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ORIGINAL ARTICLE

Feedback invariance of SISO infinite-dimensional systems

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Abstract We consider a linear single-input single-output system on a Hilbert space X, with infinitesimal generator A, bounded control element b, and bounded observation element c. We address the problem of finding the largest feedback invariant subspace of X that is in the space $c^{\perp}$ perpendicular to c. If b is not in $c^{\perp}$, we show this subspace is $c^{\perp}$. If b is in $c^{\perp}$, a number of situations may occur, depending on the relationship between b and c.

Keywords Feedback invariance · Closed loop invariance · Feedback · Infinite-dimensional systems · Zero dynamics

1 Introduction

In this paper, we consider a single-input single-output system, with bounded control and observation, on a Hilbert space X. Let the inner product on X be $\langle \cdot, \cdot \rangle$, with associated norm $\| \cdot \|$. Let A be the infinitesimal generator of a C_0-semigroup $T(t)$ on X [13, Definition 2.1]. Let b and c be elements of X. Let $U = \mathbb{C}$ and $u(t) \in U$. We consider the following system on X:

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x_0, \tag{1.1}$$

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with the observation

$$y(t) = Cx(t) := \langle x(t), c\rangle. \tag{1.2}$$

We sometimes refer to this system as (A, b, c). The transfer function for this system is $G(s) = \langle R(s, A)b, c\rangle$, where $R(s, A) := (sI - A)^{-1}$. The following is the standard definition of A-invariance.

Definition 1.1 A subspace Z of X is A-invariant if $A(Z \cap D(A)) \subset Z$.

If we allow unbounded feedback, we obtain the following definition of feedback invariance.

Definition 1.2 A subspace Z of X is (A, b) feedback invariant if it is closed and there exists an A-bounded feedback K such that Z is $A + bK$-invariant.

Our primary concern in this paper is to find the largest (A, b) feedback invariant subspace of the kernel of C. The operator K is not specified as unique in the above definition. However, if $b \notin Z$, and there are two operators K_1 and K_2 that are both (A, b) feedback invariant on Z, then $b(K_1x - K_2x) \in Z$ and so $K_1x = K_2x$ for all $x \in Z$. Even though we assume that b and c are in X, in general the feedback K is not bounded and $A + bK$ is in not the generator of a strongly continuous semigroup. For finite-dimensional systems, the largest invariant subspace in the kernel of C always exists. However, this is not the case for infinite-dimensional systems.

Feedback invariant subspaces are important in several aspects of control and systems theory. They are relevant to the topic of zero dynamics [5,15]. Feedback-invariant subspaces are critical in solving the disturbance decoupling problem; see for example [3,10–12,15,19]. In Sect. 5, we briefly discuss disturbance decoupling and give an example. Also, suppose that for a system (A, b, c) a largest feedback invariant subspace $Z \subseteq c^{\perp}$ exists, and let K be a feedback so that Z is $A + bK$-invariant. The system zeros (e.g. [8]) are identical to the eigenvalues of the operator $A + bK$ on Z.

The work in this paper builds on the results of Curtain and Zwart in the 1980s, see [2,17,18]. In [17,18] it is assumed that either the feedback K is bounded, or, if K is unbounded, it is such that $A + bK$ is a generator of a C_0-semigroup. These conditions are imposed in order to avoid difficulties about the generation of a semigroup by $A + bK$. In this paper we consider unbounded K, with no assumption on semigroup generation. This paper also extends the results in Byrnes et al. [1], where the invariance problem in solved for (A, b, c) under the assumptions that $b \in D(A)$, $c \in D(A^*)$ and $\langle b, c\rangle \neq 0$. In this paper we remove the restrictions $b \in D(A)$ and $c \in D(A^*)$, and, most significantly, also examine the case where $\langle b, c\rangle = 0$.

We denote the kernel of C by

$$c^{\perp} := \{x \in X \mid \langle x, c\rangle = 0\}.$$

If $b \notin c^{\perp}$, we show in Sect. 2 that a largest feedback invariant subspace in $c^{\perp}$ exists and it is in fact $c^{\perp}$. We give an explicit representation of a feedback operator K for which $c^{\perp}$ is $A + bK$-invariant. If $c \in D(A^*)$, the operator K is bounded. Otherwise, K is only A-bounded and so $A + bK$ need not generate a semigroup.

If $\langle b, c\rangle = 0$, then we can still find the largest feedback invariant subspace in many cases. This hinges upon the *relative degree* of (A, b, c).

Definition 1.3 (A, b, c) is of relative degree n for some positive integer n if

1. $\lim_{s\to\infty,\ s\in\mathbb{R}} s^n G(s) \neq 0$ and
2. $\lim_{s\to\infty,\ s\in\mathbb{R}} s^{n-1} G(s) = 0.$

We show that if (A, b, c) has relative degree $n + 1$ and $c \in D(A^{*n})$ then the largest invariant subspace in $c^\perp$ exists. This result is a generalization of the well-known feedback invariance result for finite-dimensional systems [15].

There is no a priori guarantee that the closed loop system has a generalized solution. Additional assumptions are required. We now give a definition of "uniform relative degree" which strengthens condition 1 in Definition 1.3 to include a specification of the behaviour of the transfer function in some right-half-plane. For $\omega \in \mathbb{R}$, let

$$C_\omega = \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega\}.$$

The space H_ω^∞ is the Hardy space of bounded analytic functions in C_ω.

Definition 1.4 (A, b, c) is of uniform relative degree n for some positive integer n if

1. the function $(s^n G(s))^{-1}$ is in H_γ^∞ for some $\gamma \in \mathbb{R}$;
2. $\lim_{s\to\infty,\ s\in\mathbb{R}} s^{n-1} G(s) = 0.$

In finite-dimensional spaces condition 1 in Definition 1.4 is equivalent to condition 1 in Definition 1.3, but they are not guaranteed to be equivalent in an infinite dimensional space. Suppose that $c \in D(A^{*n})$ and (A, b, c) is of uniform relative degree $n + 1$. Let K be an operator such that the largest feedback invariant subspace is $A+bK$-invariant. We show in Proposition 3.3 that the additional assumption of uniform relative degree is sufficient to ensure that the closed loop system

$$\dot{x}(t) = Ax(t) + bKx(t),$$

with initial data in $D(A)$, has a generalized solution which satisfies the semigroup property. Furthermore, $A + bK$ generates an integrated semigroup; see Neubrander [9] for a detailed discussion of integrated semigroups, in particular Definition 4.1 in [9] for a definition of an integrated semigroup. There is no guarantee that the closed loop operator $A + bK$ generates a strongly continuous semigroup. We also show in Sect. 3 that if $A + bK$ does generate a C_0-semigroup on X, then it generates a C_0-semigroup on the largest feedback invariant subspace of $c^\perp$.

In Sect. 4, we consider the case where $\langle b, c\rangle = 0$, but $c \notin D(A^*)$. We give an example which shows that the largest feedback invariant subspace of the kernel of C might not exist. We identify a natural feedback operator K and subspace $Z \subseteq c^\perp$ so that $(A + bK)(Z) \subset Z$, but we show that $A + bK$ is neither closed nor closable. In Sect. 5, we illustrate our results with a disturbance decoupling problem.

2 Feedback invariance

We start with some additional notation needed in this paper. Let $\omega \in \mathbb{R}$ be such that $\mathbb{C}_\omega$ is a subset of the resolvent set $\rho(A)$. For $\lambda_0 > \omega$, $R(\lambda_0, A)$ exists as a bounded operator from X into X. For any operator A, $\rho_\infty(A)$ is the largest connected subset of $\rho(A)$ that contains an interval of the form $[r, \infty)$.

The following result shows that (A, b) feedback invariance is equivalent to the notion of (A, b)-invariance, which is sometimes easier to work with.

Theorem 2.1 [18, Thm.II.26] *A closed subspace Z is (A, b) feedback invariant if and only if it is (A, b)-invariant, that is,*

$$A(Z \cap D(A)) \subseteq Z \oplus \mathrm{span}\{b\}.$$

When the operators A and b are clear we will sometimes refer to (A, b) feedback invariance simply as feedback invariance, and to a subspace as invariant.

Theorem 2.2 *If $Z \subseteq c^\perp$ is an (A, b) feedback invariant subspace and $b \in Z$, then the system transfer function is identically zero for $s \in \rho_\infty(A)$.*

Proof Since Z is feedback invariant,

$$A(Z \cap D(A)) \subset Z \oplus \mathrm{span}\{b\} \subset Z.$$

This implies that Z is A-invariant. This implies that every $z \in Z$ can be written $z = (sI - A)\xi(s)$ where $\xi(s) \in D(A) \cap Z$ [18, Lemma. I.4], and $s \in [r, \infty)$ for some $r \in \mathbb{R}$. Since $b \in Z$, $R(s, A)b \in Z$ for all $s \in [r, \infty)$. Since $Z \subset c^\perp$, the system transfer function $G(s)$ is zero for $s \in [r, \infty]$. Since G is analytic on $\rho_\infty(A)$, it must be identically zero on $\rho_\infty(A)$. □

We now show that if $b \notin c^\perp$, the largest feedback invariant subspace contained in $c^\perp$ is $c^\perp$. We do this by easily constructing a feedback operator K such that $(A + bK)(c^\perp \cup D(A)) \subseteq c^\perp$. If $c \in D(A^*)$, then the feedback K is bounded, and $A + bK$ is the generator of a semigroup on $c^\perp$. In general, $A + bK$ does not generate a C_0-semigroup.

Theorem 2.3 *Suppose that* $\langle b, c\rangle \neq 0$. *Define*

$$Kx = -\frac{\langle Ax, c\rangle}{\langle b, c\rangle}, \quad D(K) = D(A), \tag{2.3}$$

and define $(A + bK)x = Ax + bKx$ for $x \in D(A + bK) = D(A)$. Then $(A + bK)(c^\perp \cap D(A)) \subset c^\perp$ and so the largest feedback invariant subspace in $c^\perp$ is $c^\perp$ itself.

Proof The operator K is clearly A-bounded. It is straightforward to see that for $x \in D(A)$, $\langle (A + bK)x, c\rangle = \langle Ax, c\rangle - \langle Ax, c\rangle = 0$. Thus, $(A + bK)x \in c^\perp$, so $c^\perp$ is feedback invariant. □

If $\langle b, c\rangle = 0$, we can still find the largest feedback invariant subspace in many cases. In finding the largest feedback invariant subspace, a difficulty occurs using Definition 1.1 that does not occur in finite dimensions. This is because Definition 1.1 allows, roughly speaking, arbitrary elements of $D(A)$ be be "appended" to a subspace Z without changing $Z \cap D(A)$, as illustrated by the following example.

Example 2.4 Let $X = \ell^2$, $c = [1, 0, 0, 0, \ldots]^{\mathrm{T}}$ and $b = [0, 1, 0, 0, \ldots]^{\mathrm{T}}$, and

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \operatorname{diag}(ki)_{k=1}^{\infty}.$$

Let

$$v_1 = [0, 0, 1, 0, 0, \ldots]^{\mathrm{T}}$$

and v_2 be any element of $c^{\perp}$ which is not in $D(A)$, and define subsets of $c^{\perp}$ by

$$Z = \operatorname{span}\{v_1\}, \quad \tilde{Z} = \operatorname{span}\{v_1, v_2\}.$$

It is clear that Z is A-invariant. Since $v_1 \in D(A)$ and $v_2 \notin D(A)$, $z \in \tilde{Z}$ is in $D(A)$ if and only if $z = cv_1$ for some scalar c. Hence $Z \cap D(A) = \tilde{Z} \cap D(A)$, so $\tilde{Z}$ is also A-invariant, regardless of the choice of v_2.

To rule out the possibility of appending to Z arbitrary elements in $X \setminus D(A)$, as illustrated in Example 2.4, we will modify the definition of A-invariance as follows.

Definition 2.5 A subspace Z of X is A-invariant if $A(Z \cap D(A)) \subset Z$ and $Z \cap D(A)$ is dense in $D(A)$.

If $A + bK$ generates a C_0-semigroup on Z, this definition is the same as Definition 1.1, since in this case $D(A + bK) \cap Z = D(A) \cap Z$ is guaranteed to be dense in Z. In [18] the definition of a largest invariant subspace includes the assumption that $A + bK$ is the generator of a C_0-semigroup, so there is no need in [18] to include this denseness assumption.

Definition 1.2 is unchanged, except that this definition of A-invariance means that $Z \cap D(A)$ must be dense in Z in order for Z to be considered as a feedback invariant subspace.

If $c \in D(A^{*n})$ for some integer $n \geq 1$, define

$$Z_n = c^{\perp} \cap (A^* c)^{\perp} \cap \cdots (A^{*n} c)^{\perp},$$

and define $Z_0 = c^{\perp}$ and $Z_{-1} = X$.

Lemma 2.6 *$Z_n \cap D(A)$ is dense in Z_n.*

Proof We first define a projection on Z_n. Let m be the dimension of $\operatorname{span}\{c, A^* c \ldots A^{*n} c\}$. Choose $\{\alpha_j\}_{j=1}^{m}$ to be a linearly independent subset of this span. Choose an m-dimensional subspace $W_n \subset D(A)$ so that $W_n \cap Z_n = \emptyset$ and $X = Z_n \oplus W_n$. Choose $\{\beta_j\}_{j=1}^m$ to be a basis for W_n and define the projection

$$Q_n x = \sum_{j=0}^{m} \frac{\langle x, \alpha_j\rangle}{\langle \beta_j, \alpha_j\rangle} \beta_j \tag{2.4}$$

from X onto W_n. It is clear that Range$(Q_n) \subset D(A)$, and it can easily be checked that Range $(I - Q_n) = Z_n$.

For $z \in Z_n$, choose $\{z_j\} \subset D(A)$ such that $z_j \to z$. Then $(I - Q_n)z_j \in D(A)$. Since $z \in Z_n$, $Qz_j \to 0$. Hence $x_j = (I - Q)z_j \in Z_n \cap D(A)$ and $x_j \to z$. □

Theorem 2.7 *Suppose that an integer $n \geq 1$ is such that*

$$c \in D(A^{*n}), \quad b \in Z_{n-1} \tag{2.5}$$

and

$$\langle b, A^{*n} c\rangle \neq 0. \tag{2.6}$$

Then the largest feedback invariant subspace Z in $c^\perp$ is Z_n. One feedback K such that Z_n is $A + bK$-invariant is

$$Kx = \langle Ax, a\rangle, \quad a = \frac{-A^{*n} c}{\langle b, A^{*n} c\rangle}, \quad D(K) = D(A). \tag{2.7}$$

Remark 2.8 As noted after Definition 1.2, changing K on $(Z_n)^\perp$ does not change the conclusion of Theorem 2.7.

Proof We first prove that if (2.5) holds, then any feedback invariant subspace Z is contained in Z_n. We then show that Z_n is feedback invariant.

Claim If (2.5) holds and Z is a feedback invariant subspace in $c^\perp$, then $Z \subseteq Z_n$.

Proof of the claim: Assume that Z is a feedback invariant subspace and $Z \subseteq c^\perp$. We will prove the claim by induction. Suppose that (2.5) holds for $n = 1$. From Theorem 2.1, we see that

$$A(Z \cap D(A)) \subseteq Z \oplus \operatorname{span}\{b\} \subseteq c^\perp. \tag{2.8}$$

Hence for $z \in Z \cap D(A)$,

$$0 = \langle Az, c\rangle = \langle z, A^* c\rangle. \tag{2.9}$$

Since Z is $A + bK$-invariant, by Definition 2.5, $Z \cap D(A)$ is dense in Z, so (2.9) is true for all $z \in Z$, showing that $Z \subset Z_1$.

Assume the induction hypothesis that (2.5) implies that $Z \subseteq Z_n$. Suppose that $c \in D(A^{*(n+1)})$ and $b \in Z_n$, so (2.5) holds, and by the induction hypothesis $Z \subseteq Z_n$. From Theorem 2.1, we see that

$$A(Z \cap D(A)) \subseteq Z \oplus \operatorname{span}\{b\} \subseteq Z_n.$$

Therefore, for $z \in D(A) \cap Z$, $Az \in (A^{*n}c)^{\perp}$, so

$$0 = \langle Az, A^{*n}c \rangle = \langle z, A^{*(n+1)}c \rangle.$$

Since $Z \cap D(A)$ is dense in Z, this implies that $Z \subseteq Z_{n+1}$, completing the induction step, proving the claim.

We now show that Z_n is feedback invariant. Assume that (2.5) and (2.6) are true. Let P_{n-1} be an orthogonal projection of X onto Z_{n-1}. If $z \in Z_{n-1}$, then, since (2.5) and (2.6) hold,

$$\langle z, A^{*n}c \rangle = \langle P_{n-1}z, A^{*n}c \rangle = \langle z, P_{n-1}A^{*n}c \rangle,$$

so

$$Z_n = Z_{n-1} \cap (A^{*n}c)^{\perp} = Z_{n-1} \cap (P_{n-1}A^{*n}c)^{\perp}.$$

We will apply Theorem 2.3, with:

- X replaced by Z_{n-1}, which is a Hilbert space with the same inner product;
- A replaced by $P_{n-1}A|_{Z_{n-1}}$;
- The same b, which is in Z_{n-1};
- c replaced by $P_{n-1}A^{*n}c$.

Note that in general $P_{n-1}A|_{Z_{n-1}}$ does *not* generate a semigroup on Z_{n-1}, but the feedback invariance in Theorem 2.3 does not require semigroup generation of A.

We need to verify that

$$\langle b, P_{n-1}A^{*n}c \rangle \neq 0. \tag{2.10}$$

To this end, note that by using (2.5) and (2.6),

$$\langle b, P_{n-1}A^{*n}c \rangle = \langle P_{n-1}b, A^{*n}c \rangle = \langle b, A^{*n}c \rangle \neq 0.$$

For $x \in Z_{n-1} \cap D(A)$, define

$$K_n x = -\frac{\langle P_{n-1}Ax, P_{n-1}A^{n*}c \rangle}{\langle b, P_{n-1}A^{n*}c \rangle} = -\frac{\langle P_{n-1}Ax, A^{n*}c \rangle}{\langle b, A^{n*}c \rangle}.$$

Theorem 2.3 implies that the space Z_n is an invariant subspace of $P_{n-1}A|_{Z_{n-1}} + bK_n$.

Now, $A(Z_n \cap D(A)) \subseteq Z_{n-1}$, so

$$P_{n-1}A|_{Z_n} = A|_{Z_n}.$$

Hence Z_n is an invariant subspace of $A|_{Z_{n-1}} + bK_n$. Since for any $x \in Z_n \cap D(A)$, $Ax \in Z_{n-1}$, we can rewrite $K_n|_{Z_n}$ as

$$K_n x = -\frac{\langle Ax, A^{n*}c \rangle}{\langle b, A^{n*}c \rangle}. \tag{2.11}$$

We can extend $K_n|_{Z_n}$ to an operator $K \in \mathcal{B}([D(A)], U)$ by letting

$$Kx = \langle Ax, a\rangle, \quad a = \frac{-A^{n*}c}{\langle b, A^{n*}c\rangle}$$

for $x \in D(A)$. Therefore Z_n is an invariant subspace of $A + bK$. □

Note that (2.7) becomes (2.3) if $n = 0$. The operator K is A-bounded. If $a \notin D(A^*)$, K is not bounded.

Example 2.4 (continued) In this example $\langle b, c\rangle = 0$, $c \in D(A^*)$ and, since $A^*c = b$, $\langle b, A^*c\rangle = 1$. Therefore, Theorem 2.7 with $n = 1$ is applicable. Hence the largest feedback invariant subspace is $Z_1 = c^{\perp} \cap (A^*c)^{\perp} = c^{\perp} \cap b^{\perp}$, and the bounded feedback $Kx = \langle x, c\rangle$ is such that Z_1 is $A + bK$-invariant.

From this example we see why we cannot have a notion of a "largest feedback invariant subspace" while using Definition 1.1 of invariance. The subspace $\tilde{Z}$ is feedback invariant when using Definition 1.1 of invariance, but is not when using Definition 2.5. If $\langle v_2, A^*c\rangle \neq 0$, then $\tilde{Z}$ is not a subspace of Z_1, because of the elements of $\tilde{Z}$ which are not in $D(A)$ or Z_1.

We can relate conditions (2.5) and (2.6) to Definition 1.3 of relative degree. In particular, (A, b, c) is of relative degree 1 if and only if $\langle b, c\rangle \neq 0$. Also, if $c \in D(A^*)$, (A, b, c) is of relative degree 2 if and only if $\langle b, c\rangle = 0$ and $\langle b, A^*c\rangle \neq 0$.

Lemma 2.9 *For a non-negative integer n, let $c \in D(A^{*n})$. Then (A, b, c) is of relative degree $n + 1$ if and only if $b \in Z_{n-1}$ and $\langle b, A^{*n}c\rangle \neq 0$.*

Proof We first show that if $c \in D(A^{*j})$ where j is any positive integer,

$$\langle R(s, A)b, A^{*j}c\rangle = \langle -b, A^{*(j-1)}c\rangle + s\langle -b, A^{*(j-2)}c\rangle + \cdots s^{j-1}\langle -b, c\rangle + s^j G(s). \tag{2.12}$$

Since

$$\langle R(s, A)b, A^*c\rangle = \langle AR(s, A)b, c\rangle = -\langle b, c\rangle + s\langle R(s, A)b, c\rangle,$$

the statement is true for $j = 1$. It is easy to see that

$$\begin{aligned}\langle R(s, A)b, A^{*j}c\rangle &= \langle AR(s, A)b, A^{*(j-1)}c\rangle \\ &= -\langle b, A^{*(j-1)}c\rangle + s\langle R(s, A)b, A^{*(j-1)}c\rangle.\end{aligned}$$

The statement (2.12) now follows by induction.

Now assume that for a non-negative integer n, $c \in D(A^{*n})$, $b \in Z_{n-1}$ and $\langle b, A^{*n}c\rangle \neq 0$. Equation (2.12) becomes, for $j = n$,

$$\langle R(s, A)b, A^{*n}c\rangle = s^n G(s). \tag{2.13}$$

Taking limits yields,

$$\lim_{s\to\infty,\ s\in\mathbb{R}} s^n G(s) = 0.$$

For $j = n + 1$ we obtain from (2.12)

$$\lim_{s\to\infty,\ s\in\mathbb{R}} s^{n+1} G(s) = \langle b, A^{*n)} c\rangle \neq 0.$$

Thus the system has relative degree $n + 1$.

Now assume that for some non-negative integer n, the system has relative degree $n + 1$ and $c \in D(A^{*n})$. Since $\lim_{s\to\infty,\ s\in\mathbb{R}} sR(s, A)x = x$ for all $x \in X$,

$$\lim_{s\to\infty,\ s\in\mathbb{R}} sG(s) = \langle b, c\rangle.$$

This completes the proof if $n = 0$. Suppose now that $n > 0$. We obtain from (2.12), setting $j = n$ and using $\lim_{s\to\infty,\ s\in\mathbb{R}} s^n G(s) = 0$,

$$\lim_{s\to\infty,\ s\in\mathbb{R}} \langle -bA^{*(n-1)} c\rangle + s\langle -b, A^{*(n-2)} c\rangle + \cdots + s^{n-1}\langle -b, c\rangle = 0.$$

Since each coefficient of s^i is a constant, this implies that

$$\langle b, A^{*i} c\rangle = 0, \quad i = 0, \ldots, n - 1.$$

Thus, $b \in Z_{n-1}$. Now substitute $j = n + 1$ into (2.12) to obtain

$$\lim_{s\to\infty,\ s\in\mathbb{R}} s^{n+1} G(s) = \langle b, A^{*n} c\rangle \neq 0.$$

This completes the proof. □

The following theorem follows immediately from Theorem 2.7 and Lemma 2.9.

Theorem 2.10 *Suppose that* (A, b, c) *is of relative degree* $n + 1$*, where* n *is a non-negative integer, and that* $c \in D(A^{*n})$*. Then the largest feedback invariant subspace* Z *in* $c^\perp$ *is* Z_n*.*

3 Closed-loop invariance

If a feedback operator K is unbounded there is no a priori guarantee that the system obtained by setting $u(t) = Kx(t)$,

$$\dot{x}(t) = Ax(t) + bKx(t),$$

has solutions.

In Definition 1.4 we gave a definition of *uniform relative degree* that is slightly stronger than the definition of relative degree. We will see that if (A, b, c) is of uniform relative degree n for some nonnegative integer n, then the closed loop system is guaranteed to have a generalized solution which stays in the feedback invariant subspace and satisfies a semigroup property. We rely on the following result from Lasiecka and Triggiani [7].

Proposition 3.1 [7, pp. 647–649, Proposition 2.4] *Let* $Kx = \langle Ax, a\rangle$ *for* $a \in X$ *and* $D(K) = D(A)$. *If there exist some* $m > 0$ *and* $\delta \in \mathbb{R}$ *such that*

$$|1 - \langle AR(s, A)b, a\rangle|\rangle| \geq m \quad for \quad s \in \mathbb{C}_\delta, \tag{3.14}$$

then for each $x_0 \in D(A)$, *and any* $T > 0$ *there exists a unique solution* $x(t) \in C([0, T]; X)$ *of the integral equation*

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-s)}bKx(s)\,\mathrm{d}s \tag{3.15}$$

where $Kx(s) \in L_2(0, t)$ *for any* $x_0 \in D(A)$. *This solution satisfies the semigroup property:* $x(t + \tau, x_0) = x(\tau, x(t, x_0))$ *for any* $t, \tau \geq 0$. *Furthermore, the solution* $x(t)$ *is Laplace transformable with convergence in some right-half-plane.*

The solution to (3.15) does not in general yield a strongly continuous semigroup. The next result shows that if the hypotheses of Proposition 3.1 hold, then $A + bK$ generates an integrated semigroup. Integrated semgigroups are a generalization of strongly continuous semigroups. See [9] for details. In this case, if the initial data is smooth enough, then the solution given by this semigroup is a classical solution to the Cauchy problem $\dot{x}(t) = (A + bK)x(t)$; see Theorems 4.2 and 4.5 in [9] for a description of the relationship between the integrated semigroup and the solution to the Cauchy problem.

Proposition 3.2 *Let* $Kx = \langle Ax, a\rangle$ *for* $a \in X$ *and* $D(K) = D(A)$. *If there exist some* $m > 0$ *and* $\delta \in \mathbb{R}$ *such that* (3.14) *holds, then* $A + bK$ *generates an integrated semigroup.*

Proof In Theorem 4.8 of [9] it is shown that a densely defined linear operator A generates an integrated semigroup if and only if there exist real constants M, w, and $k \in \mathbb{N}_0$ such that $R(s, A)$ exists and satisfies

$$\|R(s, A)\| \leq M(1 + |s|)^k \quad \text{for all } s \in \mathbb{C}_w.$$

From [7, Eq. (2.13)], for $s \in C_\delta$ where C_δ is as in the previous proposition,

$$R(s, A + bK) = R(s, A) + \frac{R(s, A)bKR(s, A)}{1 - \langle AR(s, A)b, a\rangle}. \tag{3.16}$$

Note that

$$KR(s,A)x = \langle AR(s,A)x, a\rangle = s\langle R(s,A)x, a\rangle - \langle x, a\rangle \tag{3.17}$$

and that there exists real constants M_1 and w_1 such that

$$\|R(s,A)\| \leq \frac{M_1}{\operatorname{Re}(s) - w_1}. \tag{3.18}$$

Combining (3.16), (3.17) and (3.18),

$$\|R(s, A+bK)\| \leq M(1+|s|)^k \quad \text{for all } s \in \mathbb{C}_w,$$

is satisfied with $k = 1$, completing the proof. ◻

Proposition 3.3 *Assume that (A, b, c) has uniform relative degree $n+1$ and $c \in D(A^{*n})$ for some non-negative integer n. Defining K by (2.7), the solution to (1.1) with initial condition $x_0 \in D(A)$ and $u(t) = Kx(t)$ satisfies (3.15). Furthermore, if $x_0 \in D(A) \cap Z_n$, the solution $x(t)$ of (3.15) remains in Z_n for all t.*

Proof The first part of this result is a simple consequence of Proposition 3.1. Using the definition of K given by (2.7),

$$\begin{aligned} 1 - KR(s,A)b &= 1 - \langle AR(s,A)b, a\rangle \\ &= 1 + \frac{\langle AR(s,A)b, A^{n*}c\rangle}{\langle b, A^{n*}c\rangle} \\ &= s\frac{\langle R(s,A)b, A^{n*}c\rangle}{\langle b, A^{n*}c\rangle}. \end{aligned}$$

From (2.13),

$$s\langle R(s,A)b, A^{n*}c\rangle = s^{n+1}G(s).$$

Thus,

$$|1 - KR(s,A)b| = \frac{|s^{n+1}G(s)|}{\langle b, A^{n*}c\rangle},$$

which satisfies (3.14) since (A, b, c) has uniform relative degree $n+1$.

Indicate the unique solution of (3.15) by $S_K(t)x_0$ for any $t \geq 0$ and $x_0 \in D(A) \cap Z^n$. We will show that $\langle S_K(t)x_0, c\rangle = 0$ for all such t and x_0. This is equivalent to showing that the Laplace transform of $\langle S_K(t)x_0, c\rangle$ is identically zero in some right-half-plane. Since $\langle\cdot, c\rangle$ is a continuous operation on X we can interchange this with the Laplace transform $L(s, x_0) := \mathcal{L}(S_K(t)x_0)$. From [7, Eq. (2.13)],

$$L(s, x_0) = R(s;A)x_0 + \frac{R(s;A)b\langle AR(s;A)x_0, a\rangle}{1 - \langle AR(s,A)b, a\rangle} \tag{3.19}$$

where a is defined in (2.7). Rewriting,

$$L(s,x_0) = \frac{[R(s,A)x_0 - \langle AR(s,A)b, a\rangle R(s,A)x_0 + R(s,A)b\langle AR(s,A)x_0, a\rangle]}{1 - \langle AR(s,A)b, a\rangle}.$$

It is now straightforward to verify that if $n = 0$ in (2.7), $\langle L(s,x_0), c\rangle = 0$. Similarly, if $n > 0$, $\langle L(s,x_0), A^{*j}c\rangle = 0$ for $1 \le j \le n$. Thus, $L(s,x_0) \in Z_n$. This implies that $x(t) \in Z_n$ for all $t > 0$. □

If the conditions of Proposition 3.3 are satisfied, there is still no guarantee that that the solution semigroup is strongly continuous. It is well-known that a relatively bounded perturbation of a generator of a C_0-semigroup is not necessarily the generator of a C_0-semigroup, see for instance the example in [7, Sect. 2.2.2, p. 652]. In fact, this example can be modified in order to obtain a system with uniform relative degree 1 for which $A + bK$ generates a semigroup, yet the semigroup is not strongly continuous.

Definition 3.4 A closed subspace Z of X is *closed-loop invariant* if the closure of $Z \cap D(A)$ in X is Z, there exists an A-bounded feedback K such that $(A + bK)(Z \cap D(A)) \subseteq Z$, and the restriction of $A + bK$ to Z generates a C_0-semigroup on Z.

The condition that $(A + bK)(Z \cap D(A)) \subset Z$ allows arbitrary elements of $X \setminus D(A)$ to be appended to Z. The additional condition that the closure of $Z \cap D(A)$ is Z eliminates this ambiguity.

There are many results in the literature that give sufficient conditions for a relatively bounded perturbation of a generator of a C_0-semigroup to be the generator of a C_0-semigroup. For instance, if for any $T > 0$ and some $M_T > 0$, K satisfies for all $x_0 \in D(A)$,

$$\|KS(t)x_o\|_{L_2(0,T)} \le M_T\|x_0\|_X$$

[14, Chap. 5], or if A generates an analytic semigroup [6, Chap. 9, Sect. 2.2], then $A + bK$ generates a C_0-semigroup.

Assume now that $A + bK$ is the generator of a C_0-semigroup on X. In general, feedback invariance does not imply closed-loop invariance [18, e.g. 1.6]. However, in the case where K is given by (2.7), Z_n is closed-loop invariant under the semigroup $e^{(A+bK)t}$ generated by $A + bK$.

Theorem 3.5 *Assume that an integer $n \ge 0$ is such that (2.5) and (2.6) hold, and define K as in (2.7). Also assume that $A + bK$ generates a C_0-semigroup on X. Then the restriction of $A + bK$ to Z_n generates a C_0-semigroup on Z_n. Hence Z_n is closed-loop invariant under $A + bK$.*

Proof We will show that for $\lambda \in \rho_\infty(A+bK)$ the image of Z_n under $(\lambda I - (A+bK))$ is all of Z_n. This will imply, by [18, Lemma I.4], that Z_n is $e^{(A+bK)t}$ invariant.

We will use the projection Q_n defined in (2.4), which we will denote here by Q for convenience, to decompose X into $X_1 \oplus X_2$, where $X_1 = Z_n$ and $X_2 = W_n$. Any element of X can be written $x = x_1 + x_2$, where $x_1 = (I - Q)x \in X_1$ and

$x_2 = Qx \in X_2$. Because $Qx \in D(A)$ for every $x \in X$, if $x \in D(A)$ then $x_1 \in D(A)$ and $x_2 \in D(A)$. The operator A can be decomposed as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{3.20}$$

where

$$A_{11} = (I - Q)A|_{X_1}, \quad A_{12} = (I - Q)A|_{X_2}, \quad A_{21} = QA|_{X_1}, \quad A_{22} = QA|_{X_2}. \tag{3.21}$$

Let $b_1 = (I - Q)b$ and $b_2 = Qb$. Let K be as in (2.7), so $(A + bK)(X_1 \cap D(A)) \subset X_1$. Let $K_1 = K(I - Q)$ and $K_2 = KQ$, so with $\tilde{A}_{12} = A_{12} + b_1 K_2$ and $\tilde{A}_{22} = A_{22} + b_2 K_2$, we can write

$$(\lambda I - (A + bK))x = \begin{bmatrix} (\lambda I - A_{11} - b_1 K_1)x_1 - \tilde{A}_{12} x_2 \\ (\lambda I - \tilde{A}_{22})x_2 \end{bmatrix}. \tag{3.22}$$

Since $\lambda \in \rho(A + bK)$, the range of $(\lambda I - (A + bK))$ is all of X. Since $\{\beta_j\}_{j=1}^n$ is a basis of X_2, the image of X_2 under $\tilde{A}_{12}$ is $\operatorname{span}\{\tilde{A}_{12}\beta_j\}_{j=1}^n$. Thus, the image of X_1 under $(A + bK)$ contains X_1 if the image of X_1 under $(\lambda I - (A_{11} + b_1 K_1))$ contains $\{\tilde{A}_{12}\beta_j\}$ for each $j = 1, 2, \ldots, n$. To show this, for each $j = 1, 2, \ldots, n$ note that there exists unique x_1 and x_2 that solve

$$\begin{bmatrix} (\lambda I - A_{11} - b_1 K_1)x_1 - \tilde{A}_{12} x_2 \\ (\lambda I - \tilde{A}_{22})x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \left(\lambda I - \tilde{A}_{22}\right)\beta_j \end{bmatrix}. \tag{3.23}$$

From (3.22) we see that if $\lambda \in \rho(A + bK)$ and (3.23) holds, then $x_2 = \beta_j$. Plugging this into the first row of the matrix equation (3.23) we obtain that

$$(\lambda I - A_{11} - b_1 K_1)x_1 - \tilde{A}_{12}\beta_j = 0.$$

This shows that the image of X_1 under $A + bK$ contains $\operatorname{span}\{\tilde{A}_{12}\beta_j\}$. Hence the image of X_1 under $A + bK$ contains X_1, so X_1 is closed-loop invariant. □

Example 3.6 We consider the following one dimensional heat equation with Dirichlet boundary conditions, which was also discussed in [18, e.g. IV.22]:

$$\frac{\partial x}{\partial t}(r, t) = \frac{\partial x^2}{\partial r^2}(r, t) + b(r)u(t), \quad r \in (0, 1), \ t > 0 \tag{3.24}$$

$$x(0, t) = 0, \quad x(1, t) = 0 \tag{3.25}$$

$$y(t) = \int_0^1 x(r, t)c_i(r)\mathrm{d}r. \tag{3.26}$$

For this system, the state space is $X = L_2(0, 1)$ and the infinitesimal generator is

$$A = \frac{\partial^2}{\partial r^2}, \quad D(A) = \{x \in H_2(0,1); x(0) = x(1) = 0\}.$$

Note that for this generator $A^* = A$. We choose b to be the characteristic function on $[0, \frac{2}{\pi}]$:

$$b(r) = \chi_{[0,\frac{2}{\pi}]}(r).$$

We consider two observation elements. The first is

$$c_1(r) = \begin{cases} -100r^2 + 20r; & 0 \le r \le 0.1 \\ 1; & 0.1 < r \le \frac{1}{\pi} - 0.1 \\ 2000(r - \frac{1}{\pi})^3 + 300(r - \frac{1}{\pi})^2; & \frac{1}{\pi} - 0.1 < r \le \frac{1}{\pi} \\ 0 & \frac{1}{\pi} < r \le 1. \end{cases} \tag{3.27}$$

In [18, E.g. IV.22] it is shown that in this case the largest feedback invariant subspace in $c_1^\perp$ exists. However, these earlier results did not identify this largest subspace, nor the appropriate feedback. It is easy to check that $\langle b, c_1 \rangle \neq 0$, so the largest closed-loop invariant subspace in $c_1^\perp$ is $c_1^\perp$. Since $c_1 \in D(A^*) = D(A)$, the feedback $K_1 x = \langle Ax, c_1 \rangle / \langle b, c_1 \rangle$ is bounded, and can be written

$$K_1 x = \langle x, k_1 \rangle$$

where

$$\begin{aligned} k_1 &= \frac{-1}{\langle b, c_1 \rangle} A c_1 \\ &= \frac{-1}{\langle b, c_1 \rangle} \frac{\partial^2 c_1}{\partial r^2} \\ &= -4.25 \begin{cases} -200; & 0 \le r < 0.1 \\ 0; & 0.1 \le r < \frac{1}{\pi} - 0.1 \\ 12000(r - \frac{1}{\pi}) + 600; & \frac{1}{\pi} - 0.1 \le r \le \frac{1}{\pi} \\ 0 & \frac{1}{\pi} \le r \le 1. \end{cases} \end{aligned} \tag{3.28}$$

Consider now the observation element

$$c_2(r) = \chi_{[0,\frac{1}{\pi}]}(r),$$

which is close in the X-norm to c_1, but is not in $D(A)$. We still have that $\langle b, c_2 \rangle \neq 0$ and so the largest feedback- invariant subspace in $c_2^\perp$ is $c_2^\perp$. Since A generates an analytic semigroup, this subspace is also closed-loop invariant. However, because $c_2 \notin D(A^*)$, the feedback operator is unbounded. Numerical investigations in [18, E.g. IV.22] indicated that no largest feedback invariant subspace of $c_2^\perp$ existed, but the definition used in [18] only allowed bounded feedback operators.

4 The case when $\langle b, c\rangle = 0$ and $c \notin D(A^*)$

The previous sections dealt with invariance for relative degree $n+1$ systems that satisfy an assumption that $c \in D(A^{*n})$. If this assumption on c is not satisfied, the situation is quite different.The following example illustrates that if $\langle b, c\rangle = 0$ and $c \notin D(A^*)$ a largest feedback invariant subspace as defined in Definition 1.2 might not exist.

Example 4.1 The following example of a controlled delay equation first appeared in Pandolfi [12]:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) - x_2(t-1) \\ \dot{x}_2(t) &= u(t) \\ y(t) &= x_1(t). \end{aligned} \tag{4.29}$$

The transfer function for this system is

$$G(s) = \frac{1-e^{-s}}{s^2}. \tag{4.30}$$

The system of equations (4.29) can be written in a standard state-space form (1.1), (1.2), see [4]. Choose the state-space

$$X = R \times R \times L_2(-1,0) \times L_2(-1,0).$$

A state-space realization on X is

$$b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$D(A) := \left\{ [r_1, r_2, \phi_1, \phi_2]^{\mathrm{T}} \mid \phi_1(0) = r_1, \phi_2(0) = r_2, \phi_1 \in H^1(-1,0), \phi_2 \in H^1(-1,0) \right\},$$

and for $[r_1, r_2, \phi_1, \phi_2]^{\mathrm{T}} \in D(A)$,

$$A(r_1, r_2, \phi_1, \phi_2) = \begin{pmatrix} \phi_2(0) - \phi_2(-1) \\ 0 \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}.$$

In this example $\langle b, c\rangle = 0$ and $c \notin D(A^*)$. From the transfer function (4.30) we can see that the system has relative degree 2.

Pandolfi [12] showed that the largest feedback invariant subspace $Z \subset c^{\perp}$, if it exists, is not a delay system. We now show that this system does not have a largest feedback invariant subspace in $c^{\perp}$. Define

$$e_k(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \exp(2\pi i k t) \end{bmatrix} \in D(A) \cap c^{\perp}.$$

For each k the subspace $\operatorname{span}\{e_k\}$ is (A, b)-invariant and hence feedback invariant [19]. Define

$$V_n = \operatorname{span}_{-n \leq k \leq n} e_k.$$

Each subspace V_n is feedback invariant. Define also the union of all finite linear combinations of e_k,

$$V = \bigcup V_n.$$

By well-known properties of the exponentials, the closure of $\{\exp(2\pi i k t)\}_{k=-\infty}^{\infty}$ is $L^2(-1, 0)$. Consider a sequence of elements in V, $[0, 1, 0, z_n]$ where $z_n(0) = 1$ and $\lim_{n\to\infty} z_n = 0$. This sequence converges to $[0, 1, 0, 0]$ and so we see that the closure of V in X is $\bar{V} = 0 \times R \times 0 \times L_2(-1, 0)$. If there is a largest feedback invariant subspace Z in $c^{\perp}$, then $Z \supset \bar{V}$. The important point now is that although $b \notin V$, $b \in \bar{V}$. Since b cannot be contained in any feedback invariant subspace (Theorem 2.2), $\bar{V}$ is not feedback invariant. Hence no largest feedback invariant subspace exists for this system. □

We end this paper with further consideration of the case where $\langle b, c\rangle = 0$ and $c \notin D(A^*)$. Theorem 2.1 implies that any element $x \in D(A)$ of an (A, b)-invariant subspace of $c^{\perp}$ is contained in the set

$$Z = \{z \in c^{\perp} \cap D(A) \mid \langle Az, c\rangle = 0\}. \tag{4.31}$$

The closure of Z is a natural candidate for the largest feedback invariant subspace of $c^{\perp}$; in fact, if $c \in D(A^*)$, the closure of Z in X is $Z_1 = c^{\perp} \cap (A^*c)^{\perp}$, the largest feedback invariant subspace if $\langle b, A^*c\rangle \neq 0$. The situation if $c \notin D(A^*)$ is quite different.

Theorem 4.2 *If $c \notin D(A^*)$, the set Z is dense in $c^{\perp}$. Furthermore, $Z \neq c^{\perp} \cap D(A)$.*

Proof This will be proved by showing that if Z is not dense in $c^{\perp}$ then $c \in D(A^*)$. Let $\lambda \in \rho(A)$ and $A_\lambda = A - \lambda I$, so $D(A_\lambda) = D(A)$. $D(A)$ is a Hilbert space with the graph norm, and the graph norm is equivalent to

$$\|x\|_1 := \|A_\lambda x\|. \tag{4.32}$$

The corresponding inner product on $D(A)$ is

$$\langle x, y\rangle_1 := \langle A_\lambda x, A_\lambda y\rangle. \tag{4.33}$$

Define $e = (A_\lambda{}^*)^{-1}c \in X$. For $x \in D(A)$, the condition $\langle c, x\rangle = 0$ can be written as

$$0 = \langle x, c\rangle = \langle A_\lambda x, e\rangle = \langle A_\lambda x, A_\lambda A_\lambda{}^{-1}e\rangle = \langle x, A_\lambda{}^{-1}e\rangle_1. \tag{4.34}$$

For $x \in c^\perp \cap D(A_\lambda)$, the condition $\langle Ax, c\rangle = 0$ is equivalent to $\langle A_\lambda x, c\rangle = 0$. Hence for such x we have

$$0 = \langle A_\lambda x, c\rangle = \langle A_\lambda x, A_\lambda A_\lambda{}^{-1}c\rangle = \langle x, A_\lambda{}^{-1}c\rangle_1. \tag{4.35}$$

We can write Z as

$$\left\{x \in D(A) \mid \langle x, A_\lambda{}^{-1}e\rangle_1 = 0 \text{ and } \langle x, A_\lambda{}^{-1}c\rangle_1 = 0\right\}.$$

We now introduce the notation

$$(y)_1^\perp := \{x \in D(A) \mid \langle x, y\rangle_1 = 0\}.$$

Using this notation,

$$Z = (A_\lambda{}^{-1}e)_1^\perp \cap (A_\lambda{}^{-1}c)_1^\perp.$$

Now suppose that Z is not dense in $c^\perp$ (as a subspace of X). Then there exists $v \in c^\perp$ such that $\langle x, v\rangle = 0$ for all $x \in Z$. Define $w = (A_\lambda{}^*)^{-1}v$. As in (4.34), for $x \in D(A)$, the condition $\langle x, v\rangle = 0$ is equivalent to

$$\langle x, A_\lambda{}^{-1}w\rangle_1 = 0. \tag{4.36}$$

Hence we see that

$$Z \subseteq (A_\lambda{}^{-1}e)_1^\perp \cap (A_\lambda{}^{-1}w)_1^\perp. \tag{4.37}$$

Let R be the orthogonal projection from $D(A)$ onto $(A_\lambda{}^{-1}e)_1^\perp$ (using the inner product $\langle\cdot,\cdot\rangle_1$). Then

$$Z = (A_\lambda{}^{-1}e)_1^\perp \cap (RA_\lambda{}^{-1}c)_1^\perp$$

and

$$(A_\lambda{}^{-1}e)_1^\perp \cap (A_\lambda{}^{-1}w)_1^\perp = (A_\lambda{}^{-1}e)_1^\perp \cap (RA_\lambda{}^{-1}w)_1^\perp.$$

Hence (4.37) becomes

$$(A_\lambda{}^{-1}e)_1^\perp \cap (RA_\lambda{}^{-1}c)_1^\perp \subseteq (A_\lambda{}^{-1}e)_1^\perp \cap (RA_\lambda{}^{-1}w)_1^\perp. \tag{4.38}$$

This implies that there is a scalar γ such that

$$RA_\lambda{}^{-1}c = \gamma RA_\lambda{}^{-1}w.$$

We obtain that

$$A_\lambda{}^{-1}c = \alpha A_\lambda{}^{-1}w + \beta A_\lambda{}^{-1}e.$$

Applying A_λ to both sides of this equation,

$$c = \alpha w + \beta e.$$

Since $w = (A_\lambda{}^*)^{-1}v$ and $e = (A_\lambda{}^*)^{-1}c$, we see that $c \in D(A_\lambda{}^*) = D(A^*)$. Thus, if Z is not dense in $c^\perp$ then $c \in D(A^*)$.

Now assume that $Z = c^\perp \cap D(A)$. Then $(A_\lambda{}^{-1}e)_1^\perp \cap (A_\lambda{}^{-1}c)_1^\perp = (A_\lambda{}^{-1}e)_1^\perp$, so, as above, $c = \beta e$, which would imply that $c \in D(A^*)$. □

Lemma 4.3 *Suppose that $q \in X$ and $c \notin D(A^*)$. Then $q^\perp \cap Z$ is dense in $q^\perp \cap c^\perp$. Furthermore, $q^\perp \cap Z \neq q^\perp \cap c^\perp \cap D(A)$.*

Proof If $q = \lambda c$ for some scalar λ, then $q^\perp \cap Z = Z$ and $q^\perp \cap c^\perp = c^\perp$, and the result follows immediately from Theorem 4.2.

Assume now that q is not parallel to c. Let P be the orthogonal projection of X onto $c^\perp$, and $\tilde{q} = Pq$, so $\tilde{q} \neq 0$. Let $\tilde{X} = (\tilde{q})^\perp$, and let Q be the orthogonal projection of X onto $(\tilde{q})^\perp$. By construction, $c = Qc \in \tilde{X}$. Let

$$\tilde{A} = QA|_{\tilde{X}}, \quad D(\tilde{A}) = D(A) \cap \tilde{X}, \quad \tilde{Z} = \{x \in D(\tilde{A}) \mid \langle x, c\rangle = 0 \text{ and } \langle \tilde{A}x, c\rangle = 0\}.$$

We wish to show that $c \notin D(\tilde{A}^*)$. Note that for $x \in \tilde{X}$,

$$\langle \tilde{A}x, c\rangle = \langle \tilde{Q}Ax, c\rangle = \langle Ax, Qc\rangle = \langle Ax, c\rangle. \tag{4.39}$$

Therefore $c \notin D(A^*)$ if the functional $x \to \langle Ax, c\rangle$ is unbounded on $\tilde{X}$. To show this let $q_0 \in D(A) \cap \tilde{X}$ and let Q_0 be the (possibly not orthogonal) projection onto $\tilde{X}$ given by

$$Q_0 x = x - \frac{\langle x, \tilde{q}\rangle}{\langle q_0, \tilde{q}\rangle} q_0.$$

Then $\langle Ax, c\rangle$ is unbounded on $\tilde{X}$ if $\langle AQ_0x, c\rangle$ is unbounded on X. Note that

$$\langle AQ_0x, c\rangle = \langle Ax, c\rangle - \frac{\langle x, \tilde{q}\rangle}{\langle q_0, \tilde{q}\rangle}\langle Aq_0, c\rangle.$$

The second term on the right is clearly bounded on X, and the first term on the right is unbounded on X since $c \notin D(A^*)$, so $\langle AQ_0x, c\rangle$ is not a bounded operator on X, hence $c \notin D(\tilde{A}^*)$.

Now we can apply Theorem 4.2 to $\tilde{X}$, $\tilde{A}$, c and $\tilde{Z}$ and conclude that $\tilde{X} \cap \tilde{Z}$ is dense in $\tilde{X} \cap c^\perp$ and $\tilde{X} \cap \tilde{Z} \neq \tilde{X} \cap c^\perp \cap D(A)$.

For $x \in c^\perp$, $\langle x, Pq\rangle = \langle x, q\rangle$ and so

$$\begin{aligned}\tilde{X} \cap c^\perp &= \{x \in X \mid \langle x, c\rangle = 0,\ \langle x, Pq\rangle = 0\}\\ &= \{x \in X \mid \langle x, c\rangle = 0,\ \langle x, q\rangle = 0\}\\ &= q^\perp \cap c^\perp.\end{aligned}$$

Similarly,

$$\tilde{X} \cap \tilde{Z} = \{x \in D(A) \mid \langle x, c\rangle = 0,\ \langle x, q\rangle = 0,\ \langle \tilde{A}x, c\rangle = 0\}. \tag{4.40}$$

This can be written

$$\begin{aligned}\tilde{X} \cap \tilde{Z} &= \{x \in D(A) \mid \langle x, c\rangle = 0,\ \langle x, q\rangle = 0,\ \langle Ax, c\rangle = 0\}\\ &= q^\perp \cap Z.\end{aligned}$$

Thus we have shown that $q^\perp \cap Z$ is dense in $q^\perp \cap c^\perp$, and that the two spaces are not equal. □

If $\langle b, c\rangle = 0$, $c \in D(A^*)$, and $\langle b, A^*c\rangle \neq 0$, the largest invariant subspace in $c^\perp$ is $Z_1 = c^\perp \cap (A^*c)^\perp$, and defining $\alpha = -1/\langle b, A^*c\rangle$,

$$A + bK = A + \alpha b\langle Ax, A^*c\rangle, \quad D(A + bK) = \{z \in c^\perp \cap D(A) \mid \langle Az, c\rangle = 0\}$$

is Z_1-invariant. In many cases, this operator generates a C_0-semigroup on Z_1. It is tempting to hope that even if $c \notin D(A^*)$, the operator (with some value of α)

$$A + bK = A + \alpha b\langle A^2x, c\rangle, \quad D(A + bK) = \{z \in c^\perp \cap D(A^2) \mid \langle Az, c\rangle = 0\}$$

is a generator, or has an extension which is a generator. However, we see from the next result that this operator is not closable, so that no extension of it is a generator of a C_0-semigroup, or even an integrated semigroup (see [9, Theorem 4.5]).

Theorem 4.4 *Suppose that* $b \in X$ *and* $c \notin D(A^*)$. *Then the operator*

$$A_F x := Ax + b\langle A^2x, c\rangle, \quad D(A_F) = \{x \in c^\perp \cap D(A^2) \mid \langle Ax, c\rangle = 0\}$$

is not closable.

Proof Let $\lambda \in \rho(A)$ and $A_\lambda = A - \lambda I$, as above. From Corollary 4.3 we see that $((A_\lambda^{-1})^*c)^\perp \cap Z$ is dense in $((A_\lambda^{-1})^*c)^\perp \cap c^\perp$. Let

$$Tx := \langle A_\lambda x, c\rangle, \quad D(T) = ((A_\lambda^{-1})^*c)^\perp \cap c^\perp \cap D(A).$$

We will now show that T is not closable. From Corollary 4.3, $((A_\lambda^{-1})^*c)^\perp \cap Z \neq D(T)$. Thus we can choose $f \in D(T)$ such that $f \notin ((A_\lambda^{-1})^*c)^\perp \cap Z$, and there exists

$(f_n) \subset ((A_\lambda^{-1})^* c)^\perp \cap Z$ such that $\lim f_n = f$. From the definition of Z, $T f_n = 0$ for all n. Let $x_n = f - f_n$, so

$$\lim x_n = 0, \quad \text{and} \quad \lim T x_n = Tf \neq 0, \tag{4.41}$$

which shows that T is not closable [16, Sect. II.6, Proposition 2]. It then follows that $I + bT$ with domain $D(T)$ is not closable.

Now note that $y \in D(A_F)$ if and only if $A_\lambda y \in D(T)$, and that for $y \in D(A_F)$

$$A_F y = (I + bT) A_\lambda y + \lambda y,$$

so A_F is closable if and only if $(I + bT) A_\lambda$ is closable. Using the sequence $(x_n) \subset D(T)$ defined above, define $y_n = A_\lambda^{-1} x_n$. Note that $(y_n) \subset D(A_F)$ and

$$\lim y_n = 0 \quad \text{and} \quad \lim (I + bT) A_\lambda y_n = bTf \neq 0.$$

Hence $(I + bT) A_\lambda$ is not closable, so A_F is not closable. □

5 Disturbance decoupling

Consider the controlled, observed system with disturbance $v(t)$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) + dv(t) \\ y(t) &= \langle x(t), c \rangle, \end{aligned} \tag{5.42}$$

where b, d and c are in the state-space X.

Disturbance Decoupling Problem (DDP): Find a feedback K so that (1) $A + bK$ generates a C_0-semigroup; and (2) with $u(t) = Kx(t)$, the output $y(t)$ in (5.42) is independent of the disturbance $v(t)$.

Solution of the DDP implies the existence of a feedback such that the output y is entirely "decoupled" from the disturbance. This problem is closely connected to the invariant subspace problem considered in this paper. Previous work on the disturbance-decoupling problem for infinite-dimensional systems assumed that the feedback operator K was bounded [2,3,10,11,19]. Also, in previous work it was not known a priori which systems possessed a largest invariant subspace in the kernel of C. In [11], for instance, the existence of such a subspace was required as an additional assumption on the system. Note that although the control and observation operators are bounded we do not require the feedback K to be bounded. The use of unbounded feedback extends the class of systems for which disturbance decoupling is possible, since the results in this paper lead to a characterization of single-input single-output systems which possess a largest invariant subspace within the kernel of C.

The following theorem is an immediate consequence of the results in Sects. 2 and 3.

Theorem 5.1 *Assume that (A, b, c) has relative degree $n + 1$ for some $n \in \mathbb{N}$, $c \in D(A^{*n})$ and the operator $A + bK$ where K is defined in (2.7) generates a C_0-semigroup on X. The system can be disturbance decoupled if and only if $d \in Z_n$.*

Proof Theorem 2.10 implies that Z_n is a feedback invariant subspace inside $c^{\perp}$. The assumption that $A+bK$ generates a C_0-semigroup on X implies that Z_n is closed-loop invariant, by Theorem 3.5. Thus, if $d \in Z_n$, the closed loop system

$$\dot{x}(t) = (A + bK)x(t) + dv(t)$$

with initial condition in Z_n can be viewed as a system in Z_n, so the system state remains in Z_n. Since $Z_n \subset c^{\perp}$, the output y is identically zero.

Conversely, suppose the DDP is solvable. That is, there exists a feedback K such that (1) $A + bK$ generates a C_0-semigroup, $S_K(t)$, and (2) for all $t > 0$ and all $v \in L_2(0, t)$,

$$C \int_0^t S_K(t - s)dv(s)\mathrm{d}s = 0.$$

Equivalently, define the subspace of all reachable states $\mathbb{R}(S_K, d)$ consisting of the closure of

$$\left\{ x \in X | x = \int_0^t S_K(t - s)dv(s)\mathrm{d}s, t \geq 0, v \in L_2(0, t) \right\}.$$

Solvability of the DDP means that $\mathbb{R}(S_K, d) \subset c^{\perp}$. Also, since

$$d = \lim_{t \to 0} \frac{1}{t} \int_0^t S_K(t - s)d\mathrm{d}s,$$

$d \in \mathbb{R}(S_K, d)$. The subspace $\mathbb{R}(S_K, d)$ is invariant under the semigroup $S_K(t)$, hence $A+bK$-invariant. Thus, $\mathbb{R}(S_K, d)$ is (A, b) feedback invariant. Since Z_n is the largest (A, b) feedback invariant subspace in $c^{\perp}$, it follows that

$$Z_n \supset \mathbb{R}(S_K, d) \supset d.$$

Thus, solvability of the DDP implies that $d \in Z_n$. □

Example (3.6 continued) With both choices of observation, the control system is a relative degree 1 system. The largest feedback invariant subspace in $c^{\perp}$ is exactly $c^{\perp}$.

First consider c_1. Since the observation element $c_1 \in D(A^*)$, the feedback operator is bounded and the feedback operator is

$$K_1 x = \langle x, k_1 \rangle,$$

where $k_1 \in L_2(0, 1)$ is defined in (3.28). Since K_1 is bounded, $c^\perp$ is also closed-loop invariant. The disturbance decoupling problem has a solution if and only if $\langle d, c_1 \rangle = 0$.

Consider the second observation element $c_2 \notin D(A^*)$. The feedback operator is only A-bounded. Since A generates an analytic semigroup, $A + bK$ generates a C_0-semigroup and $c^\perp$ is again closed loop invariant. The disturbance decoupling problem is solvable for any d such that $\langle d, c_2 \rangle = 0$.

The eigenfunctions of A form a basis for the state space $L_2(0, 1)$. The operator K_2 can be calculated by computing its effect on each eigenfunction in this basis.

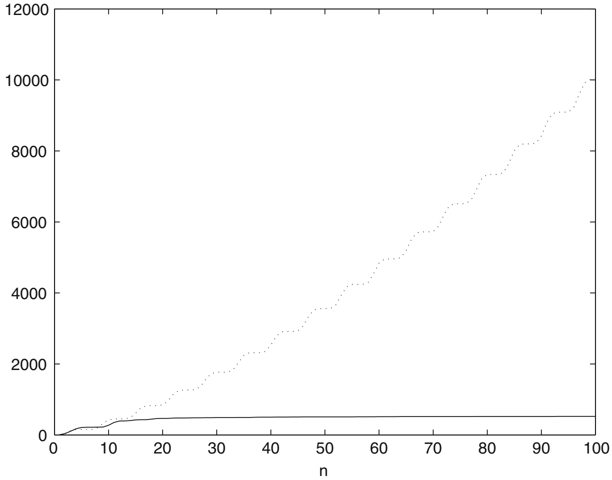

Fig. 1 Norm of feedback gain vector k_n versus approximation order n. Observation c_1 (*solid*), c_2 (*dotted*)

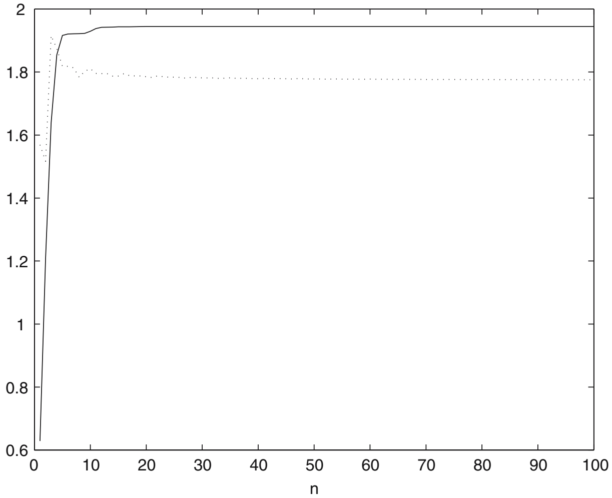

Fig. 2 Norm of feedback gain vector k_n versus approximation order n. Observation c_1 (*solid*), c_2 (*dotted*)

Projections of the system and feedback operators onto the span of the first n eigenfunctions yield a finite-dimensional model of order n. Figure 1 shows the norm of the feedback gain k_n against model order n, for both the first and second observation operator. These numerical results illustrate the theory: in the first case ($c_1 \in D(A^*)$) is bounded, while it is unbounded for the second observation operator ($c_2 \notin D(A^*)$). Figure 2 shows the norm of $A_n^{-1}k_n$ for both observation operators. As predicted by the theory, both feedback operators are A-bounded.

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References

1. Byrnes C, Lauko I, Gilliam D, Shubov V (1998) Zero dynamics for relative degree one SISO distributed parameter systems. In: 37th IEEE conference on decision and control, vol 3, pp 2390–2391
2. Curtain RF (1986) Invariance concepts in infinite dimensions. SIAM J Control Optim 24(5):1009–1031
3. Curtain RF (1986) Disturbance decoupling by measurement feedback with stability for infinite-dimensional systems. Int J Control 43(6):1723–1743
4. Curtain RF, Zwart H (1995) An Introduction to infinite-dimensional linear systems theory. Springer, Heidelberg
5. Isidori A (2001) Nonlinear control systems. Springer, Heidelberg
6. Kato T (1980) Perturbation theory for linear operators. Springer, Heidelberg
7. Lasiecka I, Triggiani R (1985) Finite rank, relatively bounded perturbations of semigroup generators, Part I: Well-posedness and boundary feedback hyperbolic dynamics. Annali Scuola Normale Superiore-Pisa, Classe di Scienze, Serie IV, Vol. XII, No. 4
8. Morris KA (2001) Introduction to feedback control. Academic, New York
9. Neubrander F (1988) Integrated semigroups and their applications to the abstract Cauchy Problem. Pacific J Math 135(1):111–155
10. Otsuka N, Hinata H (2000) Generalized Invariant Subspaces for Infinite-Dimensional Systems. J Math Anal Appl 252:325–341
11. Otsuka N, Inaba H (1990) Decoupling by state feedback in infinite-dimensional systems. IMA J Math Control Inf 7:125–141
12. Pandolfi L (1986) Disturbance decoupling and invariant subspaces for delay systems. Appl Math Optim 14:55–72
13. Pazy A (1983) Semigroups of linear operators and applications to partial differential equations. Springer, New York
14. Salamon D (1984) Control and observation of neutral systems. Pittman Advanced Publishing Program, Boston
15. Wonham WM (1985) Linear multivariable control: a geometric approach. Springer, Heidelberg
16. Yosida K (1980) Functional Analysis. Springer, New York
17. Zwart H (1988) Equivalence between open-loop and closed-loop invariance for infinite-dimensional systems: a frequency domain approach. SIAM J Control Optim 26(5):1175–1199
18. Zwart H (1989) Geometric theory for infinite dimensional systems. Lecture Notes in Control and Information Sciences, vol 115. Springer, Heidelberg
19. Zwart H (1990) On the solution of the DDP in infinite-dimensional systems. Signal processing, scattering and operator theory and numerical methods. Progr. Systems Control Theory, vol 5. Birkhauser, Boston, pp 363–372