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A Krull-Schmidt theorem for one-dimensional rings of finite Cohen-Macaulay type. (English summary)
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The Krull-Schmidt Theorem tells us that over a complete local ring R a finitely generated R-module can be written as a finite direct sum of indecomposable R-modules in an essentially unique way, i.e., the same number of terms must be used and after reordering the terms are isomorphic in pairs. The current article investigates when this property holds in the non-complete case and records some measures of failure when it does not hold.

Specifically, let R be a reduced, one-dimensional, equicharacteristic, local ring with perfect residue of characteristic not 2, 3, or 5. Further, assume that R has finite Cohen-Macaulay type, i.e., up to isomorphism there are only finitely many indecomposable maximal Cohen-Macaulay (MCM) modules. The presence of a Krull-Schmidt property for R depends upon $m := \#\mathrm{Spec}(\widehat{R}) - \#\mathrm{Spec}(R)$, which is either 0, 1, or 2. When $m = 0$, unique decompositions exist for MCM modules. When $m = 1$, one class of rings has the Krull-Schmidt property for MCM modules while the other classes have a weaker property, that the number of indecomposable MCM modules used in a decomposition is unique. Neither property holds when $m = 2$.

When the Krull-Schmidt property fails to hold, the author uses elasticity to measure the failure. Roughly, elasticity measures how large the ratio between the longest and shortest decompositions can become for all modules. When elasticity for MCM modules equals 1, the decompositions into indecomposables may not be unique, but the number of terms used for each MCM module will be unique.

The second important result of the article addresses elasticity and includes results for all finitely generated modules. When $m = 1$, the earlier result says that elasticity is 1 for MCM modules, but elasticity in general is unbounded for finitely generated modules. When $m = 2$, elasticity is $\frac{3}{2}$ for MCM modules and unbounded for general finitely generated modules.

The proofs of these results depend upon explicit computations of presentations of the completions of the rings studied and the MCM modules over these complete rings. (Many of these results were proved in [R. Wiegand, Ark. Mat. **29** (1991), no. 2, 339–357; MR1150382 (93d:13020); G.-M. Greuel and H. Knörrer, Math. Ann. **270** (1985), no. 3, 417–425; MR0774367 (86d:14025); Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, Cambridge Univ. Press, Cambridge, 1990; MR1079937 (92b:13016)].) The MCM modules in the non-complete case are then studied as a submonoid of the monoid of MCM modules over the completion of R.

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References

1. D.D. Anderson, D.F. Anderson, S. Chapman, W. Smith, Rational elasticity of factorizations in

Krull domains, Proc. Amer. Math. Soc. 117 (1993) 37–43. MR1106176 (93c:13014)
2. G. Azumaya, Corrections and supplementaties to my paper concerning Krull-Remak-Schmidt's theorem, Nagoya Math. J. 1 (1950) 231–253. MR0037832 (12,314e)
3. H. Bass, On the ubiquity of Gorenstein rings, Math. Z. 82 (1963) 8–28. MR0153708 (27 #3669)
4. N. Bourbaki, Commutative Algebra, Springer-Verlag, 1983.
5. W. Bruns, J. Herzog, Cohen-Macaulay Rings, in: Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, 1998. MR1251956 (95h:13020)
6. W. Bruns, U. Vetter, Determinantal Rings, in: Lecture Notes in Mathematics, vol. 1327, Springer-Verlag, 1988. MR0953963 (89i:13001)
7. N. Çimen, R. Wiegand, S. Wiegand, One-dimensional rings of finite Cohen-Macaulay type, in: A. Facchini, C. Menini (Eds.), Abelian Groups and Modules, Kluwer Academic, Doldrecht, 1995.
8. J.A. Dieudonné, A. Grothendieck, Eléments de géométrie algébrique, Chap. IV (part 2), Inst. Hautes Etudes Sci. Publ. Math 24 (1965) 5–231. MR0199181 (33 #7330)
9. E.G. Evans, Krull-Schmidt and cancellation over local rings, Pacific J. Math 46 (1973) 115–121. MR0323815 (48 #2170)
10. A. Geroldinger, Über nicht-eindeutige Zerlegungen in irreduzible Elemente, Math. Z. 197 (1988) 505–529. MR0932683 (89d:11096)
11. G.M. Greuel, H. Knörrer, Einfache Kurvensingularitäten und torsionfreie Moduln, Math. Ann. 270 (1985) 417–425. MR0774367 (86d:14025)
12. F. Halter-Koch, Ideal Systems, in: Monographs and Textbooks in Pure and Applied Mathematics, vol. 211, Marcel Dekker, 1998. MR1828371 (2001m:13005)
13. W. Hassler, R. Karr, L. Klingler, R. Wiegand, Indecomposable modules of large rank over commutative local rings (preprint).
14. K. Kattchee, Elasticities of Krull domains with finite divisor class group, Linear Algebra Appl. 384 (2004) 171–185. MR2055350 (2005k:13038)
15. L. Klingler, L. Levy, Representation type of commutative Noetherian rings II: Local tameness, Pacific J. Math. 200 (2001) 387–483. MR1868697 (2002i:13008b)
16. C. Lech, A method for constructing bad Noetherian local rings, in: Algebra, Algebraic Topology and their Interactions (Stockholm, 1983), in: Lecture Notes in Mathematics, vol. 1183, Springer-Verlag, New York-Tokyo, 1986, pp. 241–247. MR0846452 (87m:13010a)
17. L. Levy, C. Odenthal, Krull-Schmidt theorems in dimension 1, Trans. Amer. Math. Soc. 348 (1996) 3391–3455. MR1351492 (96m:16006a)
18. H. Matsumura, Commutative Ring Theory, Cambridge University Press, 2000. MR0879273 (88h:13001)
19. R. Pierce, Associative Algebras, in: Graduate Texts in Mathematics, vol. 88, Springer-Verlag, 1982. MR0674652 (84c:16001)
20. R.G. Swan, Induced representations and projective modules, Ann. Math. 71 (1960) 552–578. MR0138688 (25 #2131)
21. R.G. Swan, Noether's problem in Galois theory, in: B. Srinivasas, J. Sally (Eds.), Emmy Noether in Bryn Mawr, Springer-Verlag, Berlin, Heidelberg, New York, 1983, pp. 21–40. MR0713790 (84k:12013)

22. R. Wiegand, Curve singularities of finite Cohen-Macaulay type, Ark. Mat. 29 (1991) 339–357. MR1150382 (93d:13020)
23. R. Wiegand, One-dimensional local rings with finite Cohen-Macaulay type, in: C.L. Bajaj (Ed.), Algebraic Geometry and its Applications, Springer-Verlag, New York, 1994, pp. 381–389. MR1272043 (95b:13015)
24. R. Wiegand, Local rings of finite Cohen-Macaulay type, J. Algebra 203 (1998) 156–168. MR1620725 (99c:13025)
25. R. Wiegand, Failure of Krull-Schmidt for direct sums of copies of a module, in: D. Dobbs, M. Fontana, S.-E. Kabbaj (Eds.), Advances in Commutative Ring Theory, Dekker, 1999, pp. 541–547. MR1767419 (2001g:13017)
26. R. Wiegand, S. Wiegand, Bounds for one-dimensional rings of finite Cohen-Macaulay type, J. Pure Appl. Algebra 93 (1994) 311–342. MR1275969 (95c:13004)
27. Y. Yoshino, Cohen-Macaulay Modules over Cohen-Macaulay Rings, in: Lect. Notes Ser., vol. 146, London Math. Soc., 1990. MR1079937 (92b:13016)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.