  contains a field. Recall that  $R$  has finite Cohen-Macaulay type provided there are, up to isomorphism, only finitely many indecomposable maximal Cohen-Macaulay modules (finitely generated torsion-free modules in this setting). If  $R$  has finite Cohen-Macaulay type, the monoid  $\mathfrak{C}(R)$  of isomorphism classes of maximal Cohen-Macaulay (MCM) modules (together with  $[0]$ ) is isomorphic to an expanded submonoid of some  $\mathbb{N}^t$ .

For each ring  $(R, \mathfrak{m}, k)$  of finite Cohen-Macaulay type, with  $k$  perfect and of characteristic different from 2, 3 and 5, we determine exactly the defining equations for the monoid  $\mathfrak{C}(R)$ . From these defining equations we are able to determine exactly when  $\mathfrak{C}(R)$  is free, that is, direct-sum decompositions of MCM  $R$ -modules have the Krull-Schmidt uniqueness property. Further, we determine which rings have the weaker property that any two representations of a MCM module as a direct sum of indecomposables have the same number of indecomposable summands. In Theorem 5.5 we show that whenever a MCM  $R$ -module can be written both as the direct sum of  $s$  indecomposable  $R$ -modules and as the direct sum of  $t$  indecomposable  $R$ -modules then  $s/t \leq 3/2$ . In Theorem 5.7 we show that when we consider all finitely generated  $R$ -modules, it is often the case that there is no bound on the ratio  $s/t$ .

If  $R$  is complete, then  $\mathfrak{C}(R)$  is the free monoid on the set of isomorphism classes of indecomposable MCM modules. In order to describe the monoid  $\mathfrak{C}(R)$  in the non-complete case we need a detailed description of the indecomposable MCM  $\hat{R}$ -modules, together with information on their ranks at the various minimal prime ideals of  $\hat{R}$ . When  $k$  is algebraically closed, we can glean this information from the Auslander-Reiten quivers for  $\hat{R}$ , which are worked out in detail in Yoshino's book [27]. To complete our study of  $\mathfrak{C}(R)$  in the incomplete case and the case where  $k$  is perfect but not algebraically closed, we analyze the maps  $\mathfrak{C}(R) \rightarrow \mathfrak{C}(\hat{R})$  and  $\mathfrak{C}(R) \rightarrow \mathfrak{C}(S)$ , where  $(S, \mathfrak{n}, \ell)$  is a flat local extension of  $R$  and  $\ell/k$  is an algebraic extension of fields.

I am grateful to the referee, whose comments helped to improve the readability of this paper. I would also like to thank the referee for recommending the current statement of Proposition 3.6.

## 2 The Hierarchy of Complete Rings of FCMT

In this section we describe all of the complete one-dimensional equicharacteristic Cohen-Macaulay local rings of finite Cohen-Macaulay type (FCMT). We recall the classification<sup>1</sup> given in [22]. We have changed the names of the rings given in [22] in order to match the more common labels given in the literature (e.g. [11], [27]).

**Theorem 2.1.** *Let  $(R, \mathfrak{m}, k)$  be a complete one-dimensional Cohen-Macaulay local ring. Further assume that  $R$  contains a field and that  $k = R/\mathfrak{m}$  is perfect of characteristic not 2, 3, or 5. Then  $R$  has finite Cohen-Macaulay type if and only if  $R$  is isomorphic to:*

1. One of the rings  $k[[x, y]]/(f)$  listed in Table 1 or
2.  $\text{End}_S(\mathfrak{n})$  where  $(S, \mathfrak{n})$  is one of the rings listed in Table 1 not of type  $(A_1)$ .

Table 1: One-dimensional Rings with FCMT

<i>type</i>	$R$	$[K : k]$
$(A_n)$	$k[[x, y]]/(x^2 - y^{n+1})$	$(n \geq 0)$ 1
$(D_n)$	$k[[x, y]]/(x^2y - y^{n-1})$	$(n \geq 4)$ 1
$(E_6)$	$k[[x, y]]/(x^3 - y^4)$	1
$(E_7)$	$k[[x, y]]/(x^3 - xy^3)$	1
$(E_8)$	$k[[x, y]]/(x^3 - y^5)$	1
$(A2_n)$	$k[[T, \xi T^{n+1}]]$	$(n \geq 1)$ 2
$(D2_n)$	$k[[ (T, U), (\xi T^n, U), (0, U^2) ]]$	$(n \geq 1)$ 2
$(D3)$	$k[[T, \xi T]]$	3

The notation in the table deserves some explanation. We denote the integral closure of  $R$  in its total quotient ring by  $\bar{R}$  and let  $K$  be a residue field of  $\bar{R}$  with maximal degree  $[K : k]$  over  $k$ . (It is known (cf. [23]) that  $\bar{R}$  is finitely generated as an  $R$ -module whenever  $R$  has FCMT.) In the classification of the hypersurfaces with FCMT there is a natural dichotomy. When  $K = k$ ,  $R$  is a ring of type  $(A_n)$ ,  $(D_n)$ ,  $(E_6)$ ,  $(E_7)$ , or  $(E_8)$ . The remaining cases occur when  $[K : k] > 1$ . When  $[K : k] = 2$  we have the rings of type  $(A2_n)$  and  $(D2_n)$  with  $\xi \in K - k$ . As shown in [22], the isomorphism class of  $R$  is independent of the choice of  $\xi \in K$  but varies with choices of  $K$ . When  $[K : k] = 3$  we have the case  $(D3)$ . Again, the choice of  $\xi \in K - k$  does not affect the isomorphism class of  $R$ .

In addition to the above explanation, a brief apology is in order. The  $(A_n)$ ,  $(D_n)$  and  $(E_n)$  labels are usually reserved for the polynomials  $x^2 + y^{n+1}$ ,  $x^2y + y^{n-1}$ ,  $x^3 + y^4$ ,  $x^3 + xy^3$ , and  $x^3 + y^5$  in the ring of formal power series  $k[[x, y]]$  where  $k$  is an algebraically closed field. In this exposition we use these titles to refer to quotients of ring  $k[[x, y]]$  even when  $k$  is not algebraically closed. Furthermore, we have

<sup>1</sup>The classification in Theorem 2.1 contains an additional ring which was omitted in [22].

replaced the ‘+’ with a ‘-’ in each of the defining equations. Of course the change of sign does not affect the isomorphism class of the ring when  $k$  is algebraically closed. Our rationale for the sign change is to give more continuity between rings with similar module structures; for example, we will see in Section 3 that as a monoid,  $\mathfrak{C}(\mathbb{R}[[x, y]]/(x^2 - y^6)) = \mathfrak{C}(\mathbb{C}[[x, y]]/(x^2 - y^6))$  while  $\mathfrak{C}(\mathbb{R}[[x, y]]/(x^2 + y^6)) \neq \mathfrak{C}(\mathbb{C}[[x, y]]/(x^2 + y^6))$ .

We now digress momentarily to discuss an oversight of the classification in [22] and we refer the reader to Table 1 of [22]. It is easy to see that the hypersurfaces given in Table 1 are also given in Table 1 of [22]. The rings  $B_n$  and  $C_n$  in [22] are of type  $(A_n)$ , with  $n$  even and odd, respectively. The rings  $E$ ,  $F$ , and  $L$  in [22] are of types  $(E_6)$ ,  $(E_7)$ , and  $(E_8)$ , respectively. The rings  $M_n$  and  $P_n$  in [22] are rings of type  $(D_n)$ , with  $n$  odd and even, respectively. The ring of type  $A$  in [22] is a discrete valuation ring, technically a ring of type  $(A_0)$ . We now give a short construction of the remaining rings from Table 1 of [22]. We say that a ring  $S$  is an *overring* of  $R$  provided that  $S$  properly contains  $R$  and is contained in  $\bar{R}$ , the integral closure of  $R$ . The following proposition, due to Bass [3, Thm. 6.2, Prop. 7.2], gives a way of finding the remaining rings with FCMT.

**Proposition 2.2.** *Let  $(R, \mathfrak{m})$  be a one-dimensional Cohen-Macaulay local ring, not a DVR. Then  $\text{End}_R(\mathfrak{m}) = \text{Hom}_R(\mathfrak{m}, R)$  is an overring of  $R$  and  $R \subsetneq \text{End}_R(\mathfrak{m}) \subseteq \bar{R}$ .*

If, in addition,  $(R, \mathfrak{m})$  is one of the complete one-dimensional hypersurfaces with FCMT not of type  $(A_1)$  then  $\text{End}_R(\mathfrak{m})$  is local. If  $(R, \mathfrak{m})$  is a ring of type  $(A_1)$  then  $\mathfrak{m}$  is not indecomposable as an  $R$ -module and hence  $\text{End}_R(\mathfrak{m})$  is not local.

It is easy to see that in Table 1 of [22], a ring of type  $G$  (resp.  $H$ ,  $N_n$ ,  $Q_n$ ) is isomorphic to  $\text{End}_R(\mathfrak{m})$  for  $R$  a ring of type  $E$  (resp.  $F$ ,  $M_n$ ,  $P_n$ ). Now consider  $R$  to be a ring of type  $L$  from Table 1 in [22]; that is,  $R = k[[T, U^2], (0, U^3)] \subseteq k[[T]] \times k[[U]]$ . This is a ring of type  $(E_7)$  listed in Table 1. If we let  $\mathfrak{m}$  denote the maximal ideal of  $R$ , then

$$\text{End}_R(\mathfrak{m}) \cong k[[T, U^2], (0, U^3), (0, U^4)]$$

This ring is missing from the classification in [22].

We now give defining equations for the rings of type  $(A_{2n})$ ,  $(D_{2n})$  and  $(D_3)$  as we will need them in the following section.

## Rings of Type $(A_{2n})$

Let  $R$  be a ring of type  $(A_{2n})$ . That is,  $R \cong k[[T, \xi T^{n+1}]]$  with  $n \geq 1$  and  $k \subseteq K$  a field extension of degree 2 with  $\xi \in K - k$ . Letting  $x = \xi T^{n+1}$  and  $y = T$ , we have  $\xi = \frac{x}{y^{n+1}}$ . Since the isomorphism class of  $R$  is not dependent on which  $\xi$  we choose, and since  $\text{char}(k) \neq 2$ , we can assume that  $\xi^2 \in k$ . Now the minimal polynomial for  $\xi$  over  $k$  has the form  $X^2 - \xi^2$ . Thus  $\left(\frac{x}{y^{n+1}}\right)^2 - \xi^2 = 0$  and hence  $x^2 - \xi^2 y^{2n+2} = 0$ . Then

$$R \cong k[[x, y]]/(x^2 - \xi^2 y^{2n+2}).$$

## Rings of Type $(D2_n)$

Now suppose  $R$  is a ring of type  $(D2_n)$ . That is,  $R \cong k[[T, U], (\xi T^n, U), (0, U^2)]$  with  $n \geq 1$  and  $k \subseteq K$  a field extension of degree 2 with  $\xi \in K - k$ . Again we choose  $\xi \in K$  such that  $\xi^2 \in k$ . Letting  $x = (\xi T^n, U)$ ,  $y = (T, U)$  and  $z = (0, U^2)$ , we can write  $R$  as

$$R \cong \frac{k[[x, y, z]]}{(\xi^2 y^{2n} - x^2 - \xi^2 z^n + z, z^n - y^{2n-2}z, yz - xz)}.$$

Note that since  $z^n = y^{2n-2}z$  and  $\xi^2 y^{2n} - x^2 = \xi^2 z^n - z$  we have

$$z = \frac{\xi^2 y^{2n} - x^2}{\xi^2 y^{2n-2} - 1}.$$

Therefore

$$y \left( \frac{\xi^2 y^{2n} - x^2}{\xi^2 y^{2n-2} - 1} \right) - x \left( \frac{\xi^2 y^{2n} - x^2}{\xi^2 y^{2n-2} - 1} \right) = 0$$

and so  $(x - y)(x^2 - \xi^2 y^{2n}) = 0$ . Thus  $R \cong k[[x, y]]/I$  is a homomorphic image of

$$\frac{k[[x, y]]}{(x - y)(x^2 - \xi^2 y^{2n})}.$$

Since  $R$  is CM and  $\mathfrak{m}$  is two-generated we know that  $I = (g)$  for some element  $g \in k[[x, y]]$ . Then  $g \mid (x - y)(x^2 - \xi^2 y^{2n})$  and so  $(g) = ((x - y)(x^2 - \xi^2 y^{2n}))$ ,  $(g) = (x - y)$  or  $(g) = (x^2 - \xi^2 y^{2n})$ . By looking at the original relations we see that later two are impossible, and hence

$$R \cong \frac{k[[x, y]]}{(x - y)(x^2 - \xi^2 y^{2n})}.$$

## Rings of Type $(D3)$

Finally, suppose that  $R$  is a ring of type  $(D3)$ . That is,  $R \cong k[[T, \xi T]]$  with  $k \subseteq K$  a field extension of degree 3 with  $\xi \in K - k$ . Let  $X^3 + aX^2 + bX + c$  be the minimal polynomial for  $\xi$  over  $k$ . Letting  $x = T$  and  $y = \xi T$ , we have  $\xi = \frac{y}{x}$  and hence  $\left(\frac{y}{x}\right)^3 + a\left(\frac{y}{x}\right)^2 + b\left(\frac{y}{x}\right) + c = 0$ . Therefore  $y^3 + ay^2x + byx^2 + cx^3 = 0$ , and

$$R \cong k[[x, y]]/(y^3 + ay^2x + byx^2 + cx^3).$$

We adopt the following notation for the non-hypersurface rings of part 2 in Theorem 2.1: we say that  $(R', \mathfrak{m}', k) := \text{End}_R(\mathfrak{m})$  is a ring of type  $(D'_n)$  (resp.  $(E'_6)$ ,  $(E'_7)$ ,  $(E'_8)$ ,  $(D2'_n)$ ,  $(D3')$ ) if  $(R, \mathfrak{m}, k)$  is a ring of type  $(D_n)$  (resp.  $(E_6)$ ,  $(E_7)$ ,  $(E_8)$ ,  $(D2_n)$ ,  $(D3)$ ) listed in Table 1. We note that if  $R$  is a ring of type  $(A_n)$  with  $n \geq 2$  then  $R' = \text{End}_R(\mathfrak{m})$  is a ring of type  $(A_{n-2})$  and that if  $R$  is a ring of type  $(A2_n)$  with  $n \geq 1$  then  $R' = \text{End}_R(\mathfrak{m})$  is a ring of type  $(A2_{n-1})$ .

*Remark 2.3.* When  $k$  is an algebraically closed field, Theorem 2.1 gives the classification in [11]. We now express the rings from [11] as flat local extensions of rings in Table 1. The reader should be warned that the numbered equations (1)-(4) below will be referred to repeatedly throughout the rest of this paper. Let  $R$  be a ring of type  $(A_{2n})$  and let  $K$  be as in the paragraph after Theorem 2.1. Then

$$S = R \otimes_k K \cong K[[x, y]] / ((x - \xi y^{n+1})(x + \xi y^{n+1})), \quad (1)$$

which is a ring of type  $(A_{2n+1})$ . Similarly, if  $R$  is a ring of type  $(D_{2n})$ , then

$$S = R \otimes_k K \cong K[[x, y]] / ((x - y)(x - \xi y^n)(x + \xi y^n)), \quad (2)$$

which is a ring of type  $(D_{2n+2})$ . Now if  $R$  is a ring of type  $(D_3)$  and  $L$  is the Galois closure of  $K/k$  we see that

$$S = R \otimes_k L \cong L[[x, y]] / (y^3 + ay^2x + byx^2 + cx^3), \quad (3)$$

which is a ring of type  $(D_4)$  since  $y^3 + ay^2x + byx^2 + cx^3$  splits into linear factors over  $L$ .

Now if  $R \cong k[[x, y]]/(f)$  is a ring of type  $(A_n)$ ,  $(D_n)$ ,  $(E_6)$ ,  $(E_7)$ , or  $(E_8)$  with  $k$  perfect and not algebraically closed we note that

$$k[[x, y]]/(f) \longrightarrow \bar{k}[[x, y]]/(f), \quad (4)$$

where  $\bar{k}$  is the algebraic closure of  $k$ , is a flat local extension of rings. The flatness of this map follows from the fact that  $\bar{k}[[x, y]]$  is faithfully flat over  $k[[x, y]]$  and from [18, Thm. 22.4].

### 3 Indecomposable MCM Modules

In this section we classify all of the indecomposable maximal Cohen-Macaulay modules over the rings listed in Theorem 2.1 — the complete one-dimensional CM local rings with FCMT. When  $k$  is algebraically closed one can find the classification in [11] and [27, Ch. 9]. When  $k$  is not algebraically closed we consider faithfully flat extensions  $R \rightarrow (S, n, \ell)$  where  $S$  is as in (1), (2), (3), or (4) of Remark 2.3.

Let  $\mathfrak{C}(T)$  denote the set of isomorphism classes of MCM modules over a ring  $T$ . We consider the map on modules  $M \mapsto M \otimes_R S$ . Since  $S$  is faithfully flat as an  $R$ -module, this map is one-to-one up to isomorphism (*cf.* [8, 2.5.8]). Moreover, taking  $N = S$  in [5, Thm. 2.17], we see that this map takes MCM  $R$ -modules to MCM  $S$ -modules. Therefore, we have an injection  $\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(S)$ . Since we know the MCM  $S$ -modules it is enough to determine which MCM  $S$ -modules are extended ( $M \cong N \otimes_R S$  for some  $R$ -module  $N$ ) in order to classify all MCM  $R$ -modules. Moreover, the following result implies that extended  $S$ -modules with no extended proper direct summands are extended from indecomposable  $R$ -modules. We recall that the Krull-Schmidt property holds for complete local rings, [2], and direct sum cancellation holds over all local rings, [9]. We write  $M \mid N$  to indicate that  $M$  is isomorphic to a direct summand of  $N$ .

**Lemma 3.1.** *Let  $R \rightarrow S$  be a faithfully flat extension of Noetherian rings. Suppose that the Krull-Schmidt theorem holds for finitely generated modules over  $R$  and direct sum cancellation of finitely generated modules holds over  $S$ . Then, given finitely generated  $R$ -modules  $M$  and  $N$ ,*

$M \mid N$  as  $R$ -modules if and only if  $(S \otimes_R M) \mid (S \otimes_R N)$  as  $S$ -modules.

*Proof.* It is clear that  $M \mid N$  implies  $(S \otimes_R M) \mid (S \otimes_R N)$ . Now suppose that  $(S \otimes_R M) \mid (S \otimes_R N)$ . From [24, Lem. 1.2] we know that  $M \mid N^r$  for some  $r > 1$ . We induct on the length of the direct-sum decomposition of  $M$ . If  $M$  is indecomposable,  $M \mid N$  by the Krull-Schmidt theorem. Otherwise, write  $M = M_1 \oplus M_2$  where  $M_1$  is indecomposable. Then  $M_1 \mid N$ , say  $N \cong M_1 \oplus N_1$ . Now

$$[(S \otimes_R M_1) \oplus (S \otimes_R M_2)] \mid [(S \otimes_R M_1) \oplus (S \otimes_R N_2)],$$

and by cancellation over  $S$ ,  $(S \otimes_R M_2) \mid (S \otimes_R N_2)$ . By the induction hypothesis  $M_2 \mid N_2$ , and hence  $M \mid N$ .  $\square$

We first consider  $R$  to be a ring of type  $(A_n)$ ,  $(D_n)$ ,  $(E_6)$ ,  $(E_7)$ , or  $(E_8)$ . If, in addition,  $R$  has algebraically closed residue field  $k$ , we turn to [27, Ch. 9] for a complete description (by way of matrix factorizations) of each of the indecomposable MCM  $R$ -modules.

If  $R$  is a ring of type  $(A_n)$ ,  $(D_n)$ ,  $(E_6)$ ,  $(E_7)$ , or  $(E_8)$  and  $k$  is not algebraically closed, we consider the flat, local extension (4) of Remark 2.3 and the following lemma.

**Lemma 3.2.** *Let  $R = k[[x, y]]/(f)$  and  $S = \bar{k}[[x, y]]/(f)$  be rings of (the same) type  $(A_n)$ ,  $(D_n)$ ,  $(E_6)$ ,  $(E_7)$ , or  $(E_8)$ . Then there is a one-to-one correspondence between the indecomposable MCM  $R$ -modules and the indecomposable MCM  $S$ -modules given by  ${}_R M \mapsto M \otimes_R S$ .*

*Proof.* The indecomposable MCM  $S$ -modules are explicitly calculated in [27, Ch. 9] as the cokernels of certain matrices. The entries of these matrices are all monomials in  $x$  and  $y$  (where  $\mathfrak{n} = (x, y)$  is the maximal ideal of  $S$ ) with coefficients in  $\bar{k}$ .

The matrix factorizations listed in [27] are dependent on the ‘+’ in the polynomial  $f$  when  $S = \bar{k}[[x, y]]/(f)$ . However, there are obvious isomorphisms which ‘change’ the sign of the defining polynomial  $f$ . For example, the map that sends  $x$  to  $ix$  (where  $i = \sqrt{-1}$ ) and leaves  $y$  fixed induces an isomorphism of a ring of type  $(A_{2n+1})$  which sends  $f^+ = x^2 + y^{2n+2}$  to  $f^- = x^2 - y^{2n+2}$ .

It is easy to check that these maps induce maps from the matrix factorizations of  $\bar{k}[[x, y]]/(f^+)$  to matrix factorizations of  $\bar{k}[[x, y]]/(f^-)$ . Moreover, these new matrix factorizations are isomorphic (via an equivalence of matrices) to matrix factorizations whose entries are contained in  $R \cong k[[x, y]]/(f)$ . Thus every indecomposable MCM  $S$ -module  $N$  is extended from a finitely generated  $R$ -module  $M$ . Obviously  $M$  is indecomposable, and  $M$  is a MCM module by [5, Prop. 1.2.16, Thm. A.11].  $\square$

For the remainder of this section  $R$  is a ring of type  $(A_{2n})$ ,  $(D_{2n})$ , or  $(D_3)$ . Furthermore, we will always take  $S$  to be a flat extension of  $R$  defined as in (1), (2) or (3) of Remark 2.3.

We note that  $R \rightarrow S$  is a flat local extension of rings and by Lemma 3.2 we know exactly the MCM  $S$ -modules. We will now determine which of the MCM  $S$ -modules are extended from MCM  $R$ -modules and thus determine the MCM  $R$ -modules.

We begin by stating the following proposition, whose proof consists of the essential details in the proof of the Krull-Schmidt theorem (*cf.* [19, Sec. 5.4]). Recall that a ring  $E$  is “local” in the non-commutative sense provided that  $E/J(E)$  (where  $J(E)$  is the Jacobson radical of  $E$ ) is a division ring, equivalently,  $E$  has a unique maximal left ideal.

**Proposition 3.3.** *Let  $(R, \mathfrak{m})$  be a local ring and suppose  $X_1 \oplus \cdots \oplus X_s \cong Y_1 \oplus \cdots \oplus Y_t$ , where  $X_i$  and  $Y_j$  are indecomposable finitely generated  $R$ -modules. Suppose  $\text{End}_R(X_1)$  has a unique maximal left ideal. Then  $X_1 \cong Y_j$  for some  $j$ .*

Since direct sum cancellation holds over local rings [9] we have:

**Corollary 3.4.** *Let  $(R, \mathfrak{m})$  be a local ring and  $M$  a finitely generated  $R$ -module. If  $\text{End}_R(X)$  has a unique maximal left ideal and  $X$  occurs with multiplicity  $\mu$  in some direct sum decomposition of  $M$  then  $X$  occurs with multiplicity  $\mu$  in every direct sum decomposition of  $M$ .*

We note in particular that if  $C$  is a cyclic module over a local ring  $R$  then  $\text{End}_R(C)$  is local. We define the *multiplicity of  $C$  in  $M$* ,  $\mu_M(C)$ , to be the number of copies of  $C$  that occur in a direct sum decomposition of  $M$ . By Corollary 3.4,  $\mu_M(C)$  is well-defined.

To simplify the notation, we write  $L$  in place of  $K$  when  $R$  is a ring of type  $(A2_n)$  or  $(D2_n)$ . Then, when  $R$  is a ring of type  $(A2_n)$ ,  $(D2_n)$  or  $(D3)$ , we have that  $R \rightarrow S := R \otimes_k L$  is a Galois extension of rings with Galois group  $G := \text{Gal}_R(S) = \text{Gal}_k(L)$  (*cf.* [21, Prop. 2.4]). That is, (1)  $R$  is the  $G$ -invariant subring of  $S$  and (2) for all subgroups  $H$  of  $G$  and all  $H$ -stable ideals  $I$  of  $S$  with  $I \neq S$ ,  $H$  acts faithfully on  $S/I$ . Recall the following fact about Galois extensions. A proof may be found in [4, Sec. 2.2].

**Proposition 3.5.** *Let  $A \subseteq B$  be a finite Galois extension of rings with Galois group  $G$ . Let  $Q$  be a prime ideal of  $A$  and let  $\mathcal{P} = \{P_1, \dots, P_s\}$  be the set of primes of  $B$  lying over  $Q$ . Then  $G$  acts transitively on  $\mathcal{P}$ .*

We say that an action of the Galois group  $G$  on an  $S$ -module  $M$  is *semi-linear* if for each  $g \in G$ ,

$$g(sm) = g(s)g(m)$$

for all  $s \in S$  and  $m \in M$ . We know that the  $S$ -module  $M$  is extended if and only if  $G$  acts semi-linearly on  $M$ . Indeed, if  $G$  acts semi-linearly on  $M$ , then  $M \cong S \otimes_R M^G$  where  $M^G$  is the fixed module (*cf.* [21, Prop. 2.5]). Conversely, if  $M = S \otimes_R N$  for some  $R$ -module  $N$ , then  $G$  acts semi-linearly on  $M$  via  $g(s \otimes n) = g(s) \otimes n$ . The following proposition allows us to determine the cyclic MCM  $R$ -modules.

**Proposition 3.6.** *Let  $A \subset B$  be a Galois extension of rings with Galois group  $G$  ( $A = B^G$ ). Let  $Q$  be a prime of  $A$  and let  $\mathcal{P} = \{P_1, \dots, P_s\}$  be the set of primes of  $B$  lying over  $Q$ . Also, let  $I$  be an ideal of  $B$  which is stable under the action of  $G$ .*

Let  $n_1, n_2, \dots, n_s$  be non-negative integers. Also, set  $X_i = B/P_i$ ,  $Y_i = B/(P_i \cap I)$ , and  $Z_i = B/\left(\bigcap_{j \neq i} P_j\right)$ .

1. The  $B$ -module  $\bigoplus_{i=1}^s X_i^{n_i}$  is extended if and only if  $n_1 = n_2 = \dots = n_s$ .
2. The  $B$ -module  $\bigoplus_{i=1}^s Y_i^{n_i}$  is extended if and only if  $n_1 = n_2 = \dots = n_s$ .
3. The  $B$ -module  $\bigoplus_{i=1}^s Z_i^{n_i}$  is extended if and only if  $n_1 = n_2 = \dots = n_s$ .
4. If  $M$  is an extended  $B$ -module then  $\mu_M(X_1) = \mu_M(X_2) = \dots = \mu_M(X_s)$ ,  $\mu_M(Y_1) = \mu_M(Y_2) = \dots = \mu_M(Y_s)$  and  $\mu_M(Z_1) = \mu_M(Z_2) = \dots = \mu_M(Z_s)$ .

*Proof.* We now address the proof of 1 along with the first part of 4. To show that the  $B$ -module  $M = \bigoplus_{i=1}^s (B/P_i)^{n_i}$  is extended whenever  $n_1 = n_2 = \dots = n_s$  it suffices to assume that  $n_1 = n_2 = \dots = n_s = 1$ . Let  $g \in G$ . For each  $i \in \{1, 2, \dots, s\}$  there is a unique index  $g(i) \in \{1, 2, \dots, s\}$  such that  $g(P_i) = P_{g(i)}$ . Then  $G$  has a natural action on the  $B$ -module  $M$  defined by

$$g(b_1 + P_1, b_2 + P_2, \dots, b_s + P_s) \mapsto (g(b_{g(1)}) + P_1, g(b_{g(2)}) + P_2, \dots, g(b_{g(s)}) + P_s).$$

This action is clearly semi-linear and thus  $M$  is an extended module.

If  $M$  is any extended  $B$ -module then  $G = \langle g \rangle$  acts semi-linearly on  $M$ . Suppose that  ${}_B M$  decomposes into a direct sum of  $B$ -modules  $N \oplus N'$ . It is easily checked that  $g(N)$  and  $g(N')$  are  $B$ -submodules of  $M$  and that  $g(N) \oplus g(N') = M$ .

Using Proposition 3.5, choose  $g_1, g_2, \dots, g_{s-1} \in G$  such that  $g_i(P_i) = P_{i+1}$ . If  $N \cong B/P_1$ , the compositions

$$\begin{aligned} g_1(N) &\xrightarrow{g_1^{-1}} N \xrightarrow{\cong} B/P_1 \xrightarrow{g_1} B/P_2 \\ g_2 g_1(N) &\xrightarrow{(g_2 g_1)^{-1}} N \xrightarrow{\cong} B/P_1 \xrightarrow{g_2 g_1} B/P_3 \\ &\vdots \\ g_s \cdots g_2 g_1(N) &\xrightarrow{(g_s \cdots g_2 g_1)^{-1}} N \xrightarrow{\cong} B/P_1 \xrightarrow{g_s \cdots g_2 g_1} B/P_s \end{aligned}$$

give isomorphisms  $g_s \cdots g_1(N) \cong B/P_s$  for all  $i \in \{1, 2, \dots, s\}$ . Thus, if  $M$  has a direct summand isomorphic to  $B/P_1$ ,  $M$  also has direct summands isomorphic to  $B/P_i$  for all  $i \in \{1, 2, \dots, s\}$ . Using symmetry we conclude that  $\mu_M(B/P_1) = \mu_M(B/P_2) = \dots = \mu_M(B/P_s)$ . This proves the first part of 4. and the converse of 1. follows immediately. The other assertions are proved similarly.  $\square$

In what follows we exhibit matrix factorizations over  $S$  which are also matrix factorizations over  $R$ . Thus we show that the MCM  $S$ -modules corresponding to

the cokernels of these matrices are extended from MCM  $R$ -modules. The next proposition summarizes several basic facts about matrix factorizations that we will need when classifying MCM modules. Recall that a matrix factorization  $(\phi, \psi)$  is *reduced* provided the entries of  $\phi$  and  $\psi$  are not units (cf. [27, Ch. 7]).

**Proposition 3.7.** *Let  $B$  be a regular local ring, let  $f$  be a non-zero non-unit of  $B$ , and let  $A = B/(f)$ , a hypersurface. Let  $(\phi, \psi)$  be a reduced matrix factorization of  $f$ .*

1.  $\text{coker}(\psi) \cong \text{syz}_R^1(\text{coker}(\phi))$ .
2.  $\text{syz}_R^1(\text{coker}(\phi)) \cong \text{coker}(\psi)$ .
3. *If  $\text{coker}(\phi)$  is indecomposable, then so is  $\text{coker}(\psi)$ .*
4.  $B^n \xrightarrow{\phi} B^n \rightarrow \text{coker}(\phi) \rightarrow 0$  is a minimal free presentation for  $\text{coker}(\phi)$ . *If  $(\phi', \psi')$  is another reduced matrix factorization of  $f$  with  $\text{coker}(\phi) \cong \text{coker}(\phi')$ , then  $I_j(\text{coker}(\phi)) = I_j(\text{coker}(\phi'))$  for all  $j$ , where  $I_j$  denotes the  $j$ th Fitting ideal of a module.*
5. *If  $M = \text{coker}(\phi)$  and  $N = \text{coker}(\psi)$ , then  $M \oplus N = \text{coker} \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}$ .*

*Proof.* Proofs of (1), (2) and (3) can be found in [27] and a proof of (4) can be found in [6]. The proof of part (5) is trivial.  $\square$

**Proposition 3.8.** *Let  $R$  be a reduced one-dimensional local ring. Let  $M_R$  be a MCM  $R$ -module. Then  $(0 :_R M) = \cap \text{Ass}(M)$ . In particular, if  $M$  is cyclic, then  $M \cong R/I$  where  $I$  is an intersection of minimal primes of  $R$ .*

*Proof.* Let  $r \in (0 :_R M)$  and let  $P \in \text{Ass}(M)$ . Then there is an injection  $R/P \hookrightarrow M$  and thus  $r(R/P) = 0$ ; that is,  $r \in P$ . As  $P \in \text{Ass}(M)$  was arbitrary,  $r \in \cap \text{Ass}(M)$ . Now let  $r \in \cap \text{Ass}(M)$ . We want to show that  $r \in (0 :_R M)$ . Since  $M$  is torsion-free it is enough to show that  $(rM)_Q = 0$  for all minimal primes  $Q$  of  $R$ . Since  $R$  is one-dimensional and  $\mathfrak{m} \notin \text{Ass}(M)$ , we know that  $\text{Ass}(M) = \text{Supp}(M)$ . If  $Q \notin \text{Ass}(M)$  then  $M_Q = 0$  and thus  $(rM)_Q = 0$ . On the other hand, if  $Q \in \text{Ass}(M)$  then  $r \in Q$ , and as  $R_Q$  is a field,  $rR_Q = 0$ . Therefore  $(rM)_Q = 0$  and hence  $r \in (0 : M)$ .  $\square$

We are now ready to determine the MCM  $R$ -modules when  $R$  is a ring of type  $(A_{2n})$ ,  $(D_{2n})$ , or  $(D_3)$ .

## Rings of type $(A_{2n})$

Suppose that  $R$  is a ring of type  $(A_{2n})$ , a domain. Then, by (1) of Remark 2.3,  $S := L \otimes_k R$  is a ring of type  $(A_{2n+1})$  with minimal primes  $P_1 = (x - \xi y^{n+1})$  and  $P_2 = (x + \xi y^{n+1})$ . We know, from [27, 9.9] and Lemma 3.2, that there are  $n$  non-cyclic indecomposable MCM  $S$ -modules given by  $2 \times 2$  matrix factorizations of  $x^2 - \xi^2 y^{2n+2}$ . Consider the matrix factorizations  $(\phi_j, \psi_j)$  for  $x^2 - \xi^2 y^{2n+2}$ :

$$\phi_j = \begin{bmatrix} -\xi^2 y^{2n+2-j} & x \\ x & -y^j \end{bmatrix} \quad (1 \leq j \leq 2n+2) \quad (5)$$

$$\psi_j = \begin{bmatrix} y^j & x \\ x & \xi^2 y^{2n+2-j} \end{bmatrix} \quad (1 \leq j \leq 2n+2)$$

Let  $M_i = \text{coker}(\phi_j)$  and  $N_j = \text{coker}(\psi_j)$ . It is easy to see that  $M_j \cong N_j$ ,  $M_j \cong M_{2n+2-j}$ , and that  $M_{n+1} \cong S/(x - \xi^2 y^{n+1}) \oplus S/(x + \xi y^{n+1})$ . For  $1 \leq j \leq n$ , we have  $I_1(M_j) = (x, y^j)$ . It follows that  $M_i \not\cong M_j$  unless  $i = j$ . If  $M_j$  were decomposable, it would be a direct sum of two cyclic modules. Since, by Proposition 3.8, the only cyclic MCM  $S$ -modules are  $S$ ,  $S/P_1$  and  $S/P_2$ , Proposition 3.7 shows that such a decomposition is impossible. We note that each  $M_j$  is extended from an  $R$ -module, since the entries of  $\phi_j$  are monomials in generators for  $\mathfrak{m}$  with coefficients in  $k$ . From Proposition 3.6,  $S/P_1$  and  $S/P_2$  are not extended and  $\mu_M(S/P_1) = \mu_M(S/P_2)$  for all extended  $S$ -modules  $M$ . Thus the indecomposable MCM  $R$ -modules extend to the following  $S$ -modules:

$$S, M_1, \dots, M_n, \text{ and } M_{n+1} \cong S/P_1 \oplus S/P_2. \quad (6)$$

## Rings of type $(D2_n)$

Now suppose that  $R$  is a ring of type  $(D2_n)$ . Then  $R$  has two minimal primes,  $Q_1 = (x - y)$  and  $Q_2 = (x^2 - \xi^2 y^{2n})$ . From (2) of Remark 2.3,  $S$  is a ring of type  $(D2_{2n+2})$  with three minimal primes  $P_1 = (x - y)$ ,  $P_2 = (x - \xi y^n)$  and  $P_3 = (x + \xi y^n)$  satisfying  $P_2 \cap R = P_3 \cap R = Q_2$ . We now give matrix factorizations of  $(x - y)(\xi^2 y^{2n} - x^2)$  which will in turn give us indecomposable MCM  $S$ -modules. Let  $j$  run from 1 to  $n$  and let  $i$  run from 1 to  $n - 1$ . Now consider the following  $4n - 2$  matrix factorizations  $(\alpha_j, \beta_j)$ ,  $(\beta_j, \alpha_j)$ ,  $(\phi_i, \psi_i)$ , and  $(\psi_i, \phi_i)$ :

$$\alpha_j = \begin{bmatrix} \xi^2 y^{2n+1-j} & x(x - y) \\ x & y^{j-1}(x - y) \end{bmatrix} \quad \beta_j = \begin{bmatrix} y^{j-1}(x - y) & -x(x - y) \\ -x & \xi^2 y^{2n+1-j} \end{bmatrix} \quad (7)$$

$$\phi_i = \begin{bmatrix} \xi^2 y^{2n-i} & x \\ x & y^i \end{bmatrix} \quad \psi_i = \begin{bmatrix} y^i(x - y) & -x(x - y) \\ -x(x - y) & \xi^2 y^{2n-i}(x - y) \end{bmatrix}.$$

Now let  $X_j = \text{coker}(\alpha_j)$ ,  $Y_j = \text{coker}(\beta_j)$ ,  $M_i = \text{coker}(\phi_i)$ , and  $N_i = \text{coker}(\psi_i)$ . Note that when  $n = 1$  we only consider the modules  $X_1$  and  $Y_1$ . We know from [27, 9.12] and Lemma 3.2 that  $S$  has  $4n - 2$  non-cyclic indecomposable MCM modules, and thus we need to justify that the  $4n - 2$  modules given above are distinct and indecomposable.

We note that  $I_2(\alpha_j) = I_2(\beta_j) = (\xi^2 y^{2n} - x^2)(x - y)$  for all  $j$ . Also,  $I_2(\phi_i) = (\xi^2 y^{2n} - x^2)$  and  $I_2(\psi_i) = (\xi^2 y^{2n} - x^2)(x - y)^2$  for all  $i$ . By Proposition 3.7, for any  $i \in \{1, \dots, n - 1\}$  and  $j \in \{1, \dots, n\}$  we have  $X_j \not\cong M_i$ ,  $X_j \not\cong N_i$ ,  $Y_j \not\cong M_i$ , and  $M_i \not\cong N_j$ . From Proposition 3.7,  $Y_j \cong \text{syz}_S^1(X_j)$  and [27, 9.12] implies that  $X_j \not\cong Y_j$  for all  $j$  since no indecomposable MCM module is isomorphic to its AR-translation.

Note that  $I_1(\alpha_j) = I_1(\beta_j) = I_1(\phi_j) = (x, y^j)$  and that  $I_1(\psi_j) = (x, y^j)(x - y)$ . Then again by Proposition 3.7 we can conclude that the  $X_j$ ,  $Y_j$ ,  $M_i$ , and  $N_i$  are pairwise non-isomorphic. Thus the  $4n - 2$  MCM  $S$ -modules given by the matrix factorizations in (7) are distinct, and we need only show they are indecomposable.

The cyclic MCM  $S$ -modules are  $S$  and the cokernels of the following six  $1 \times 1$  matrices:

$$\begin{aligned} A &= [x - y] & B &= [\xi y^n - x] & C &= [\xi y^n + x] \\ D &= [(x - y)(\xi y^n - x)] & E &= [(x - y)(\xi y^n + x)] & F &= [\xi^2 y^{2n} - x^2] \end{aligned}$$

Suppose, temporarily, that  $n \geq 2$ . To see that each of these  $4n - 2$  modules is indecomposable it is enough, by Proposition 3.7, to show that the  $X_j$  and  $M_i$  are all indecomposable. Note that  $x$  belongs to the first Fitting ideal of each of these modules. On the other hand, the sum of the first Fitting ideals of all of the cyclic MCM modules save  $B$  (respectively  $C$ ) is  $(x - y, \xi y^n + x)$ , respectively,  $(x - y, \xi y^n - x)$ . Since neither of these ideals contains  $x$ , we conclude that the only possible decomposition of one of our MCM modules, say  $Z$ , is  $Z \cong \text{coker}(B) \oplus \text{coker}(C)$ . Consideration of the second Fitting ideals rules out all cases except possibly  $M_i$  for some  $i \leq n - 1$ . But  $I_1(B) + I_1(C) = (x, y^n)$  while  $I_1(M_i) = (x, y^i)$ , so the possibility  $M_i \cong \text{coker}(B) \oplus \text{coker}(C)$  is ruled out as well.

We now deal with the case  $n = 1$ . By Proposition 3.7, we only need to show that  $\text{coker}(\alpha_1)$  is indecomposable. By considering the second Fitting ideals we see that the only possibilities for a decomposition of  $\text{coker}(\alpha_1)$  are as  $\text{coker}(A) \oplus \text{coker}(F)$ ,  $\text{coker}(B) \oplus \text{coker}(E)$  or  $\text{coker}(C) \oplus \text{coker}(D)$ . A consideration of the first Fitting ideals of these modules rules out each of these cases. Thus  $\text{coker}(\alpha_1)$  and  $\text{coker}(\beta_1)$  along with the seven cyclic modules forms a complete list of the indecomposable MCM  $S$ -modules.

Note that each of these  $4n - 2$  modules is extended from a MCM  $R$ -module since the entries of the matrices  $\phi_j$ ,  $\psi_j$ ,  $\alpha_j$ , and  $\beta_j$  are defined over  $R$ . Applying Proposition 3.6, we see that the indecomposable MCM  $R$ -modules extend to the following  $S$ -modules:

$$M_1, \dots, M_{n-1}, N_1, \dots, N_{n-1}, X_1, \dots, X_n, Y_1, \dots, Y_n, \quad (8)$$

$$S, S/P_1, S/(P_2 \cap P_3), A \cong S/P_2 \oplus S/P_3, \text{ and } B \cong S/(P_1 \cap P_2) \oplus S/(P_1 \cap P_3)$$

## Rings of type (D3)

Finally, suppose that  $R$  has type (D3). Then by (3) of Remark 2.3  $S$  is a ring of type (D4). Then  $S$  has two non-cyclic indecomposable MCM modules given by  $2 \times 2$  matrix factorizations of  $y^3 + ay^2x + byx^2 + cx^3$ . Consider the matrix factorization  $(\phi, \psi)$ :

$$\phi = \begin{bmatrix} y^2 & x^2 \\ -by - cx & y + ax \end{bmatrix} \text{ and } \psi = \begin{bmatrix} y + ax & -x^2 \\ by + cx & y^2 \end{bmatrix} \quad (9)$$

Then  $X = \text{coker}(\phi)$  and  $Y = \text{coker}(\psi)$  are the non-cyclic indecomposable MCM  $S$ -modules. We note that  $X$  and  $Y$  are extended since the entries of  $\phi$  and  $\psi$  are contained in  $\mathfrak{m}$  with coefficients in  $k$ . Applying Proposition 3.6 we see that the indecomposable MCM  $R$ -modules extend to the following  $S$ -modules

$$S, X, Y, A \cong S/P_1 \oplus S/P_2 \oplus S/P_3, \text{ and} \quad (10)$$

$$B \cong S/(P_1 \cap P_2) \oplus S/(P_1 \cap P_3) \oplus S/(P_2 \cap P_3)$$

We have now determined the indecomposable MCM  $R$ -modules when  $R$  is as in part 1 of Theorem 2.1. We turn now to the non-hypersurfaces. That is, we take  $R' \cong \text{End}_S(\mathfrak{n})$  as in part 2 of Theorem 2.1.

Note that if  $M$  is a MCM  $R'$ -module,  $M$  has a natural  $R$ -module structure since  $R \subset R'$ . Then as  $M$  is torsion-free as an  $R'$ -module,  $M$  must be torsion-free as an  $R$ -module. We also point out that each  $R$ -isomorphism is an  $R'$ -isomorphism and that if  $M$  is indecomposable as a  $R$ -module, then  $M$  is indecomposable as an  $R'$ -module. The following result gives us a complete list of indecomposable MCM  $R'$ -modules.

**Proposition 3.9.** *Let  $R$  be a one-dimensional Gorenstein ring and assume that  $\text{End}_R(\mathfrak{m})$  is local.*

1. [3, Prop. 7.2] *Let  $M$  be a MCM  $R$ -module and assume that  $M$  has no direct summand isomorphic to  $R$ . Then  $M$  is a MCM  $\text{End}_R(\mathfrak{m})$ -module.*
2. [27, Lemma 9.4] *Every indecomposable MCM  $\text{End}_R(\mathfrak{m})$ -module is an indecomposable MCM  $R$ -module.*

*Note that  $R$  has no structure as a module over  $\text{End}_R(\mathfrak{m})$  compatible with the natural inclusion  $R \hookrightarrow \text{End}_R(\mathfrak{m})$ .*

Thus the indecomposable MCM  $\text{End}_R(\mathfrak{m})$ -modules are exactly the indecomposable MCM  $R$ -modules other than  $R$  itself.

Now we have classified all of the indecomposable MCM modules over the rings classified in Theorem 2.1. When  $R$  is a hypersurface with algebraically closed residue field, the modules are listed in [27, Chap. 9]. For the rings of type  $(A_n)$ ,  $(D_n)$ ,  $(E_6)$ ,  $(E_7)$ , and  $(E_8)$  where the residue field is not necessarily algebraically closed we appeal to Lemma 3.2 and the classification in [27]. When  $R$  is a ring of type  $(A_{2n})$ ,  $(D_{2n})$ , or  $(D_3)$  the indecomposable modules are those that extend to the modules listed in (6), (8), and (10). When  $R$  is a non-hypersurface, Proposition 3.9 and the classifications of the modules over the hypersurfaces give us the complete list of indecomposable MCM  $R$ -modules.

## 4 Ranks of Indecomposables

Throughout this section we assume that  $(R, \mathfrak{m}, k)$  is one of the complete local rings from Theorem 2.1. Since  $R$  is a one-dimensional CM local ring of finite CM type,  $R$  is reduced (*cf.* [7]) and hence  $R_P$  is a field for all minimal primes  $P$  of  $R$ . Recall that if  $P$  is a minimal prime of  $R$  and  $M$  is an  $R$ -module, the *rank*  $\text{rank}_P(M)$  of  $M$  at  $P$  is defined to be the vector space dimension of  $M_P$  over the field  $R_P$ . For the remainder of this section we take  $P_1, \dots, P_s$  to be the minimal primes of  $R$  (as listed in the previous section). Then the *rank* of an  $R$ -module  $M$  is the  $s$ -tuple  $\text{rank}(M) = (r_1, \dots, r_s)$  where  $r_i = \text{rank}_{P_i}(M)$ .

We begin by calculating the ranks of the indecomposable modules over rings of types  $(A_n)$ ,  $(D_n)$ ,  $(E_6)$ ,  $(E_7)$ , and  $(E_8)$  using the AR-sequences given in [27]. We illustrate this by computing the ranks for a ring  $R$  of type  $(D_6)$ . The remaining

calculations can be carried out in a similar fashion and are summarized in Theorem 4.2 below.

Let  $R$  be a ring isomorphic to  $k[[x, y]]/(x^2y - y^5)$ . Then from [27, 9.12] we have the following AR-sequences:

$$\begin{array}{ll} 0 \rightarrow R/P_1 \rightarrow X_1 \rightarrow R/(P_2 \cap P_3) \rightarrow 0 & 0 \rightarrow R/(P_2 \cap P_3) \rightarrow Y_1 \rightarrow R/P_1 \rightarrow 0 \\ 0 \rightarrow X_1 \rightarrow R/(P_2 \cap P_3) \oplus N_1 \rightarrow Y_1 \rightarrow 0 & 0 \rightarrow Y_1 \rightarrow R/P_1 \oplus M_1 \rightarrow X_1 \rightarrow 0 \\ 0 \rightarrow M_1 \rightarrow X_1 \oplus Y_2 \rightarrow N_1 \rightarrow 0 & 0 \rightarrow N_1 \rightarrow Y_1 \oplus X_2 \rightarrow M_1 \rightarrow 0 \end{array}$$

For our purposes we need only know that these are short exact sequences and that the rank function is additive along such sequences. It is easy to see that  $\text{rank}(R) = (1, 1, 1)$  and that:

$$\begin{array}{ll} \text{rank}(R/P_1) = (1, 0, 0) & \text{rank}(R/(P_2 \cap P_3)) = (0, 1, 1) \\ \text{rank}(R/P_2) = (0, 1, 0) & \text{rank}(R/(P_1 \cap P_3)) = (1, 0, 1) \\ \text{rank}(R/P_3) = (0, 0, 1) & \text{rank}(R/(P_1 \cap P_2)) = (1, 1, 0) \end{array}$$

Using this information and the additivity of the rank function along the AR-sequences, we see that  $\text{rank}(X_1) = \text{rank}(X_2) = \text{rank}(Y_1) = \text{rank}(Y_2) = (1, 1, 1)$ ,  $\text{rank}(M_1) = (0, 1, 1)$ , and  $\text{rank}(N_1) = (2, 1, 1)$ .

We next determine the ranks of the indecomposable modules described in the previous section; that is, we assume that  $R$  is a ring of type  $(A_{2n})$ ,  $(D_{2n})$  or  $(D_3)$ .

**Lemma 4.1.** *Let  $R \rightarrow S$  be an extension of one-dimensional reduced local rings. Let  $P$  be a minimal prime of  $R$  and let  $Q$  be a minimal prime of  $S$  lying over  $P$ . If  ${}_R N$  is a finitely generated torsion-free  $R$ -module, then*

$$\text{rank}_P(N) = \text{rank}_Q(N \otimes_R S)$$

*Proof.* Let  $r = \text{rank}_P(N)$  and  $s = \text{rank}_Q(N \otimes_R S)$ . Then  $N_P \cong R_P^r$  and  $(N \otimes_R S)_Q \cong S_Q^s$ . Now  $(N \otimes_R S)_Q \cong N_P \otimes_{R_P} S_Q \cong R_P^r \otimes_{R_P} S_Q \cong S_Q^r$  and hence  $S_Q^r \cong S_Q^s$ . Thus  $r = s$ .  $\square$

If  $R$  is a ring of type  $(A_n)$ ,  $(D_n)$ ,  $(E_6)$ ,  $(E_7)$ , or  $(E_8)$  with a non-algebraically closed residue field we appeal to Lemma 3.2 and Lemma 4.1 to see that the ranks of the indecomposable MCM modules are exactly the same as when the residue field is algebraically closed. Now we need to use Lemma 4.1 to calculate the ranks of the indecomposable MCM modules for the rings of type  $(A_{2n})$ ,  $(D_{2n})$  and  $(D_3)$ .

## Rings of Type $(A_{2n})$

Let  $(R, \mathfrak{m}, k)$  be a ring of type  $(A_{2n})$ . Then  $S := R \otimes_k K$  is a ring of type  $(A_{2n+1})$ . Then the indecomposable  $R$ -modules extend to the following  $S$ -modules:

$$S, M_1, \dots, M_n, \text{ and } M_{n+1} \cong S/P_1 \oplus S/P_2,$$

each having rank  $(1, 1)$  over  $S$ . Since  $R$  is a domain, both minimal primes of  $S$  lie over  $(0)_R$ . By Lemma 4.1 we know that each indecomposable MCM  $R$ -module has rank one.

## Rings of Type $(D2_n)$

Now suppose that  $(R, \mathfrak{m}, k)$  is a ring of type  $(D2_n)$ . Then  $S := R \otimes_k K$  is a ring of type  $(D_{2n+2})$ . The indecomposable  $R$ -modules extend to the  $S$ -modules  $M_1, \dots, M_{n-1}; N_1, \dots, N_{n-1}; X_1, \dots, X_n; Y_1, \dots, Y_n; S, S/P_1, S/(P_2 \cap P_3), A \cong S/P_2 \oplus S/P_3$ , and  $B \cong S/(P_1 \cap P_2) \oplus S/(P_1 \cap P_3)$ . The ranks of these  $S$ -modules are  $(0, 1, 1), \dots, (0, 1, 1); (2, 1, 1), \dots, (2, 1, 1); (1, 1, 1), \dots, (1, 1, 1); (1, 1, 1), \dots, (1, 1, 1); (1, 1, 1), (1, 0, 0), (0, 1, 1), (0, 1, 1)$ , and  $(2, 1, 1)$ , respectively. The ordering of the minimal primes is such that the second and third lie over a common prime of  $R$ . Applying Lemma 4.1, we see that the ranks of the corresponding indecomposable MCM  $R$ -modules are  $(0, 1), \dots, (0, 1); (2, 1), \dots, (2, 1); (1, 1), \dots, (1, 1); (1, 1), \dots, (1, 1); (1, 1), (1, 0), (0, 1), (0, 1)$ , and  $(2, 1)$ , respectively.

## Rings of Type $(D3)$

Finally, suppose that  $R$  is a ring of type  $(D3)$ , so that  $S := R \otimes_k L$  is a ring of type  $(D4)$ . Then we know from the previous section that the indecomposable MCM  $R$ -module extend to the  $S$ -modules  $S, X, Y, A \cong S/P_1 \oplus S/P_2 \oplus S/P_3$ , and  $B \cong S/(P_1 \cap P_2) \oplus S/(P_1 \cap P_3) \oplus S/(P_2 \cap P_3)$ . As  $S$ -modules,  $S, X, Y$ , and  $A$  have constant rank 1 and  $B$  has constant rank 2. Lemma 4.1 implies that the corresponding indecomposable MCM  $R$ -modules have the same ranks.

## Non-hypersurface Rings

Now suppose that  $R'$  is one of the non-hypersurfaces of finite CM type (Case 2 of Theorem 2.1). Then  $R' \cong \text{End}_R(\mathfrak{m})$  where  $(R, \mathfrak{m})$  is a hypersurface of finite CM type. By Proposition 3.9 the indecomposable MCM  $R'$ -modules are exactly the indecomposable MCM  $R$ -modules other than  $R$ . Thus we know the ranks of the indecomposable modules over the non-hypersurfaces. The following theorem summarizes these calculations.

**Theorem 4.2.** *Let  $R$  be one of the complete hypersurfaces listed in Table 1. The ranks of all indecomposable MCM modules over these rings are given in the following table. In addition we give the number of indecomposables of each rank.*

one minimal prime			two minimal primes			three minimal primes		
type	rank	#	type	rank	#	type	rank	#
$(A_{2n})$	1	$n+1$	$(A_{2n+1})$	(1, 0)	1	$(D_{2n+2})$	(1, 0, 0)	1
$(E_6)$	1	5		(0, 1)	1		(0, 1, 0)	1
	2	2		(1, 1)	$n+1$		(0, 0, 1)	1
$(E_8)$	1	7	$(D_{2n+3})$	(1, 0)	1		(1, 1, 0)	1
	2	7		(0, 1)	$n+1$		(1, 0, 1)	1
	3	3		(1, 1)	$2n+2$		(0, 1, 1)	$n$
$(A_{2n})$	1	$n+2$		(2, 1)	$n$		(1, 1, 1)	$2n+1$
$(D_3)$	1	4	$(E_7)$	(1, 0)	1		(2, 1, 1)	$n-1$
	2	1		(0, 1)	2			
				(1, 1)	6			
				(1, 2)	1			
				(2, 1)	2			
				(2, 2)	3			
			$(D_{2n})$	(1, 0)	1			
				(0, 1)	$n+1$			
				(1, 1)	$2n+1$			
				(2, 1)	$n$			

The reader may be disturbed by the lack of symmetry. For each of the rings in Table 1 we have fixed an order on the minimal primes. This gives us, for example, an indecomposable module of rank (2, 1) but no indecomposable module of rank (1, 2) in the case when  $R$  is a ring of type  $(D_5)$ .

## 5 Non-complete Rings and a Krull-Schmidt Theorem

Throughout this section  $(R, \mathfrak{m}, k)$  is an equicharacteristic one-dimensional CM local ring with  $k$  perfect of characteristic not 2, 3 or 5. We assume throughout that  $R$  has FCMT, equivalently [23], the  $\mathfrak{m}$ -adic completion  $\hat{R}$  has FCMT. Thus  $\hat{R}$  is isomorphic to a ring listed in Theorem 2.1. Since we do not have explicit formulas for the rings  $R$  we cannot give concrete descriptions of the modules as in the complete case. Instead we represent  $\mathfrak{C}(R)$  as a submonoid of the monoid  $\mathfrak{C}(\hat{R})$  via the map  $M \mapsto \hat{M}$ . By identifying the irreducible elements of the monoid  $\mathfrak{C}(R)$  with the isomorphism classes of indecomposable MCM  $R$ -modules we are able to use monoid-theoretic techniques to describe the different ways in which a MCM  $R$ -module can be written as a direct sum of indecomposables.

In this paper we consider a *monoid* to be a commutative, cancellative, additive semigroup with 0. We further restrict our attention to *reduced* monoids — monoids in which 0 is the only invertible element. Recall that  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

We put a monoid structure on  $\mathfrak{C}(R)$  by declaring  $[N] + [M] = [N \oplus M]$ . As direct sum cancellation holds for  $R$ -modules ([9]),  $\mathfrak{C}(R)$  satisfies our definition of a monoid. As  $\hat{R}$  is a complete local ring the decomposition of  $\hat{R}$ -modules is unique up to isomorphism, [20], and hence  $\mathfrak{C}(\hat{R}) \cong \mathbb{N}^t$  where  $t$  is the number of isomorphism

classes of indecomposable MCM  $\hat{R}$ -modules. It is shown in [24] that the natural map taking  $M$  to  $\hat{R} \otimes_R M$  induces a *divisor homomorphism*  $\mathfrak{C}(R) \rightarrow \mathfrak{C}(\hat{R})$ , that is, for any two MCM  $R$ -modules  $M$  and  $N$ , if  $\hat{M} \mid \hat{N}$  then  $M \mid N$ . Thus we may consider  $\mathfrak{C}(R)$  as a *full submonoid* of  $\mathbb{N}^t$ ; that is, a submonoid that satisfies, for any  $a, b \in \mathfrak{C}(R)$ , if  $b = a + c$  for some  $c \in \mathbb{N}^t$  then  $c \in \mathfrak{C}(R)$ .

Since we have a list (Theorem 4.2) of all of the indecomposable MCM  $\hat{R}$ -modules as well as their ranks at each of the minimal primes of  $\hat{R}$ , we can determine  $\mathfrak{C}(R)$  using the following result, which is an immediate corollary of [17, Thm. 6.2]:

**Proposition 5.1.** *Let  $R$  and  $\hat{R}$  be as above. In particular,  $R$  is one-dimensional and  $\hat{R}$  is reduced. Let  $M$  be a finitely generated  $\hat{R}$ -module. Then  $M$  is extended from an  $R$ -module ( ${}_{\hat{R}}M \cong N \otimes_R \hat{R}$  for some  $R$ -module  $N$ ) if and only if  $\text{rank}_P(M) = \text{rank}_Q(M)$  whenever  $P$  and  $Q$  are minimal primes of  $\hat{R}$  lying over the same prime of  $R$ .*

We will apply Proposition 5.1 to determine which of the MCM  $\hat{R}$ -modules are extended from MCM  $R$ -modules and then use this information to determine  $\mathfrak{C}(R)$  as a full submonoid of  $\mathfrak{C}(\hat{R})$ . In order to efficiently determine the structure of  $\mathfrak{C}(R)$  we need to introduce some additional terminology.

If there exists a divisor homomorphism  $H \hookrightarrow \mathbb{N}^t$  we say that  $H$  is a *Krull monoid*. If  $H \hookrightarrow \mathbb{N}^t$  is a divisor homomorphism and each element of  $\mathbb{N}^t$  is the greatest lower bound of a finite set of elements of  $\phi(H)$  then we say that  $H \hookrightarrow \mathbb{N}^t$  is a *divisor theory* for  $H$ . (It is known, [12], that every Krull monoid has a divisor theory.) In this case, we can define the *class group*  $Cl(H)$  of  $H$  to be the cokernel of the induced map  $\mathcal{Q}(H) \hookrightarrow \mathbb{Z}^t$ , where  $\mathcal{Q}(H)$  is the group of formal differences of elements of  $H$ .

Let  $H$  be a monoid and let  $h \in H$ . Then

$$L(h) = \{n \mid h = a_1 + a_2 + \cdots + a_n \text{ for irreducible } a_i\}$$

is the *set of lengths for the element  $h$* . A monoid  $H$  is said to be *factorial* if each element can be written uniquely (up to order of the terms) as a sum of irreducible elements of  $H$ . We note that this occurs exactly when  $H$  is free. A monoid  $H$  is said to be *half-factorial* if  $L(h)$  is a singleton for each  $h \in H$ . The *elasticity of an element  $h \in H$*  is  $\rho(h) = \frac{\sup\{L(h)\}}{\inf\{L(h)\}}$ . The *elasticity of the monoid  $H$*  is  $\rho(H) = \sup\{\rho(h) \mid h \in H - \{0\}\}$ . We note that  $\rho(H) = 1$  if and only if  $H$  is half-factorial. It is known (cf. [1]), using our definition of monoid, that for every monoid  $H$  there exists  $h \in H$  such that  $\rho(H) = \rho(h) \in \mathbb{Q}$ .

The *prime divisor classes*  $G_0$  in  $G = Cl(H)$  are the elements  $q \in Cl(H)$  such that  $q = p + \mathcal{Q}(H)$  for some irreducible element  $p \in \mathbb{N}^t$ . Note that the irreducible elements of  $\mathbb{N}^t$  are just the unit vectors  $e_j$ ,  $j = 1, \dots, t$ . Consider the map

$$c : \begin{array}{ccc} \mathcal{F}(G_0) & \longrightarrow & G \\ \prod_{g \in G_0} g^{n_g} & \longmapsto & \sum_{g \in G_0} n_g g \end{array}$$

where  $\mathcal{F}(G_0)$  is the free abelian monoid (written multiplicatively) on the set  $G_0$ . The submonoid  $\mathcal{B}(G_0) = \{s \in \mathcal{F}(G_0) : c(s) = 0\} \subseteq \mathcal{F}(G_0)$  is called the *block*

monoid of  $G_0$ . It is shown in [10] that the set of lengths of  $H$  is the same as the set of lengths of  $\mathcal{B}(G_0)$ .

We now state and prove a lemma which allow us to calculate  $\mathcal{Cl}(\mathfrak{C}(R))$  as well as the set of lengths of  $\mathfrak{C}(R)$ .

**Lemma 5.2.** *1. The divisor class group  $\mathcal{Cl}(H)$  of a finitely generated reduced Krull monoid  $H$  is trivial if and only if  $H \cong \mathbb{N}^t$  for some  $t$ ; i.e.,  $H$  is free.*

*2. Let  $R$  and  $\hat{R}$  be as above. If  $\#\text{Spec}(\hat{R}) = \#\text{Spec}(R)$ , then  $\mathfrak{C}(R) \cong \mathfrak{C}(\hat{R})$ , and hence  $\mathcal{Cl}(\mathfrak{C}(R)) = 0$ .*

*3. If a monoid  $H$  contains a  $\mathbb{Z}$ -basis for a group  $G$  containing  $H$ , then  $G = \mathcal{Q}(H)$ .*

*4. Let  $H = \ker(\mathcal{A}) \cap \mathbb{N}^t \subseteq \mathbb{N}^t$  where  $\mathcal{A}$  is a  $t \times s$  matrix with entries in  $\mathbb{Z}$ . Further assume that  $H$  contains a  $\mathbb{Z}$ -basis for  $\ker(\mathcal{A})$  and that the natural inclusion  $i : H \rightarrow \mathbb{N}^t$  is a divisor theory. Then  $\mathcal{Cl}(H)$  is isomorphic to the image of  $\mathcal{A} : \mathbb{Z}^s \rightarrow \mathbb{Z}^t$ . Furthermore, the prime divisor classes in  $\mathcal{Cl}(H)$  are the elements  $\{\mathcal{A}e_j\}_{j=1}^t$ .*

*Proof.* Suppose first that  $\mathcal{Cl}(H) = 0$ . Then there is a divisor theory  $\phi : H \hookrightarrow \mathbb{N}^k$  with  $\mathcal{Q}(H) = \mathbb{Z}^k$ . Because  $\phi$  is a divisor homomorphism we have

$$\phi(H) = \mathcal{Q}(\phi(H)) \cap \mathbb{N}^k = \mathbb{Z}^k \cap \mathbb{N}^k = \mathbb{N}^k.$$

Suppose now that  $H$  is free. Then there exists an isomorphism  $\phi : H \rightarrow \mathbb{N}^k$ . Clearly  $\phi$  is a divisor theory and  $\mathcal{Q}(\phi) : \mathcal{Q}(H) \rightarrow \mathbb{Z}^k$  is also an isomorphism. Thus  $\mathcal{Cl}(H) = \text{coker}(\mathcal{Q}(\phi)) = 0$ .

Suppose that  $\#\text{Spec}(\hat{R}) = \#\text{Spec}(R)$ . Then each minimal prime of  $\hat{R}$  lies over a unique minimal prime of  $R$ . Thus it is clear from Proposition 5.1 that all finitely generated  $\hat{R}$ -modules are extended from  $R$ -modules. Thus  $\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(\hat{R})$  is an isomorphism. Then, as  $\mathfrak{C}(\hat{R})$  is free, so is  $\mathfrak{C}(R)$ . By part 1 we have that  $\mathcal{Cl}(\mathfrak{C}(R)) = 0$ .

It is clear that  $\mathcal{Q}(H) \subseteq G$ . To prove the reverse inclusion, let  $\{h_1, \dots, h_s\}$  be a  $\mathbb{Z}$ -basis for  $G$  contained in  $H$ . Given  $g \in G$  we can write  $g = a_1h_1 + \dots + a_sh_s$  with  $a_i \in \mathbb{Z}$ . Re-index so that for some  $t \leq s$  we have  $a_i \geq 0$  for  $0 \leq i \leq t$  and  $a_i < 0$  for  $t < i \leq s$ . Now we can write  $g = (a_1h_1 + \dots + a_th_t) - ((-a_{t+1}h_{t+1}) + \dots + (-a_sh_s)) \in \mathcal{Q}(H)$ .

By part 3 we have  $\mathcal{Q}(H) = \ker(\mathcal{A})$ . Thus  $\mathcal{Cl}(H) \cong \mathbb{Z}^t / \ker(\mathcal{A}) \cong \text{Im}(\mathcal{A})$ . It is now clear that  $\{\mathcal{A}e_j\}_{j=1}^t$  is the set of prime divisor classes in  $\mathcal{Cl}(H)$ .  $\square$

We now use Lemma 5.2 to calculate  $\mathfrak{C}(R)$  and  $\mathcal{Cl}(\mathfrak{C}(R))$  when  $R$  is a non-complete local ring with FCMT. Here we provide the calculations when  $R$  is a local domain whose completion is either a ring of type  $(A_{2n+1})$  or a ring of type  $(D_{2n+2})$ . The remaining calculations can be worked out similarly and are summarized in Theorem 5.4. Before we begin we recall the following result, [16, Thm. 1], of C. Lech.

**Proposition 5.3.** *Let  $S$  be a complete Noetherian local ring. Then  $S$  is the completion of a Noetherian local domain if and only if the following conditions hold.*

1. The prime ring  $\pi$  of  $S$  is a domain and  $S$  is a torsion-free  $\pi$ -module.
2. Either  $S$  is a field or  $\text{depth}(S) \geq 1$ .

Since we are dealing only with equicharacteristic CM local rings of dimension one these conditions are automatically satisfied. Thus there is merit in the calculations that follow.

( $A_{2n+1}$ )

Let  $R$  be a local domain whose completion is isomorphic to the ring  $k[[x, y]]/(x^2 - y^{2n})$ . For example,  $R$  could be isomorphic to  $\frac{k[x, y]_{(x, y)}}{(x^3 + x^2 - y^{2n+2})}$ . Referring to Proposition 4.2 we see that  $\hat{R}$  has one indecomposable MCM module each of rank  $(0, 1)$  and  $(1, 0)$  and  $n + 1$  indecomposable MCM modules of rank  $(1, 1)$ . Let  $A$  be the indecomposable of rank  $(0, 1)$ , let  $B$  be the indecomposable of rank  $(1, 0)$  and let  $M_0, \dots, M_n$  be the indecomposables of constant rank one. If  $L$  is any MCM  $\hat{R}$ -module, write

$$L = \left( \bigoplus_{i=0}^n M_i^{m_i} \right) \oplus A^a \oplus B^b$$

where  $m_i$ ,  $a$ , and  $b$  are nonnegative integers. Then

$$\text{rank}(L) = \left( \sum_{j=0}^n m_j + a, \sum_{j=0}^n m_j + b \right)$$

By Proposition 5.1,  $L$  is extended from a MCM  $R$ -module if and only if  $a = b$ . Therefore the monoid of MCM  $R$ -modules is  $\mathfrak{C}(R) \cong \mathbb{N}^{n+2}$ . As  $\mathfrak{C}(R)$  is free,  $\text{Cl}(\mathfrak{C}(R)) = 0$  by Lemma 5.2.

( $D_{2n+2}$ )

Suppose that  $R$  is a local domain whose completion is isomorphic to  $k[[x, y]]/(x^2y - y^{2n+1})$ . Referring to Proposition 4.2, we find the following indecomposable MCM  $\hat{R}$ -modules:

module	rank	module	rank
$A$	$(1, 0, 0)$	$E$	$(1, 0, 1)$
$B$	$(0, 1, 0)$	$F_j$	$(0, 1, 1) \quad 1 \leq j \leq n$
$C$	$(0, 0, 1)$	$G_j$	$(2, 1, 1) \quad 1 \leq j \leq n - 1$
$D$	$(1, 1, 0)$	$H_j$	$(1, 1, 1) \quad 1 \leq j \leq 2n + 1$

Now, if  $L$  is any MCM  $\hat{R}$ -module,

$$L = A^a \oplus B^b \oplus C^c \oplus D^d \oplus E^e \oplus \left( \bigoplus_{j=1}^n F_j^{f_j} \right) \oplus \left( \bigoplus_{j=1}^{n-1} G_j^{g_j} \right) \oplus \left( \bigoplus_{j=1}^{2n+1} H_j^{h_j} \right)$$

Since  $R$  is a domain,  $L$  is extended if and only if  $\text{rank}(L)$  is constant; i.e., if and only if

$$\sum_{j=1}^{n-1} g_j + a + e = \sum_{j=1}^n f_j + b \quad \text{and} \quad b + d = c + e.$$

Thus we have  $\mathfrak{C}(R) = (\ker(\mathcal{A}) \cap \mathbb{N}^{2n+4}) \oplus \mathbb{N}^{2n+1}$  where

$$\mathcal{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 & -1 & \cdots & -1 & 1 & \cdots & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

is a  $2 \times (2n + 4)$  matrix whose first row consists of  $n + 1$  ones and  $n + 1$  negative ones.

I claim that the following  $4n + 5$  elements form a  $\mathbb{Z}$ -basis for  $\ker(\mathcal{A}) \oplus \mathbb{Z}^{2n+1}$  that is contained in  $\mathfrak{C}(R)$ .

$$\begin{aligned} \mathcal{B} = \{ & e_1 + e_6, e_2 + e_5, e_3 + e_4, e_4 + e_5 + e_6 \} \cup \{e_1 + e_j \mid 7 \leq j \leq n + 5\} \\ & \cup \{e_6 + e_j \mid n + 6 \leq j \leq 2n + 4\} \\ & \cup \{e_j \mid 2n + 5 \leq j \leq 4n + 5\} \end{aligned}$$

Clearly  $\mathcal{B} \subseteq \ker(\mathcal{A}) \oplus \mathbb{Z}^{2n+1}$ . Now, if  $h = \sum_{i=1}^{4n+5} h_i e_i \in \ker(\mathcal{A}) \oplus \mathbb{Z}^{2n+1}$ , then  $h_2 + h_4 = h_3 + h_5$  and

$$h_1 + h_5 + \sum_{i=n+5}^{2n+4} h_i = h_2 + h_6 + \sum_{i=7}^{n+5} h_i$$

Thus we can write  $h$  uniquely as

$$\begin{aligned} h &= \sum_{i=7}^{n+5} h_i (e_1 + e_i) + \sum_{i=n+6}^{2n+4} h_i (e_6 + e_i) \\ &+ \left( h_1 - \sum_{i=7}^{n+5} h_i \right) (e_1 + e_6) \\ &+ \left( h_6 - h_1 + \sum_{i=7}^{n+5} h_i - \sum_{i=n+6}^{2n+4} h_i \right) (e_4 + e_5 + e_6) \\ &+ \left( h_4 - h_6 + h_1 - \sum_{i=7}^{n+5} h_i + \sum_{i=n+6}^{2n+4} h_i \right) (e_3 + e_4) \\ &+ h_2 (e_2 + e_5) + \sum_{i=2n+5}^{4n+5} h_i (e_i) \end{aligned}$$

Thus  $\mathcal{B}$  is a  $\mathbb{Z}$ -basis for  $\ker(\mathcal{A}) \oplus \mathbb{Z}^{2n+1}$ . We now show that each irreducible element of  $\mathbb{N}^{4n+5}$  is the greatest lower bound of two or three elements of  $\mathfrak{C}(R)$ .

Then we will have shown that  $\mathfrak{C}(R) \subseteq \ker(\mathcal{A}) \oplus \mathbb{Z}^{2n+1}$  is a divisor theory. Since  $e_j \in \mathfrak{C}(R)$  for  $2n+5 \leq j \leq 4n+5$ , we need only consider  $e_j$  for  $j \leq 2n+4$ . Let  $j$  and  $k$  be such that  $7 \leq j \leq n+5$ , and  $n+6 \leq k \leq 2n+4$ . Then  $e_1 = \text{glb}(e_1 + e_6, e_1 + e_2 + e_3)$ ,  $e_2 = \text{glb}(e_2 + e_5, e_1 + e_2 + e_3)$ ,  $e_3 = \text{glb}(e_3 + e_4, e_1 + e_2 + e_3)$ ,  $e_4 = \text{glb}(e_3 + e_4, e_4 + e_5 + e_6)$ ,  $e_5 = \text{glb}(e_2 + e_5, e_4 + e_5 + e_6)$ ,  $e_6 = \text{glb}(e_1 + e_6, e_4 + e_5 + e_6)$ ,  $e_j = \text{glb}(e_1 + e_j, e_j + e_4 + e_5)$ , and  $e_k = \text{glb}(e_6 + e_k, e_k + e_2 + e_3)$ . Thus the natural inclusion  $\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(\hat{R})$  is a divisor theory. By Lemma 5.2,  $Cl(\mathfrak{C}(R)) \cong \mathbb{Z} \oplus \mathbb{Z}$ .

The remaining calculations may be worked out in a similar fashion, and the following proposition summarizes the results.

**Proposition 5.4.** *Let  $(R, \mathfrak{m})$  be a one-dimensional equicharacteristic CM local ring with perfect residue field of characteristic different from 2, 3 and 5. Let  $\hat{R}$  denote the  $\mathfrak{m}$ -adic completion of  $R$ . Then  $Cl(\mathfrak{C}(R))$  depends only on the singularity type of  $\hat{R}$  and on  $m := \#\text{Spec}(\hat{R}) - \#\text{Spec}(R)$ . The results are summarized in the following table. We list the results only when  $m > 0$ , since the case  $m = 0$  was taken care of in Lemma 5.2.*

Table 3: Monoids of MCM Modules

$\hat{R}$	$m$	$\mathfrak{C}(\hat{R})$	$\mathfrak{C}(R)$	$Cl(\mathfrak{C}(R))$
$(A_{2n+1})$	1	$\mathbb{N}^{n+3}$	$\mathbb{N}^{n+2}$	0
$(D_{2n+3})$	1	$\mathbb{N}^{4n+4}$	$(\ker(\mathcal{A}_1) \cap \mathbb{N}^{2n+2}) \oplus \mathbb{N}^{2n+2}$	$\mathbb{Z}$
$(D'_{2n+3})$	1	$\mathbb{N}^{4n+3}$	$(\ker(\mathcal{A}_1) \cap \mathbb{N}^{2n+2}) \oplus \mathbb{N}^{2n+1}$	$\mathbb{Z}$
$(D_{2n+2})^1$	1	$\mathbb{N}^{4n+5}$	$(\ker(\mathcal{A}_1) \cap \mathbb{N}^{2n+2}) \oplus \mathbb{N}^{2n+3}$	$\mathbb{Z}$
$(D'_{2n+2})^1$	1	$\mathbb{N}^{4n+4}$	$(\ker(\mathcal{A}_1) \cap \mathbb{N}^{2n+2}) \oplus \mathbb{N}^{2n+2}$	$\mathbb{Z}$
$(D_{2n+2})^2$	1	$\mathbb{N}^{4n+5}$	$(\ker(\mathcal{A}_2) \cap \mathbb{N}^4) \oplus \mathbb{N}^{4n+1}$	$\mathbb{Z}$
$(D'_{2n+2})^2$	1	$\mathbb{N}^{4n+4}$	$(\ker(\mathcal{A}_2) \cap \mathbb{N}^4) \oplus \mathbb{N}^{4n}$	$\mathbb{Z}$
$(D_{2n+2})$	2	$\mathbb{N}^{4n+5}$	$(\ker(\mathcal{A}_3) \cap \mathbb{N}^{2n+4}) \oplus \mathbb{N}^{2n+1}$	$\mathbb{Z} \oplus \mathbb{Z}$
$(D'_{2n+2})$	2	$\mathbb{N}^{4n+4}$	$(\ker(\mathcal{A}_3) \cap \mathbb{N}^{2n+4}) \oplus \mathbb{N}^{2n}$	$\mathbb{Z} \oplus \mathbb{Z}$
$(E_7)$	1	$\mathbb{N}^{15}$	$(\ker(\mathcal{A}_4) \cap \mathbb{N}^6) \oplus \mathbb{N}^9$	$\mathbb{Z}$
$(E'_7)$	1	$\mathbb{N}^{14}$	$(\ker(\mathcal{A}_4) \cap \mathbb{N}^6) \oplus \mathbb{N}^8$	$\mathbb{Z}$
$(D_{2n})$	1	$\mathbb{N}^{4n+3}$	$(\ker(\mathcal{A}_1) \cap \mathbb{N}^{2n+2}) \oplus \mathbb{N}^{2n+1}$	$\mathbb{Z}$
$(D'_{2n})$	1	$\mathbb{N}^{4n+2}$	$(\ker(\mathcal{A}_1) \cap \mathbb{N}^{2n+2}) \oplus \mathbb{N}^{2n}$	$\mathbb{Z}$

The matrices in the table above are as follows:

- $\mathcal{A}_1 = [1 \ 1 \ \cdots \ 1 \ -1 \ -1 \ \cdots \ -1]_{1 \times (2n+2)}$
- $\mathcal{A}_2 = [1 \ -1 \ -1 \ 1]_{1 \times 4}$
- $\mathcal{A}_3 = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 & -1 & \cdots & -1 & 1 & \cdots & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}_{2 \times (2n+4)}$
- $\mathcal{A}_4 = [1 \ 1 \ 1 \ -1 \ -1 \ -1]_{1 \times 6}$

**Note:** The two cases for  $(D_{2n+2})$  and  $(D'_{2n+2})$  and  $m = 1$  correspond to the cases 1:  $P_1 \cap R = P_2 \cap R$  (or  $P_1 \cap R = P_3 \cap R$ ) and 2:  $P_2 \cap R = P_3 \cap R$ . As expected, the two cases coincide when  $n = 4$ .

We are now ready to prove our main result.

**Theorem 5.5.** *Let  $R$  be a reduced one-dimensional equicharacteristic local ring with perfect residue field of characteristic not 2, 3, or 5. Suppose further that  $R$  has finite Cohen-Macaulay type. Let  $m = \#\text{Spec}(\hat{R}) - \#\text{Spec}(R)$ .*

1. *If  $m = 0$  then  $\mathfrak{C}(R)$  is factorial and Krull-Schmidt holds for torsion-free  $R$ -modules.*
2. *Let  $m = 1$ . If  $\hat{R}$  is a ring of type  $(A_{2n+1})$  then  $\mathfrak{C}(R)$  is factorial; otherwise  $\mathfrak{C}(R)$  is half-factorial but not factorial.*
3. *If  $m = 2$  then  $\mathfrak{C}(R)$  is not half-factorial.*

*Proof.* First recall from part 2 of Lemma 5.2 that  $\mathcal{Cl}(\mathfrak{C}(R)) = 0$  if and only if  $\mathfrak{C}(R)$  is free and hence factorial.

We now turn to the case where  $\mathcal{Cl}(\mathfrak{C}(R)) \cong \mathbb{Z}$ . This is the case when  $R$  is a domain and  $\hat{R}$  is a ring of type  $(D_{2n+2})$ ,  $(D_{2n+3})$ ,  $(E_7)$ , or  $(D_{2n})$  with  $\#\text{Spec}(\hat{R}) - \#\text{Spec}(R) = 1$ . From Proposition 5.4 we know that in each of these cases  $\mathfrak{C}(R) \cong \ker(\mathcal{A}) \cap \mathbb{N}^s \oplus \mathbb{N}^t$  for some  $1 \times s$  integer matrix  $\mathcal{A}$  where  $s$  and  $t$  are non-negative integers.

From Lemma 5.2 we know that in order to determine the block monoid of  $\mathfrak{C}(R)$  we need only compute  $\mathcal{A}'(e_j)$  for all  $j$ ,  $1 \leq j \leq s$ , where  $\mathcal{A}' = [\mathcal{A} \mid 0 \cdots 0]_{1 \times (s+t)}$ . Referring to the matrices  $\mathcal{A}$  in Proposition 5.4 we see that the set of prime divisor classes of  $\mathcal{Cl}(\mathfrak{C}(R))$  is  $G_0 = \{0, +1, -1\}$  and hence  $\mathcal{B}(\mathfrak{C}(R)) \cong \mathbb{N}^2$ . Since  $L(\mathbb{N}^2) = \{\{n\} \mid n \in \mathbb{N}\}$  and as the set of lengths for  $\mathfrak{C}(R)$  is necessarily the same as for  $\mathcal{B}(\mathfrak{C}(R))$ ,  $\mathfrak{C}(R)$  is half-factorial. Since  $\mathcal{Cl}(\mathfrak{C}(R)) \neq 0$ ,  $\mathfrak{C}(R)$  is not factorial by part 1 of Lemma 5.2.

We now compute the block monoid associated to  $\mathcal{Cl}(\mathfrak{C}(R)) \cong \mathbb{Z} \oplus \mathbb{Z}$ , which occurs when  $R$  is a domain and  $\hat{R}$  is of type  $(D_{2n+2})$ . Recall that  $\mathfrak{C}(R) \cong \ker(\mathcal{A}) \cap \mathbb{N}^{2n+4} \oplus \mathbb{N}^{2n+1}$  where

$$\mathcal{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 & -1 & \cdots & -1 & 1 & \cdots & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (11)$$

By Lemma 5.2 the set of the prime divisor classes is

$$\{(1, 0), (-1, 0), (0, 1), (0, -1), (1, -1), (-1, 1)\}$$

and thus the block monoid is

$$\mathcal{B}(\mathfrak{C}(R)) \cong \ker \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{bmatrix} \cap \mathbb{N}^6.$$

The irreducible elements of this monoid are:

$$\begin{aligned} h_1 &= (1, 0, 0, 0, 0, 1) & h_2 &= (0, 1, 0, 0, 1, 0) & h_3 &= (0, 0, 1, 1, 0, 0) \\ h_4 &= (1, 1, 1, 0, 0, 0) & h_5 &= (0, 0, 0, 1, 1, 1) \end{aligned}$$

Note that  $h_1 + h_2 + h_3 = h_4 + h_5$  and hence this monoid is not half-factorial.  $\square$

We conclude this section with a result which shows that the class of torsion-free modules behaves much better than the class of arbitrary finitely generated modules. This is exhibited using the elasticity function  $\rho$ . We first make the following remark.

*Remark 5.6.* In Theorem 5.7 we consider, in addition to the monoid  $\mathfrak{C}(R)$  of isomorphism classes of MCM  $R$ -modules, the monoid  $\mathfrak{M}(R)$  of isomorphism classes of all finitely generated  $R$ -modules. Note that  $\mathfrak{M}(R)$  is not a monoid in the sense that we previously defined since it is not finitely generated. Since we are only interested in calculating the elasticity of elements in  $\mathfrak{M}(R)$ , our current definitions will suffice. We note that the elasticity of a non-finitely generated monoid  $H$  need not be rational nor accepted. One says that the elasticity is *accepted* provided  $\rho(H) = \rho(h)$  for some  $h \in H$ .

**Theorem 5.7.** *Let  $R$ ,  $\hat{R}$ , and  $m$  be as in Theorem 5.5. Let  $\mathfrak{M}(R)$  denote the monoid of all finitely generated  $R$ -modules.*

1. *If  $m = 0$  then the Krull-Schmidt property holds for the class of all finitely generated  $R$ -modules.*
2. *Let  $m = 1$ .*
  - (a) *If  $\hat{R}$  is a ring of type  $(A_{2n+1})$  for some  $n \geq 0$ , then  $\mathfrak{C}(R)$  is factorial; otherwise  $\rho(\mathfrak{C}(R)) = 1$  but  $\mathfrak{C}(R)$  is not factorial.*
  - (b) *If  $\hat{R}$  is a ring of type  $(A_1)$  then  $\rho(\mathfrak{M}(R)) = 1$  but  $\mathfrak{M}(R)$  is not factorial; otherwise  $\rho(\mathfrak{M}(R)) = \infty$ .*
3. *If  $m = 2$ , then  $\rho(\mathfrak{C}(R)) = \frac{3}{2}$  and  $\rho(\mathfrak{M}(R)) = \infty$ .*

*Proof.* Since Proposition 5.1 holds for all finitely generated modules, part 2 of Lemma 5.2 gives 1.

We now suppose that  $m > 0$ . A result in [13] says that if  $\hat{R}$  is a ring from Table 1 not of type  $(A_1)$  (in other words, not Dedekind-like) then for any  $s$ -tuple of natural numbers  $(r_1, \dots, r_s)$  (where  $s$  is the number of minimal primes of  $\hat{R}$ ) there exists an indecomposable finitely generated  $\hat{R}$ -module  $M$  such that the torsion-free rank of  $M$  is  $(r_1, \dots, r_s)$ .

Suppose now that  $R$  is a domain and  $\hat{R}$  has two minimal primes. Let  $n \in \mathbb{N}$  and set  $N = 1 + 2 + \dots + n$ . By [13], for all  $i \in \{1, 2, \dots, n, N\}$  there exist indecomposable finitely generated  $\hat{R}$ -modules  $M_{0,i}$  and  $M_{i,0}$  of rank  $(0, i)$  and  $(i, 0)$ , respectively. Since  $R$  is a domain, none of these modules is extended. However,  $M_{0,1} \oplus M_{1,0}, \dots, M_{0,n} \oplus M_{n,0}$ ,  $\left(\bigoplus_{i=1}^n M_{i,0}\right) \oplus M_{0,N}$ , and  $\left(\bigoplus_{i=1}^n M_{0,i}\right) \oplus M_{N,0}$  are extended. Moreover, none of these extended modules has a direct summand which is extended. Therefore, they are extended from indecomposable  $R$ -modules. Now

since

$$\begin{aligned} M &= \left( \left( \bigoplus_{i=1}^n M_{i,0} \right) \oplus M_{0,N} \right) \oplus \left( \left( \bigoplus_{i=1}^n M_{0,i} \right) \oplus M_{N,0} \right) \\ &\cong \bigoplus_{i=1}^n (M_{0,i} \oplus M_{i,0}) \oplus (M_{0,N} \oplus M_{N,0}), \end{aligned}$$

we see that  $2, n+1 \in L([M])$  and hence the elasticity of the element  $[M] \in \mathfrak{M}(R)$  is at least  $\frac{n+1}{2}$ . As  $n$  was arbitrary the elasticity of  $\mathfrak{M}(R)$  is infinite. A similar argument takes care of the remaining cases when  $m > 0$  and  $\hat{R}$  is not of type  $(A_1)$ .

Now suppose that  $R$  is a local domain whose completion  $\hat{R}$  is of type  $(A_1)$ . Then  $\hat{R}$  is split Dedekind-like and hence has infinitely many non-isomorphic indecomposable finitely generated modules of ranks  $(1, 0)$  and  $(0, 1)$ . Moreover, all indecomposable finitely generated  $\hat{R}$ -modules have ranks  $(1, 0)$ ,  $(0, 1)$ , or  $(1, 1)$  (cf. [15]). Thus there are non-isomorphic finitely generated indecomposable modules  $M$  and  $M'$  with rank  $(1, 0)$  and non-isomorphic indecomposable modules  $N$  and  $N'$  with rank  $(0, 1)$ . None of these modules is extended, but  $M \oplus N$ ,  $M' \oplus N'$ ,  $M \oplus N'$ , and  $M' \oplus N$  are extended. Thus we have

$$(M \oplus M') \oplus (N \oplus N') \cong (M \oplus N') \oplus (M' \oplus N)$$

which exhibits the failure of Krull-Schmidt for finitely generated  $R$ -modules. However, since each extended  $\hat{R}$ -module  $M$  must have constant rank, any direct sum decomposition of  $M$  must have the same number of indecomposable summands of rank  $(1, 0)$  as it has of rank  $(0, 1)$ . Therefore,  $\mathfrak{M}(R)$  is half-factorial and  $\rho(\mathfrak{M}(R)) = 1$ .

Now we deal with the monoid  $\mathfrak{C}(R)$ . When  $m \leq 1$  Theorem 5.5 implies that  $\mathfrak{C}(R)$  is half-factorial and hence  $\rho(\mathfrak{C}(R)) = 1$ . Now suppose  $m = 2$ . Recall that the set of lengths of  $\mathfrak{C}(R)$  is the same as the set of lengths of  $H \cong \ker(\mathcal{A}) \cap \mathbb{N}^6$ , where the matrix  $\mathcal{A}$  is as in (11). Using the algorithm from [14, Sec. 2] we easily determine that  $\rho(\mathfrak{C}(R)) = \rho(H) = \frac{3}{2}$ .  $\square$

## 6 No Indecomposable of Rank 4

In the previous section we computed the monoid of MCM  $R$ -modules if  $R$  is an equicharacteristic one-dimensional local domain of finite representation type. We now restrict our attention to the case when  $R$  is a domain. Then from the descriptions of the monoids given it is easy to see that the possible ranks of indecomposable MCM  $R$ -modules are 1, 2, and 3 whenever  $R$  is a domain. This result contradicts a result in [26], which states the existence of an indecomposable module of rank 4. We note that in the equicharacteristic case, this ring is of type  $(D3')$  as in Table 1. In this section we show that the module of rank 4 in [26] decomposes into two modules of rank 2. We first recall the basic set-up in [26].

Let  $R$  be a reduced one-dimensional local ring with module-finite integral closure  $\bar{R} \neq R$ . Let  $\mathfrak{f}$  be the conductor ideal. Then  $R$  can be represented as the pullback in the following diagram:

$$\begin{array}{ccc}
R & \longrightarrow & \bar{R} \\
\downarrow & & \downarrow \\
R/\mathfrak{f} & \longrightarrow & \bar{R}/\mathfrak{f}
\end{array}$$

The bottom line of the pullback diagram is called an *Artinian pair*. (The rings are Artinian since  $\mathfrak{f}$  contains a non-zero-divisor.) A module over an Artinian pair  $A \rightarrow B$  is a pair  $V \rightarrow W$  such that  $W$  is a finitely generated projective  $B$ -module and  $V$  is an  $A$ -submodule of  $B$  satisfying  $BV = W$ . A direct sum decomposition of  $V \rightarrow W$  is a direct sum decomposition of  $W$  as a  $B$ -module,  $W = W_1 \oplus \cdots \oplus W_n$ , such that  $V = (W_1 \cap V) \oplus \cdots \oplus (W_n \cap V)$ . By Proposition 2.2 of [26] there is a bijective correspondence between the indecomposable modules over this Artinian pair and the indecomposable MCM  $R$ -modules.

We now show how the purported indecomposable  $M$  of rank 4 in [26] decomposes as a direct sum of two modules of rank 2. We have the following setup:  $k$  is a perfect field,  $K$  is a degree-three non-Galois extension of  $k$ , and  $L$  is the Galois closure of  $K/k$ . The Galois group of  $L/k$  is generated by an element  $\sigma$  of order two and an element  $\tau$  of order three. We wish to decompose the module  $N = (L \xrightarrow{\delta} L \times L)$  as a  $k \rightarrow K$  module, where  $\delta(x) = (x, x)$  and the  $K$ -action on  $L \times L$  is defined to be  $s \cdot (x, y) = (s^\tau x, sy)$  for all  $s \in K$  and  $(x, y) \in L \times L$ .

First suppose that the characteristic of  $k$  is not 2. Then this module decomposes as:

$$(\{z \in L \mid z^{\sigma\tau} = z\} \rightarrow \{(z^{\sigma\tau}, z) \mid z \in L\}) \oplus (\{z \in L \mid z^{\sigma\tau} = -z\} \rightarrow \{(z^{\sigma\tau}, -z) \mid z \in L\})$$

Since  $\{(z^{\sigma\tau}, z) \mid z \in L\} \cap \{(z^{\sigma\tau}, -z) \mid z \in L\} = 0$  and as each of these summands has  $K$ -dimension 2 we see that this is indeed a direct sum decomposition of  $L \times L$ . Since  $\sigma\tau$  is an element of order 2 in the Galois group, the fixed field  $F$  of  $\sigma\tau$  has  $k$ -dimension 3. We now prove that  $\{z \in L \mid z^{\sigma\tau} = -z\}$  also has  $k$ -dimension 3. This will show that the above decomposition of  $L \times L$  induces a decomposition of the diagonal, finishing the proof.

Choose  $\alpha \in L$  such that  $F(\alpha) = L$ . If  $z \in L$  and  $z^{\sigma\tau} = z$ , write  $z = x + \alpha y$  with  $x, y \in F$ . Then  $-(x + \alpha y) = (x + \alpha y)^{\sigma\tau} = x + \alpha^{\sigma\tau} y$ , whence  $x = -\frac{1}{2}(\alpha + \alpha^{\sigma\tau})y$ . Rearranging terms and multiplying by 2, we see that  $\{z \in L \mid z^{\sigma\tau} = -z\} = \{(\alpha - \alpha^{\sigma\tau})y \mid y \in F\}$ . Since  $[F : k] = 3$ , we are done.

Suppose now that the characteristic of  $k$  is 2. Let  $\iota$  denote the natural inclusion

$$(\{z \in L \mid z = z^\tau\} \rightarrow \{(z^\tau, z) \mid z \in L\}) \hookrightarrow (L \rightarrow L \times L)$$

We will now exhibit a splitting of  $\iota$  thus, proving that  $\{z \in L \mid z = z^\tau\} \rightarrow \{(z^\tau, z) \mid z \in L\}$  is a direct summand of  $L \rightarrow L \times L$ .

Let  $\nu : L \times L \rightarrow \{(z^\tau, z) \mid z \in L\}$  be defined by  $(x, y) \mapsto (x^{\tau^2} + y + y^\tau, x^\tau + y^{\tau^2} + y)$ . Then  $\nu(x, x) = (x + x^\tau + x^{\tau^2}, x + x^\tau + x^{\tau^2})$  and so  $\nu$  sends the diagonal of  $L \times L$  to the diagonal of  $\{(z^\tau, z) \mid z \in L\}$ . Furthermore, since the characteristic of  $k$  is 2,  $\nu(x^\tau, x) = (x^\tau, x)$  and so  $\nu$  is indeed a splitting of  $\iota$ .

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