STRONG SOLUTIONS FOR SEMILINEAR WAVE EQUATIONS WITH DAMPING AND SOURCE TERMS

PETRONELA RADU

Abstract. This paper deals with local existence of strong solutions for semilinear wave equations with power-like interior damping and source terms. A long standing restriction on the range of exponents for the two nonlinearities governs the literature on wellposedness of weak solutions of finite energy. We show that this restriction may be eliminated for the existence of higher regularity solutions by employing natural methods that use the physics of the problem. This approach applies to the Cauchy problem posed on the entire $\mathbb{R}^n$ as well as for initial boundary problems with homogeneous Dirichlet boundary conditions.
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1. Introduction

1.1. **Description of the model.** Consider the following semilinear wave equation which exhibits a competitive interaction between a nonlinear source and a damping term:

$$\text{(SW)} \qquad \begin{cases} u_{tt} - \Delta u + f(u) + g(u_t) = 0, \quad (x,t) \in \Omega \times [0,\infty) \\ (u, u_t)|_{t=0} = (u_0, u_1), \quad x \in \Omega \\ u = 0, \quad (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

The domain $\Omega \subset \mathbb{R}^n$ may be a *bounded or unbounded* regular domain in $\mathbb{R}^n$ (possibly the entire $\mathbb{R}^n$), for $3 \leq n < 6$. The focus here is on *strong* solutions $(u, u_t) \in H^2 \times H^1$ for which we show local in time existence.

The nonlinear terms are usually thought of as being powers of u, respectively u_t, with a prototype equation given by:

$$u_{tt} - \Delta u + au|u|^{p-1} + bu_t|u_t|^{m-1} = 0.$$

Thus, $a = 0, b = 1$ yields the nonlinear damped wave equation, while for $a = 1, b = 0$ we obtain the nonlinear Klein-Gordon equation. For our purposes, the more interesting case to be analyzed is $a < 0$ and $b > 0$ when the two nonlinearities compete with each other.

The considerable interest that the system has received over the past decade is attributed mainly to the interaction between the two nonlinearities. In other models the nonlinearities appear as boundary conditions, instead of affecting the system internally; this brings in

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other interesting (and often) technical aspects, but regardless of where the terms arise (internally or on the boundary), the crux of the problem lies in the question:

Does the energy lost through damping compensate for the energy brought into the system by f, thus extending the lifespan of the solution?

The intuition is that an increase in energy will lead to blow-up of solutions in finite time, while a decrease would lead to global existence of solutions (unless the system experiences overdamping). Roughly speaking, g extends the life span of the solution while taking energy out of the system, whereas f has the opposite effect.

Mathematically, it has been shown (for certain ranges of exponents p and m) that the interplay between these two power-like nonlinearities is exhibited through their exponents. Thus, if the exponent p of the energy accretive source $f(u) = -|u|^{p-1}u$ is higher than the exponent m of the damping term $g(u_t) = u_t|u_t|^{m-1}$, then solutions cease to exist after a finite time, whereas if the exponent of the damping term dominates the interaction, then the system will have a solution for arbitrarily large times ([8], [5]). Thus, by juxtaposing the two nonlinear terms to the wave operator one expects a system where the issue of existence vacillates between global existence and blow up in finite time. The blow-up phenomenon occurs when "bad" sources pump energy into the system, so the norm of the solution becomes unbounded in finite time. For "good" sources (e.g., $f(u) = |u|^{p-1}u$) global existence of solutions is expected in the subcritical case $p < 5$ in $n = 3$ dimensions (see [13]); for supercritical exponents $p > 5$ see the discussion on Lebeau's ill-posedness results below.

The interaction of the nonlinearities plays the determinant role not only for the analysis of solutions for large times, but it is foremost critical for the local existence of solutions, as we demonstrate in this paper. For certain ranges of exponents p and m the local (in time) existence of weak solutions remains an open problem. The following restriction for the range of exponents p and m governed previous existence results concerning (SW)

$$\text{(R)} \qquad p + \frac{p}{m} < \frac{2n}{n-2} \qquad \text{for } \frac{n}{n-2} < p < \frac{n+2}{n-2}.$$

This bound describes a trade-off mechanism regarding higher integrability between the source term and the damping term (for more on this see Remark 5.1 in section 5). The barrier imposed by the above condition has been considered inherent to the problem and not necessarily to the methods used ([2, 16]). The author showed, however, in [13, 14] that one can allow some other exponents, in particular $1 < p < \frac{n+2}{n-2}$ in the linear damping case ($m = 1$) or in the absence of damping ($m = 0$). Also, if the source has the right sign the author showed that the result holds for the entire range $1 < p < \frac{n+2}{n-2}, m \geq 0$. Recently, the understanding of wellposedness has moved into the supercritical range (above energy level) for source terms. Thus, in [3, 4] the authors showed existence of solutions for $5 < p < 6$ in $n = 3$ dimensions when Ω is a bounded domain, while [6] settles the local existence problem on the entire space. These existence results for weak solutions are obtained under the same restriction (R). It remains an open problem to show local existence of *weak* solutions even for $3 < p < 6$ (for $n = 3$) outside of the range given by the condition (R). The case $p \geq 6$

is completely open, except for the case of absorption sources ($f(u) = |u|^{p-1}u$), for which local existence was shown in [13].

In this paper we continue the work done in [13, 14] and we show local existence of *strong* H^2 solutions in the absence of the restriction (R), more precisely, for all exponents p, m such that:

$$1 \leq p \leq \frac{n+2}{n-2}, \quad m \geq 0.$$

We illustrate the extension of the range of exponents given by this work with the following graphs in Figure 1 below.

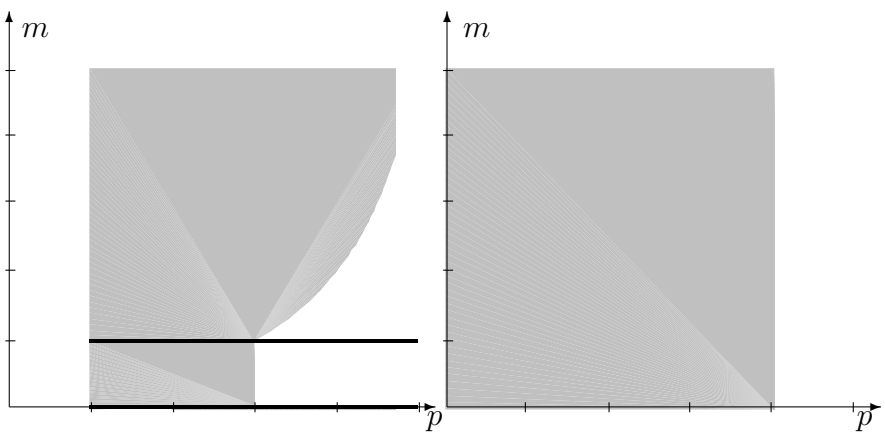

FIGURE 1. Ranges of exponents for $n = 3$. On the left the restriction (R) is imposed, but linear ($m = 1$) and no damping ($m = 0$) terms are allowed ([13]); on the right the full range is allowed for strong solutions (this work).

Regarding the methods of investigation used, one quickly notes that the problem is beset with difficulties in connection with passing to the limit in the two competing nonlinear terms; to overcome these challenges one usually employs compactness and monotonicity arguments. By using these arguments alone one can tackle the problem in the case when $1 \leq p \leq \frac{n}{n-2}$, but the methods will not suffice for handling $p > \frac{n}{n-2}$. The theory of Orlicz spaces was employed in [16] to allow source and damping terms that satisfy the restriction (R). These results are obtained for the Cauchy problem on $\mathbb{R}^n$. The author showed that local existence holds on bounded domains with homogeneous Dirichlet data in [14], while the works [3, 4, 5, 6] extended the results further to supercritical sources $5 < p < 6$ in $n = 3$ on bounded and unbounded domains.

Our approach here uses the *patching argument* first used by Tartar and Crandall in [17, 18] for the Broadwell model, and later redeveloped by the author for semilinear wave equations in [13, 14]. This method turns out to be an extremely valuable tool that has great potential for many problems for which some energy (or "entropy") estimates are available, and for which a finite speed of propagation property holds. Eliminating the restriction (R) for strong solutions requires additional work, such as the derivation of higher order estimates.

Once these higher order estimates are obtained we employ a *second order* potential well argument to obtained bounds on the higher order derivatives of the solution. This idea is complemented by the robustness of the patching argument to yield local existence of strong solutions.

The methods used are natural from a physical point of view and they apply to the investigation of problems on both, bounded and unbounded domains, in contrast with other methods that are available and which apply exclusively to only one of the setups. Also, earlier works [2, 8, 16] used the smoothing effect of the damping term, but our method circumvents this step. By making stronger use of the monotonicity method we are able to show local wellposedness for the problem which contains only source terms ($g = 0$ or $m = 0$). Finally, we remark that our results hold for finite energy initial data, not necessarily with compact support as it is the case in [16, 19].

Note that although we will focus on the Cauchy problem associated with this semilinear wave equation, the wellposedness also holds on a bounded domain. The applicability of the method to the boundary value problems was shown in [14], but for the range of exponents described by the restriction (R). We will only sketch the necessary adjustments, without going through the complete arguments which follow closely [14].

Regarding the behavior of solutions to this problem a few more facts should be pointed out in order to give a better picture of existence of regular solutions. First, one may not be guaranteed even *local* existence of strong solutions for $p \geq \frac{2n}{n-2}$ as Lebeau showed in [9, 10] that for odd supercritical exponents the problem is ill-posed in all spaces H^s for $s \in (1, \frac{n}{2} - \frac{2}{p-1})$ when $f(u) = |u|^{p-1}u$ and $g(u_t) = 0$ (note that this is the case of a "good" absorption source term so the energy of the system decreases). He constructed solutions that blow-up immediately in these H^s norms. However, in the presence of some special damping terms, positive results regarding global existence of solutions have been established. Thus, some preliminary results [20] seem to indicate global existence of H^2 solutions in the presence of arbitrarily large polynomial absorption sources and cubic damping term $g(u_t) = u_t^3$.

For easy navigation through the paper we provide here a brief outline. The setup of the problem is done in the next section where we specify notations, conditions satisfied by the problem, and recall the definitions for weak and strong solutions. The proof is built on results (most of them classical) for the problem with Lipschitz source terms. These results are the energy identity, global existence of solutions, and two versions of the finite speed of propagation property; they are presented in section 3 where we also provide references for the proofs. The main theorem which gives local existence of strong solutions is presented in section 4 with the main steps organized in subsections. We conclude the paper with some final remarks and conclusions. Many arguments of the proof coincide with arguments used in [13, 14] but we reproduce only the essential ones here.

2. Notation, assumptions, definitions

In the sequel we use the following notation: $|\cdot|_{q,\Omega}$ is the norm in $L^q(\Omega)$. For $q=2$ we take the liberty to drop the subscript 2, so $|\cdot|_\Omega$ is the norm in $L^2(\Omega)$, while for $\Omega=\mathbb{R}^n$ we drop the subscript Ω; thus, $|\cdot|_q$ denotes the norm in $L^q(\mathbb{R}^n)$.

The source and damping terms satisfy the following assumptions:

(A_f) $f\in C^2(\mathbb{R})$, $f(0)=f'(0)=0$, and there exist $f_1,f_2>0$ such that:

$$|f''(u)|\le \max\{f_1|u|^{p-2},f_2\},\ \text{for some } p\in\left(1,\frac{n+2}{n-2}\right).$$

(A_g) $g\in C^1(\mathbb{R})$, $g(0)=0$, $g'(0)\ge 0$, and g satisfies the growth assumptions:

$$vg(v)\ge g_1|v|^{m+1},\quad |g(v)|\le g_2|v|^m,$$

for some $m\ge 0$ and $g_1,g_2>0$.

Remark 2.1. As a consequence of (A_f) we have the following growth conditions for $f(u)$ and $F(u)=\int_0^u f(y)dy$:

$$|f(u)|\le \max\{f_1|u|^p,f_2|u|^2\},\tag{2.1}$$
$$|F(u)|\le \max\{\frac{f_1}{p+1}|u|^{p+1},\frac{f_2}{3}|u|^3\},$$

for $p\in\left(1,\frac{n+2}{n-2}\right)$.

We also make the following convention. The problem (SW) will be denoted by (SWB) when we specifically impose that Ω is a *bounded* domain. Whenever the problem is posed on an arbitrary domain that is not required to be bounded (possibly $\mathbb{R}^n$) we will use the label (SW).

The meanings of weak and strong solutions for (SW) are classical, but we include the two definitions below for clarity and preciseness reasons.

Definition 2.2 (Weak solutions)**.** Let $\Omega_T:=\Omega\times(0,T)$, where $\Omega\subset\mathbb{R}^n$ is an open connected set with smooth boundary $\partial\Omega$. Suppose the functions f and g satisfy the assumptions (A_f) and (A_g), and further suppose that $u_0\in H_0^1(\Omega)\cap L^{p+1}(\Omega)$ and $u_1\in L^2(\Omega)\cap L^{m+1}(\Omega)$.

A *weak solution* on Ω_T of the boundary value problem (SW) is any function u satisfying

$$u\in C(0,T;H_0^1(\Omega))\cap L^{p+1}(\Omega_T),\quad u_t\in L^2(\Omega_T)\cap L^{m+1}(\Omega_T),$$

and

$$\int_{\Omega_T}\Big(u(x,s)\phi_{tt}(x,s)+\nabla u(x,s)\cdot\nabla\phi(x,s)+f(u)\phi(x,s)$$
$$+g(u_t)\phi(x,s)\Big)\,dxds=\int_\Omega\Big(u_1(x)\phi(x,0)-u_0(x)\phi_t(x,0)\Big)dx$$

for every $\phi\in C_c^\infty(\Omega\times(-\infty,T))$.

Definition 2.3 (Strong solutions)**.** Let Ω_T, f, and g satisfy the assumptions from the above definition. Assume $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H^1(\Omega) \cap L^{m+1}(\Omega)$. A *strong solution* u for the problem (SW) is a weak solution in the sense of Definition 2.2 with the additional regularity

$$u \in C^1([0,T]; L^2(\Omega)) \quad \text{and} \quad u_{tt},\, \Delta u \in L^2(0,T; L^2(\Omega)).$$

Remark 2.4. The additional regularity required by a strong solution of (SW) as described above implies that

$$(u(\cdot,t), u_t(\cdot,t)) \in (H^2(\Omega) \cap H_0^1)(\Omega) \times H^1(\Omega) \text{ for all } t > 0.$$

Finally, we denote by C all constants whose values do not affect the proof. We will specify and keep track only of the constants that are important in our arguments.

3. Preliminary results. Lipschitz source terms and general damping

In this section we will recall some results regarding strong solutions for (SW) with Lipschitz source terms and damping when the initial data $(u_0, u_1) \in (H^2 \cap H_0^1) \times H^1$. Throughout the section we assume that there exists $L > 0$ such that:

$$|f(u) - f(v)| \leq L|u - v|, \tag{3.1}$$

for all $u, v \in \mathbb{R}$. The first result gives us the basic bound for the semilinear wave equation (SW); for a proof see [13].

Proposition 3.1 (The energy identity)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H^1(\Omega)$. If u is a strong solution of (SW), where f, g satisfy (A_f), (A_g), and (3.1), then we have the following equality:*

$$E(t) + \int_\Omega F(u(x,t))\,dx + \int_0^t \int_\Omega g(u_t(x,s))u_t(x,s)\,dx\,ds = E(0), \tag{3.2}$$

where $F(v) = \int_0^v f(y)dy$ and

$$E(t) := \frac{1}{2}|u_t(t)|_\Omega^2 + \frac{1}{2}|\nabla u(t)|_\Omega^2.$$

Existence of strong solutions in the presence of globally Lipschitz sources is a well known result; see [1] for a proof using monotone semigroup theory, or [13] for a monotonicity argument involving a-priori estimates.

Theorem 3.2 (Existence and uniqueness of strong solutions for dissipative wave equations with Lipschitz source terms)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary, and let the functions f and g satisfy (A_f) and (A_g). Additionally, f is Lipschitz. Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ with $G(u_1) \in L^1(\Omega)$, where G is defined by the formula $G(v) = \int_0^v g(y)dy$. Then for any $T > 0$ the initial boundary-value problem:*

$$\begin{cases} u_{tt} - \Delta u + f(u) + g(u_t) = 0 \text{ in } \Omega \times (0,T); \\ (u, u_t)|_{t=0} = (u_0, u_1) \text{ in } \Omega; \\ u = 0 \text{ on } \partial\Omega \times (0,T). \end{cases} \tag{SWB}$$

admits a unique strong solution u on the time interval $[0,T]$ in the sense of Definition 2.3.

Remark 3.3. By using monotone semigroup theory one can prove that this theorem holds for $(u_0,u_1) \in [H^k(\Omega)\cap H_0^1(\Omega)] \times H^{k-1}(\Omega)$ which generate a solution $u \in H^k(\Omega)\cap H_0^1(\Omega)$, for $k \geq 2$.

Finally, the following result gives us the finite speed of propagation property as seen in two different ways. The first statement describes how the compact support of solutions grows in time, while the second is a uniqueness type result. Note that these two statements are not equivalent since we are dealing with possible nonlinear terms. Both results will be used in the proof of the main result. For a proof of the theorem see [13].

Theorem 3.4. (Finite speed of propagation) *Consider the problem (SW) under the hypothesis of Theorem* 3.2. *Then*

(1) *if the initial data u_0, u_1 is compactly supported inside the domain $B(x_0,R)\cap\Omega$, then $u(x,t)=0$ outside $B(x_0,\ R+t)\cap\Omega$;*

(2) *if $(u_0,u_1),(v_0,v_1)$ are two pairs of initial data with compact support, with the corresponding solutions $u(x,t)$, respectively $v(x,t)$, with*

$$(u_0(x),u_1(x)) = (v_0(x),v_1(x)) \text{ for } x \in B(x_0,R)\cap\Omega,$$

then $u(x,t)=v(x,t)$ inside $B(x_0,\ R-t)\cap\Omega$ for any $t<R$.

4. Local existence of strong solutions

This section contains our main result which will guarantee existence of H^2 solutions for the problem (SW) up to some time $T>0$. The removal of the restriction (R) on the range of exponents p,m that control the growth of the source and damping terms is obtained by employing higher order energy estimates.

Theorem 4.1 (Local existence in time of strong solutions)**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded or unbounded domain in $\mathbb{R}^n$ with smooth C^2 boundary (possibly the entire $\mathbb{R}^n$), where $3\leq n<6$. Let f,g satisfy the assumptions (A_f) and (A_g). Then there exists a time $T>0$ such that the problem*

$$\text{(SW)} \qquad \begin{cases} u_{tt}-\Delta u+f(u)+g(u_t)=0,\ (x,t)\in\Omega\times[0,\infty) \\ (u,u_t)|_{t=0}=(u_0,u_1),\ x\in\Omega \\ u=0,\ (x,t)\in\partial\Omega\times(0,T). \end{cases}$$

admits a strong solution on $(0,T)$.

Remark 4.2. In case $\Omega=\mathbb{R}^n$ the boundary condition for the above problem is removed.

Proof. We start with a brief outline of the proof. First we consider the problem (SW) on a bounded domain Ω. We approximate the source term f with Lipschitz functions f_ε and obtain a sequence of approximating solutions u^ε. We show that for *small* initial data on these bounded domains, one has local existence in time of solutions. Next, we show that

the smallness assumption is satisfied by appropriate truncations of our initial data on every small ball $B_{x_0}(\rho)$ for every point $x_0 \in \Omega$, where the radius ρ is independent of x_0 and ε. Thus we show that on every small ball the following bounds hold independently of ε:

$$|\nabla u^\varepsilon|_{B_{x_0}(\rho)} < C,\ |\nabla u_t^\varepsilon|_{B_{x_0}(\rho)} < C,\ |\Delta u^\varepsilon|_{B_{x_0}(\rho)} < C,\ \ |u_{tt}^\varepsilon|_{B_{x_0}(\rho)} < C, \tag{4.1}$$

for all $0 < t < 1$. We use these bounds and compactness arguments to pass to the limit in the source terms, and we use a monotonicity argument to pass to the limit in the damping terms. Thus, one obtains local existence of solutions on these small domains. Finally, we "patch" these small solutions to construct a solution on the whole domain Ω. The higher order estimates will yield the H^2 regularity of the solutions.

4.1. **Compactness arguments.** The goal of this section is to derive the bounds (4.1) which will allow us to pass to the limit in the sequence of approximations u^ε, and consequently, to obtain the local existence result on the bounded small domain $B_{x_0}(\rho)$ (which is yet to be constructed).

In a first step we assume that we are working with the following assumptions:

(a1) Lipschitz source terms f_ε (the Lipschitz approximations f_ε are constructed exactly as in [13] and they satisfy the assumptions (A_f); more details will be provided later);
(a2) on a bounded regular domain denoted by Ω;
(a3) the initial data u_0, u_1 are compactly supported inside a ball of radius R centered at a point x_0 denoted by $B(R)$ (since the equation (SW) is invariant by spatial translations one may assume w.l.o.g. that x_0 is the origin). Thus, u_0 has zero trace on $\partial\Omega$, and hence so does the solution u^ε.
(a4) the initial data has the improved regularity $(u_0, u_1) \in H^3(\Omega) \times H^2(\Omega)$. According to Theorem 3.2 and Remark 3.3 this data will generate more regular solutions $u^\varepsilon \in H^3(\Omega) \cap H_0^1(\Omega)$ and $u_{ttt}^\varepsilon(\cdot, t) \in L^2(\Omega)$, for all $t > 0$. The additional regularity assumption is eliminated through a standard approximation argument (see [13] for details on this);
(a5) "smallness" hypothesis of initial data:

$$|\nabla u_0|_\Omega < \lambda, \quad \frac{1}{2}|u_1|_\Omega^2 + \frac{1}{2}|\nabla u_0|_\Omega^2 + \int_\Omega F_\varepsilon(u_0(x))\,dx < \Phi_1(\lambda), \tag{4.2}$$

where the function ϕ_1 and the value λ are chosen in the sequel.

Next we will outline the "first order" potential well argument (as developed by Payne and Sattinger in [12, 15]) that gives bounds for $|\nabla u^\varepsilon|_\Omega$ (we use the terminology of "first order" potential for obtaining bounds for the first order derivatives, to distinguish it from "second order" potential which estimates norms of second order derivatives), then we show that $|\nabla u_t^\varepsilon|_{B_{x_0}(\rho^*)} < C$ for $\rho^* \le \rho$ sufficiently small. These bounds for the gradient will give us higher L^p regularity for u^ε and will also help us show the other two estimates from (4.1).

4.1.1. *First order estimates; "first order" potential well.* We begin with the following remark. By the initial assumption (a3) and by part (1) of the finite speed of propagation property (Theorem 3.4) we have that $u(t) \in H_0^1(B(R+t))$. Hence, for $t < 1$ we have that $u(t) \in H_0^1(\Omega)$, where $\Omega := B(R+1)$. (The upper bound for the time $t < 1$ is irrelevant since we will prove local existence in time.)

For $q < \frac{2n}{n-2}$ and any $v \in H_0^1(\Omega)$, we have by Hölder's inequality and Sobolev's embedding theorem that the following inequality holds:

$$|v|_{q,\Omega} \leq C_q (R+1)^{\frac{2n-q(n-2)}{2q}} |\nabla v|_\Omega, \tag{4.3}$$

where

$$C_q = C^* \left(\frac{\omega_n}{n}\right)^{\frac{2n-q(n-2)}{2nq}}, \tag{4.4}$$

and the constant C^* is the (optimal) Sobolev's constant that depends only on n and $\frac{\omega_n}{n}$ is the volume of the unit sphere in $\mathbb{R}^n$.

Hence, the inequality (4.3) holds for $u^\varepsilon(t)$, for all $0 < t < 1$, where we choose $q = p+1 < \frac{2n}{n-2}$.

The first estimate that will be essential to the first order potential well argument follows from the energy identity:

$$\begin{aligned} \frac{1}{2}|u_t^\varepsilon(t)|_\Omega^2 + \frac{1}{2}|\nabla u^\varepsilon(t)|_\Omega^2 + \int_\Omega F_\varepsilon(u^\varepsilon(x,t))\,dx \\ \leq \frac{1}{2}|u_1|_\Omega^2 + \frac{1}{2}|\nabla u_0|_\Omega^2 + \int_\Omega F_\varepsilon(u_0(x))\,dx = E(0). \end{aligned} \tag{4.5}$$

We bound the term containing the source term by using the growth assumption for F (as a consequence of (A_f)) and the inequality (4.3):

$$\begin{aligned} &\frac{1}{2}|\nabla u^\varepsilon(t)|_\Omega^2 + \int_\Omega F_\varepsilon(u^\varepsilon(x,t))\,dx \\ &\geq \frac{1}{2}|\nabla u^\varepsilon(t)|_\Omega^2 - \max\left\{f_1|u^\varepsilon(t)|_{p+1,\Omega}^{p+1}, f_2|u^\varepsilon(t)|_{3,\Omega}^3\right\} \\ &\geq \frac{1}{2}|\nabla u^\varepsilon(t)|_\Omega^2 - \max\Big\{f_1 C_{p+1}^{p+1}|\nabla u^\varepsilon(t)|_\Omega^{p+1}(R+1)^{\frac{2n-(p+1)(n-2)}{2}}, \\ &\qquad\qquad f_2 C_3^3|\nabla u^\varepsilon(t)|_\Omega^3 (R+1)^{\frac{6-n}{2}}\Big\}. \end{aligned} \tag{4.6}$$

From the above inequality we will extract bounds for the norm of the gradient by using the following slightly modified potential well [1]

$$\Phi_1(x) := \frac{x^2}{2} - \max\left\{Ax^{p+1}, Bx^3\right\}, \quad x \geq 0. \tag{4.7}$$

[1] A "classical" potential well is generated by a function $\Phi(x) = \dfrac{x^2}{2} - A_0 x^q$, for some constants $A_0 > 0, q > 2$. This is a C^1 function and it has one local maximum at $x = (A_0 q)^{-1/q-1}$, whereas the above function ϕ_1 may not be differentiable at the local maximum, or it may have two "humps". Nevertheless, as we show in the sequel the arguments follow the same way for this slightly modified function.

where by the constants $A, B > 0$ we denote:

$$A = f_1 C_{p+1}^{p+1}(R+1)^{\frac{2n-(p+1)(n-2)}{2}}, \quad B = f_2 C_3^3 (R+1)^{\frac{6-n}{2}}.$$

We are interested in detecting the height of this generalized potential well generated by the function ϕ_1 and where its maximum value is attained. Observe that for $p > 1$, Φ_1 has at most three critical points on the positive semiaxis: $x = 0$ and possibly two local maxima, denoted by x_{c_1} and x_{c_2}. Let $\lambda := \min\{x_{c_1}, x_{c_2}\}$. Then, we deduce that the first local maximum for Φ_1 is attained at λ, by considering the following cases:

(i) If $x < (B/A)^{\frac{1}{p-2}}$, then $\max\{Ax^{p+1}, Bx^3\} = Ax^{p+1}$; hence, $\Phi_1(x) = \dfrac{x^2}{2} - Ax^{p+1}$. For this function, the maximum is attained at $x_{c_1} = [(p+1)A]^{-1/(p-1)}$. Hence, for the interval $[0, (B/A)^{\frac{1}{p-2}}]$ the highest value will be attained by Φ_1 at

$x = \max\left\{x_{c_1}, (B/A)^{\frac{1}{p-2}}\right\}$.

(i) If $x \geq (B/A)^{\frac{1}{p-2}}$ then we have a similar analysis for the branch $\Phi_1(x) = \dfrac{x^2}{2} - Bx^3$, whose maximum on $[0, \infty)$ is attained at $x_{c_2} = 1/(3B)$. On the interval $[(B/A)^{\frac{1}{p-2}}, \infty)$ the maximum will be attained at at $x = \max\left\{x_{c_2}, (B/A)^{\frac{1}{p-2}}\right\}$.

For our later analysis it is necessary to remark that the critical value λ depends on R, the radius of the ball which contains the support of the initial data.

From (4.6) and the definition of Φ_1 we obtain:

$$\frac{1}{2}|\nabla u^\varepsilon(t)|_\Omega^2 + \int_\Omega F_\varepsilon(u^\varepsilon(x,t))\,dx \geq \Phi_1(|\nabla u^\varepsilon(t)|_\Omega).$$

Combining the above inquality, along with (4.5) and assumption (a5) we deduce:

$$\begin{aligned} &\frac{1}{2}|u_t^\varepsilon(t)|_\Omega^2 + \Phi_1(|\nabla u^\varepsilon(t)|_\Omega) \\ &\leq \frac{1}{2}|u_t^\varepsilon(t)|_\Omega^2 + \frac{1}{2}|\nabla u^\varepsilon(t)|_\Omega^2 + \int_\Omega F_\varepsilon(u^\varepsilon(x,t))\,dx \\ &\leq \frac{1}{2}|u_1|_\Omega^2 + \frac{1}{2}|\nabla u_0|_\Omega^2 + \int_\Omega F_\varepsilon(u_0(x))\,dx < \Phi_1(\lambda), \end{aligned} \tag{4.8}$$

from which we extract the bound

$$\Phi_1(|\nabla u^\varepsilon(t)|_\Omega) < \Phi_1(\lambda). \tag{4.9}$$

We have that $t \to |\nabla u^\varepsilon(t)|_\Omega$ is a continuous mapping; hence, should one assume that $|\nabla u^\varepsilon(t)|_\Omega \geq \lambda$ for some times $t < 1$ one would get $\Phi_1(|\nabla u^\varepsilon(t)|_\Omega) \geq \Phi_1(\lambda)$ for those times t, due to the fact that $|\nabla u_0|_\Omega < \lambda$ (in other words, we would start in the well with the initial data, but the solution would reach and surpass the maximum height, so it would leave the well). This comes in contradiction with (4.9), hence

$$|\nabla u^\varepsilon(t)|_\Omega < \lambda \text{ for all } t < 1. \tag{4.10}$$

4.1.2. *Second order estimates.* By differentiating (SW), multiplying by u^ε_{tt}, and integrating over Ω (see footnote [2]) we obtain:

$$\int_\Omega u^\varepsilon_{tt}u^\varepsilon_{ttt} - \Delta u^\varepsilon_t u^\varepsilon_{tt} + f'_\varepsilon(u^\varepsilon)u^\varepsilon_t u^\varepsilon_{tt} + g'(u^\varepsilon_t)(u^\varepsilon_{tt})^2 dx = 0$$

After a standard integration by parts we obtain:

$$\int_\Omega \frac{d}{2dt}\left[(u^\varepsilon_{tt})^2 + |\nabla u^\varepsilon_t|^2\right] + f'_\varepsilon(u^\varepsilon)u^\varepsilon_t u^\varepsilon_{tt} + g'(u^\varepsilon_t)(u^\varepsilon_{tt})^2 dx = 0.$$

Since the damping term g is represented by an increasing differentiable function, we have that $g' \geq 0$, so after integrating in time we obtain the following inequality:

$$\int_\Omega (u^\varepsilon_{tt})^2 + |\nabla u^\varepsilon_t|^2 dx + 2\int_0^t \int_\Omega f'_\varepsilon(u^\varepsilon)u^\varepsilon_t u^\varepsilon_{tt} dxds \leq \int_\Omega [u_{tt}(0)]^2 + |\nabla u_1|^2 dx \tag{4.11}$$

We focus on the term involving the source $f'_\varepsilon(u^\varepsilon)$ and estimate it by using Hölder's inequality for $0 < t < 1$ to obtain the following:

$$\begin{aligned}
2\int_0^t \int_\Omega f'_\varepsilon(u^\varepsilon)u^\varepsilon_t u^\varepsilon_{tt} dxds &= \int_0^t \int_\Omega f'_\varepsilon(u^\varepsilon)\frac{d}{dt}(u^\varepsilon_t)^2 dxds \\
&= \int_\Omega f'_\varepsilon(u^\varepsilon)(u^\varepsilon_t)^2 dx|_{s=0}^{s=t} - \int_0^t \int_\Omega f''_\varepsilon(u^\varepsilon)(u^\varepsilon_t)^3 dxds \\
&\gtrsim -\int_\Omega |u^\varepsilon|^{p-1}(u^\varepsilon_t)^2 dx - \int_\Omega |f'_\varepsilon(u_0)|u_1^2 dx - \int_0^t \int_\Omega |u^\varepsilon|^{p-2}|u^\varepsilon_t|^3 dxds \\
&\gtrsim -\left(\int_\Omega |u^\varepsilon|^{\frac{n(p-1)}{2}} dx\right)^{\frac{2}{n}} \left(\int_\Omega |u^\varepsilon_t|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}} - \int_\Omega |f'_\varepsilon(u_0)|u_1^2 dx \\
&- \left(\int_\Omega |u^\varepsilon|^{\frac{2n(p-2)}{6-n}} dx\right)^{\frac{6-n}{2n}} \left(\int_\Omega |u^\varepsilon_t|^{\frac{2n}{n-2}} dx\right)^{\frac{3(n-2)}{2n}}
\end{aligned} \tag{4.12}$$

Note that the Hölder exponents are valid (i.e. they are greater than 1). Indeed, $\frac{n}{2}$ and $\frac{2n}{3(n-2)}$ together with their conjugates $\frac{n}{n-2}$, respectively, $\frac{2n}{6-n}$ are all greater than 1, since $3 \leq n < 6$.

We further estimate the terms in the above inequality and since $p < \frac{n+2}{n-2}$ we have by (4.3) that

$$\begin{aligned}
\int_\Omega |u^\varepsilon|^{\frac{n(p-1)}{2}} dx &\leq C_{\frac{n(p-1)}{2}}(R+1)^{\frac{4-(p-1)(n-2)}{2(p-1)}} \left(|\nabla u^\varepsilon|_\Omega\right)^{\frac{n(p-1)}{2}}, \\
\int_\Omega |u^\varepsilon|^{\frac{2n(p-2)}{6-n}} dx &\leq C_{\frac{2n(p-2)}{6-n}}(R+1)^{\frac{6-n-(n-2)(p-2)}{2(p-2)}} \left(|\nabla u^\varepsilon|_\Omega\right)^{\frac{2n(p-2)}{6-n}},
\end{aligned} \tag{4.13}$$

where the constants on the right hand sides are computed with (4.4). Due to the fact that the initial data was chosen small enough so that the solution lives inside the first order

[2]Note that all these operations are legitimate due to our higher regularity assumption (iv): $(u, u_t) \in (H^3(\Omega) \cap H^1_0(\Omega) \times H^2(\Omega)$

potential well given by the function Φ_1 for all times $t < 1$, the norms of the gradients above are bounded as given by inequality (4.10); hence, from (4.13) we have

$$\int_\Omega |u^\varepsilon|^{\frac{n(p-1)}{2}} dx < M_1, \quad \int_\Omega |u^\varepsilon|^{\frac{2n(p-2)}{6-n}} dx < M_2, \tag{4.14}$$

where M_1, M_2 are given by

$$M_1 := C_{\frac{n(p-1)}{2}} (R+1)^{\frac{4-(p-1)(n-2)}{2(p-1)}} \lambda^{\frac{n(p-1)}{2}}$$
$$M_2 := C_{\frac{2n(p-2)}{6-n}} (R+1)^{\frac{6-n-(n-2)(p-2)}{2(p-2)}} \lambda^{\frac{2n(p-2)}{6-n}}.$$

From inequality (4.3) applied to u_t^ε we also have

$$\int_\Omega |u_t^\varepsilon|^{\frac{2n}{n-2}} dx \leq C^* \left(\int_\Omega |\nabla u_t^\varepsilon|^2 dx \right)^{\frac{n}{n-2}}. \tag{4.15}$$

We return to (4.11) and use the estimate (4.12) together with the bounds (4.14), (4.15) to obtain

$$\begin{aligned} \int_\Omega |\nabla u_t^\varepsilon|^2 dx - A_2 \int_\Omega |\nabla u_t^\varepsilon|^2 dx - B_2 \left(\int_\Omega |\nabla u_t^\varepsilon|^2 dx \right)^{3/2} \\ \leq \int_\Omega |f'(u_0)| u_1^2 dx + \int_\Omega [u_{tt}(0)]^2 + |\nabla u_1|^2 dx, \end{aligned} \tag{4.16}$$

where

$$A_2, B_2 > 0$$

Based on the above estimate we construct the second order potential well function

$$\Phi_2(x) = (1 - A_2)x^2 - B_2 x^3. \tag{4.17}$$

We state the following:

Claim 1. The constant $A_2 > 0$ can be chosen such that $A_2 < 1$.

Assuming that Claim 1 holds true, the estimate (4.16) implies that

$$\Phi_2(|\nabla u_t^\varepsilon|_\Omega) < C.$$

Since $\Phi_2(y) \to \infty$ as $y \to \infty$ we obtain the desired bound

$$|\nabla u_t^\varepsilon|_\Omega < C. \tag{4.18}$$

Proof. (Claim 1). The constant A_2 is generated by Sobolev's embedding constants and by the $L^{\frac{n(p-1)}{2}}$ and $L^{2(p-2)}$ norms of u on the domain Ω. The Sobolev constant itself depends on the domain Ω, but since these domains will be chosen to be balls, the constant does not increase when the size of the domain decreases. The $L^{\frac{n(p-1)}{2}}$ and $L^{2(p-2)}$ norms of u are bounded by $|\nabla u^\varepsilon|_\Omega$ according to the Sobolev embedding theorem, so one could decrease both these norms (and, consequently, decrease A_2) by decreasing the radius of the domain

(this follows from the equi-integrability of the ∇u^ε on Ω). Note, also, that $|\nabla u^\varepsilon|_\Omega \to 0$ if $|\Omega| \to 0$. Hence

$$A_2 \to 0 \text{ as } |\Omega| \to 0.$$

In section 4.2 we will construct small domains $B_{x_0}(\rho)$ where the smallness assumptions on the initial data are satisfied; smallness of the initial data implies smallness for $|\nabla u^\varepsilon|_\Omega$, so we will choose those domains small enough so that we also have $A_2 < 1$.

□

We next use the multipliers $-\Delta u_t^\varepsilon$ and u_{tt}^ε to complete our list of a priori estimates. Multiply the equation by $-\Delta u_t^\varepsilon$ and integrate by parts to obtain:

$$\int_\Omega \nabla u_{tt}^\varepsilon \cdot \nabla u_t^\varepsilon + \Delta u^\varepsilon \cdot \Delta u_t^\varepsilon + f_\varepsilon'(u^\varepsilon)\nabla u^\varepsilon \cdot \nabla u_t^\varepsilon - g(u_t^\varepsilon)\Delta u_t^\varepsilon dx = 0.$$

Note that

$$-\int_\Omega g(u_t^\varepsilon)\Delta u_t^\varepsilon dx = \int_\Omega g(u_t^\varepsilon)|\nabla u_t^\varepsilon|^2 dx \geq 0,$$

hence, by Hölder's inequality we obtain the following estimates:

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla u_t^\varepsilon|^2 + |\Delta u^\varepsilon|^2 dx \leq C\int_\Omega |u^\varepsilon|^{p-1}|\nabla u^\varepsilon||\nabla u_t^\varepsilon| dx \\
&\leq C\left(\int_\Omega |u^\varepsilon|^{(p-1)\alpha}|dx\right)^{1/\alpha}\left(\int_\Omega |\nabla u^\varepsilon|^\beta dx\right)^{1/\beta}\cdot \\
&\left(\int_\Omega |\nabla u_t^\varepsilon|^{\frac{2n}{n-2}\gamma} dx\right)^{1/\gamma}
\end{aligned}$$

with $\alpha = n$, $\beta = 2$, $\gamma = \frac{2n}{n-2}$. Since $p < \frac{n+2}{n-2}$ we clearly have that $(p-1)n < \frac{2n}{n-2}$, so by Sobolev's emebedding theorems and the bounds (4.10) and (4.18) we obtain from the above inequality that for all $t < 1$

$$|\Delta u^\varepsilon|_\Omega < C. \tag{4.19}$$

By differentiating (SW) with respect to time and using the multiplier u_{tt}^ε we obtain:

$$\frac{d}{dt}\int_\Omega |u_{tt}^\varepsilon|^2 + |\nabla u_t^\varepsilon|^2 dx \leq -2\int_\Omega f_\varepsilon'(u^\varepsilon)u_t^\varepsilon u_{tt}^\varepsilon dx - 2\int_\Omega g'(u_t^\varepsilon)(u_{tt}^\varepsilon)^2 dx.$$

After integrating in time from 0 to $t < 1$ and using the positivity of g' we obtain:

$$\int_\Omega |u_{tt}^\varepsilon|^2 dx \leq -2\int_{\Omega\times(0,1)} f_\varepsilon'(u^\varepsilon)u_t^\varepsilon u_{tt}^\varepsilon dx ds$$

By using the estimate (4.12) together with the bounds (4.10), (4.18) we obtain

$$|u_{tt}^\varepsilon|_\Omega < C. \tag{4.20}$$

4.2. **Generating the "small" solutions.** In this section we will show how to find the domains $B_{x_0}(\rho)$ and how to construct on these domains pairs of initial data such that the assumptions (a2)-(a5) made in section 4.1 are satisfied. We start by finding a domain $B_{x_0}(\rho)$ small enough, and construct a new pair of initial data $(u_0^{x_0}, u_1^{x_0})$ such that $(u_0^{x_0}, u_1^{x_0})$ satisfy the smallness assumptions (4.2) on $B_{x_0}(\rho)$ and $u_0^{x_0}$ vanishes on the boundary of the ball. The balls $B_{x_0}(\rho)$ have the radius $\rho < 1$ independent of x_0 and small enough such that

$$
\begin{aligned}
&|\nabla u_0|_{B(x_0,\rho)} < \frac{\lambda}{2}, \quad \text{and} \quad |\nabla u_0|^2_{B(x_0,\rho)} \le \frac{\Phi(\lambda)}{8}, \\
&2(C^*\omega_n)^{\frac{1}{n}}\left(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)}\right) \le \min\Big\{\frac{\lambda}{2}, \frac{\sqrt{\Phi(\lambda)}}{8}\Big\} \text{ and} \\
&4{C^*}^2\omega_n^{\frac{n+4}{2n}}|u_0|^2_{B(x_0,\rho)} \le \frac{\Phi(\lambda)}{8}, \quad \frac{1}{2}|u_1|^2_{B(x_0,\rho)} \le \frac{\Phi(\lambda)}{4}, \\
&\frac{f_1}{p+1}(C^*)^{p+1}\Big(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)}\big(\frac{2}{\rho}+1\big)\Big)^{p+1} \le \frac{\Phi(\lambda)}{8}, \\
&\frac{f_2}{3}(C^*)^3\Big(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)}\big(\frac{2}{\rho}+1\big)\Big)^{3} \le \frac{\Phi(\lambda)}{8},
\end{aligned}
\tag{4.21}
$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and C^* is the constant from the Sobolev inequality. It can be easily shown that the above inequalities are satisfied by ρ such that

$$
\begin{aligned}
|\nabla u_0|_{B(x_0,\rho)} < \min\Bigg\{&\frac{\lambda}{2}, \sqrt{\frac{\Phi_1(\lambda)}{8}}, \frac{\lambda}{4(C^*\omega_n)^{\frac{1}{n}}}, \\
&\frac{1}{2C^*}\left(\frac{(p+1)\Phi_1(\lambda)}{8m_1}\right)^{1/(p+1)}, \frac{1}{2C^*}\left(\frac{3\Phi_1(\lambda)}{8f_2}\right)^{1/3}\Bigg\},
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
|u_0|_{B(x_0,\rho)} < \min\Bigg\{&\frac{\lambda}{4(C^*\omega_n)^{\frac{1}{n}}}, \frac{1}{\omega_n C^*}\sqrt{\frac{\Phi(\lambda)}{32\omega_n^{\frac{n+4}{2n}}}}, \\
&\frac{1}{8C^*}\left(\frac{\Phi_1(\lambda)}{8m_1}\right)^{1/(p+1)}, \frac{1}{8C^*}\left(\frac{3\Phi_1(\lambda)}{8m_2}\right)^{1/3}\Bigg\},
\end{aligned}
\tag{4.23}
$$

and

$$
|u_1|^2_{B(x_0,\rho)} \le \sqrt{\frac{\Phi_1(\lambda)}{8}}. \tag{4.24}
$$

The fact that ρ can be chosen independently of x_0 is motivated by the equi-integrability of the functions $u_0, \nabla u_0, u_1$ (see [7]), which also gives that ρ does not vary with x_0.

Next we will show that these smallness assumptions remained satisfied when one truncates the initial data to obtain zero trace on the boundary.

4.2.1. *Truncation of the initial data.* The bound (4.10) gives us by Sobolev's embedding theorem that $u_t^\varepsilon \in L^{\frac{2n}{n-2}}$, provided we have that $u^\varepsilon = u_t^\varepsilon = 0$ on $\partial\Omega$. Hence, we will

construct new initial data u_0 by multiplying it by a smooth cut-off function θ such that

$$\theta(x) = \begin{cases} 1, & |x - x_0| \leq \rho/2 \\ 0, & |x - x_0| \geq \rho \end{cases}$$

and

$$|\theta|_{\infty, B(x_0,\rho)} \leq 1, \quad |\nabla\theta|_{\infty, B(x_0,\rho)} \leq \frac{2}{\rho}. \tag{4.25}$$

On $B_{x_0}(\rho)$ we denote

$$u_0^{x_0} = \theta u_0, \quad u_1^{x_0} = u_1,$$

and by u^{x_0} the solution generated by $(u_0^{x_0}, u_1^{x_0})$. Then (4.10) and (4.18) imply by Sobolev's embedding theorem that $u, u_t \in L^{\frac{2n}{n-2}}(B_{x_0}(\rho))$.

In order to show that the smallness conditions are satisfied we start with the following estimate

$$|\nabla u_0^{x_0}|_{B(x_0,\rho)} \leq |\theta|_{\infty, B(x_0,\rho)} |\nabla u_0|_{B(x_0,\rho)} + |\nabla\theta|_{\infty, B(x_0,\rho)} |u_0|_{B(x_0,\rho)}.$$

By (4.25), (4.21), Hölder's inequality, followed by Sobolev's inequality we conclude that:

$$\begin{aligned} |\nabla u_0^{x_0}|_{B(x_0,\rho)} &< \frac{\lambda}{2} + |B(x_0,\rho)|^{\frac{1}{n}} \frac{2}{\rho} |u_0|_{\frac{2n}{n-2}, B(x_0,\rho)} \\ &\leq \frac{\lambda}{2} + 2(C^*\omega_n)^{\frac{1}{n}} \left(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)} \right) \overset{(4.21)}{\leq} \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda. \end{aligned}$$

so the first part of (4.2) is satisfied by $u_0^{x_0}$. For the second part, first note that by (4.21) we have

$$\frac{1}{2} |u_1^{x_0}|^2_{B(x_0,\rho)} < \frac{\Phi(\lambda)}{4}$$

since $u_1 = u_1^{x_0}$ on $B(x_0,\rho)$. Also with a similar argument as above we have the estimates

$$\begin{aligned} |\nabla u_0^{x_0}|^2_{B(x_0,\rho)} &\leq |\theta|^2_{\infty, B(x_0,\rho)} |\nabla u_0|^2_{B(x_0,\rho)} + |\nabla\theta|^2_{\infty, B(x_0,\rho)} |u_0|^2_{B(x_0,\rho)} \\ &< \frac{\Phi(\lambda)}{8} + 4(C^*\omega_n)^{\frac{2}{n}} \left(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)} \right)^2 \overset{(4.21)}{\leq} \frac{\Phi(\lambda)}{4}. \end{aligned} \tag{4.26}$$

To finish proving (4.2), we use the growth assumption from (A_f), the Sobolev embedding theorem and the restrictions for ρ in (4.21) to obtain:

$$\begin{aligned} &\int_{B(x_0,\rho)} F(u_0^{x_0}(x))\, dx \leq \int_{B(x_0,\rho)} \max\left\{ \frac{f_1}{p+1} |u_0^{x_0}(x)|^{p+1}, \frac{f_2}{3} |u_0^{x_0}(x)|^3 \right\} dx \\ &\leq \max\left\{ \frac{f_1}{p+1} (C^*)^{p+1} \left(|\nabla u_0^{x_0}|_{B(x_0,\rho)} + |u_0^{x_0}|_{B(x_0,\rho)} \right)^{p+1}, \frac{f_2}{3} (C^*)^3 \left(|\nabla u_0^{x_0}|_{B(x_0,\rho)} + |u_0^{x_0}|_{B(x_0,\rho)} \right)^3 \right\} \\ &\leq \max\left\{ \frac{f_1}{p+1} (C^*)^{p+1} \left(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)} \left(\frac{2}{\rho} + 1 \right) \right)^{p+1}, \right. \\ &\left. \frac{f_2}{3} (C^*)^3 \left(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)} \left(\frac{2}{\rho} + 1 \right) \right)^3 \right\} \end{aligned}$$

$$< \frac{\Phi(\lambda)}{8} + \frac{\Phi(\lambda)}{8} = \frac{\Phi(\lambda)}{4}.$$

By summing the above inequalities we have that $(u_0^{x_0}, u_1^{x_0})$ satisfies the second inequality of (4.2) i.e.

$$\frac{1}{2}|u_1^{x_0}|^2_{B(x_0,\rho)} + \frac{1}{2}|\nabla u_0^{x_0}|^2_{B(x_0,\rho)} + \int_{B(x_0,\rho)} F(u_0^{x_0}(x))\,dx < \Phi_1(\lambda). \tag{4.27}$$

4.3. **Convergence of approximations of weak solutions.** At this point we construct Lipschitz approximations f_ε for f such that all f_ε satisfy the assumptions (A_f) with constants f_1, f_2 (see [13] for details). It is easy to see that the coefficients A, B from (4.7), corresponding to f_ε, as well as the root λ, and the radius ρ chosen above, will not depend on ε. By solving the problem with initial data $(u_0^{x_0}, u_1^{x_0})$ on $B_{x_0}(\rho)$ we obtain a solution u^{x_0} which satisfies the estimate:

$$|\nabla u_\varepsilon^{x_0}(t)|_{B_{x_0}(\rho)} < \lambda, \tag{4.28}$$

for all $t > 0$. The maximum time of existence will be restricted through the patching argument (see section 4.4) by $\rho/2$ so in the sequel all our estimates will be considered for $0 \le t \le \rho/2$.

The energy identity (3.2), (A_f), and the fact that g is increasing imply that:

$$|u_{\varepsilon_t}^{x_0}(t)|^2_{B_{x_0}(\rho)} + |\nabla u_\varepsilon^{x_0}(t)|^2_{B_{x_0}(\rho)} + \int_{B_{x_0}(\rho)} F_\varepsilon(u_\varepsilon^{x_0}(x,t))dx \le E(0),$$

so, with the growth condition on F_ε, Sobolev's inequality and (4.28) we obtain the bound:

$$\begin{aligned} &|u_{\varepsilon_t}^{x_0}(t)|^2_{B_{x_0}(\rho)} + |\nabla u_\varepsilon^{x_0}(t)|^2_{B_{x_0}(\rho)} \\ &\le E(0) + \int_{B_{x_0}(\rho)} \max\left\{\frac{f_1}{p+1}|u_\varepsilon^{x_0}(x,s)|^{p+1}, \frac{f_2}{3}|u_\varepsilon^{x_0}(x,s)|^3)dx\right\} \\ &\le E(0) + \max\left\{C(\rho,f_1)|\nabla u_\varepsilon^{x_0}(t)|^{p+1}_{B_{x_0}(\rho)}, C(\rho,f_2)|\nabla u_\varepsilon^{x_0}(t)|^3_{B_{x_0}(\rho)}\right\} \\ &< C, \end{aligned} \tag{4.29}$$

for all $0 \le t \le \rho/2$. By integrating in time (4.29) up to $\rho/2$, we deduce from Alaoglu's Theorem the existence of a subsequence, denoted also by $u_\varepsilon^{x_0}$, for which we have the convergences:

$$\begin{aligned} &u_\varepsilon^{x_0} \to u^{x_0} \text{ weakly star in } L^\infty(0,\rho/2; H_0^1(B_{x_0}(\rho))) \\ &u_{\varepsilon_t}^{x_0} \to u_t^{x_0} \text{ weakly star in } L^\infty(0,\rho/2; L^2(B_{x_0}(\rho))). \end{aligned} \tag{4.30}$$

Also, by Aubin's Theorem and Rellich-Kondrachov compactness embedding theorem we have the convergence

$$u_\varepsilon^{x_0} \to u^{x_0} \text{ strongly in } L^2((0,\rho/2)\times B_{x_0}(\rho))$$

so for a subsequence we have

$$u_\varepsilon^{x_0}(x,t) \to u^{x_0}(x,t) \text{ a.e. } (x,t) \in B_{x_0}(\rho)\times(0,\rho/2). \tag{4.31}$$

It is shown in [13] that this implies

$$f_\varepsilon(u_\varepsilon^{x_0}(x,t)) \to f(u^{x_0}(x,t)) \text{ a.e. } (x,t) \in B_{x_0}(\rho) \times (0,\rho/2), \tag{4.32}$$

so we finally have (after using the subcritical growth of f)

$$f_\varepsilon(u_\varepsilon^{x_0}(x,t)) \to f(u^{x_0}(x,t)) \text{ in } L^1(B_{x_0}(\rho) \times (0,\rho/2)), \tag{4.33}$$

hence, also in the sense of distributions.

A monotonicity argument will be applied in order to pass to the limit in the nonlinear dissipative term $g(u_{\varepsilon_t}^{x_0})$. From ($\mathrm{A}_g$), the energy identity and the bounds on F_ε obtained above, we have:

$$\begin{aligned} &|u_{\varepsilon_t}^{x_0}(t)|^2_{B_{x_0}(\rho)} + |\nabla u_\varepsilon^{x_0}(t)|^2_{B_{x_0}(\rho)} + \int_0^t \int_{B_{x_0}(\rho)} |u_{\varepsilon_t}^{x_0}(x,s)|^{m+1} dxds \\ &\quad \le |u_{\varepsilon_t}^{x_0}(t)|^2_{B_{x_0}(\rho)} + |\nabla u_\varepsilon^{x_0}(t)|^2_{B_{x_0}(\rho)} + \int_0^t \int_{B_{x_0}(\rho)} g(u_{\varepsilon_t}^{x_0}(x,s)) u_{\varepsilon_t}^{x_0}(x,s) dxds \le C. \end{aligned} \tag{4.34}$$

Therefore, we can again extract a subsequence $u_\varepsilon^{x_0}$ such that:

$$\begin{aligned} &u_{\varepsilon_t}^{x_0} \rightharpoonup u_t^{x_0} \text{ in } L^{m+1}((0,\rho/2) \times B_{x_0}(\rho)) \\ &g(u_{\varepsilon_t}^{x_0}) \rightharpoonup \xi \text{ in } L^{(m+1)/m}((0,\rho/2) \times B_{x_0}(\rho)). \end{aligned} \tag{4.35}$$

Passing to the limit in ε we obtain (we drop the ^{x_0} superscript for u in the sequel)

$$u_{tt} - \Delta u + f(u) + \xi = 0 \text{ in the sense of distributions.} \tag{4.36}$$

To verify that $\xi = g(u_t)$ let u^ε, u^η be two terms in the sequence $(u^\varepsilon)_{\varepsilon>0}$. We subtract the equations satisfied by u^ε, u^η and we obtain :

$$\begin{aligned} &\int_{B_{x_0}(\rho)} |u_t^\varepsilon - u_t^\eta|^2 + |\nabla u^\varepsilon - \nabla u^\eta|^2 dx + \int_0^t \int_{B_{x_0}(\rho)} (g(u_t^\varepsilon) - g(u_t^\eta))(u_t^\varepsilon - u_t^\eta) dxds \\ &= -\int_0^t \int_{B_{x_0}(\rho)} (f_\varepsilon(u^\varepsilon) - f_\eta(u^\eta))(u_t^\varepsilon - u_t^\eta) dxds. \end{aligned} \tag{4.37}$$

The standard process through which we obtained the above equality (multiplication by $u_t^\varepsilon - u_t^\eta$, integration by parts etc.) is motivated exactly the same way as in obtaining the energy identity for equations with Lipschitz source terms with nonlinear damping (see [13] for more details).

In the spirit of the monotonicity argument found in [11] on pg. 518 (although we are here in a different setup with approximations on the source terms, not on the damping) we obtain

$$\lim_{\eta,\varepsilon\to 0} \int_0^t \int_{B_{x_0}(\rho)} (g(u_t^\varepsilon) - g(u_t^\eta))(u_t^\varepsilon - u_t^\eta) dxds = 0. \tag{4.38}$$

This equality is obtained by passing to the limit as $\eta, \varepsilon \to 0$ in (4.37). The convergence to zero of the term on the RHS of (4.37) is shown below. After using the positivity of all

of the terms on the LHS the conclusion of (4.38) follows. We repeat the argument used in [13] to show the convergence to zero of the difference of source terms, i.e.

$$\lim_{\eta,\varepsilon\to 0}\int_0^t\int_{B_{x_0}(\rho)}(f_\varepsilon(u^\varepsilon)-f_\eta(u^\eta))(u_t^\varepsilon-u_t^\eta)dxds=0. \tag{4.39}$$

First we multiply out the quantities in the integrand and show convergence for each of them. We start with the study of the "non-mixed" product $f_\varepsilon(u^\varepsilon)u_{\varepsilon_t}$ (identical analysis for $f_\eta(u^\eta)u_t^\eta$). We have the equality

$$\int_0^t\int_{B_{x_0}(\rho)}f_\varepsilon(u^\varepsilon(x,s))u_t^\varepsilon(x,s)dxds=\int_{B_{x_0}(\rho)}f_\varepsilon(u^\varepsilon(x,s))dx|_{s=0}^{s=t}, \tag{4.40}$$

where we notice that we can pass to the limit by (4.10) combined with the condition of subcritical growth for F.

The analysis of the "mixed" terms (which are $f_\varepsilon(u^\varepsilon)u_t^\eta$ and $f_\eta(u^\eta)u_t^\varepsilon$ will require a finer analysis. We first analyze $f_\varepsilon(u^\varepsilon)u_t^\eta$ which converges a.e. to $f(u)u_t^\eta$ as $\varepsilon\to 0$ by (4.32). By Egoroff's Theorem for every $\delta>0$ there exists a set $A\subset(0,t)\times B_{x_0}(\rho)$ with $|A|<\delta$ such that $f_\varepsilon(u^\varepsilon)u_t^\eta\to f(u)u_t^\eta$ uniformly (hence, in L^1) on $(0,t)\times B_{x_0}(\rho)\backslash A$.

We write

$$\int_0^t\int_{B_{x_0}(\rho)}f_\varepsilon(u^\varepsilon)u_t^\eta dxds=\int_{(0,t)\times B_{x_0}(\rho)\backslash A}f_\varepsilon(u^\varepsilon)u_t^\eta dxds+\int_A f_\varepsilon(u^\varepsilon)u_t^\eta dxds. \tag{4.41}$$

Due to the uniform convergence of $f_\varepsilon(u^\varepsilon)u_t^\eta\to f(u)u_t^\eta$ (as $\varepsilon\to 0$) on $(0,t)\times B_{x_0}(\rho)\backslash A$, we have

$$\lim_{\varepsilon\to 0}\int_{(0,t)\times B_{x_0}(\rho)\backslash A}f_\varepsilon(u^\varepsilon)u_t^\eta dxds=\int_{(0,t)\times B_{x_0}(\rho)\backslash A}f(u)u_t^\eta dxds. \tag{4.42}$$

In order to analyze the integral on A from (4.41) we apply Hölder's inequality with conjugate exponents λ,β, and γ:

$$\int_A|f_\varepsilon(u^\varepsilon)u_t^\eta|dxds\le C\left(\int_A|u^\varepsilon|^{\lambda p}dxds\right)^{\frac{1}{\lambda}}\left(\int_A|u_t^\eta|^\beta dxds\right)^{\frac{1}{\beta}}|A|^{\frac{1}{\gamma}}. \tag{4.43}$$

We choose α, β, and γ such that:

$$\alpha=\frac{2n}{p(n-2)},\quad \beta=\frac{2n}{n-2},\quad \gamma=\frac{2n}{2n-(n-2)(p+1)},$$

and by Sobolev embedding theorem, (4.10), and (4.18) we have the desired bounds in (4.43) if $\gamma>0$. The positivity of γ amounts to $p<\dfrac{n+2}{n-2}$ which is guaranteed by assumption (A_f).

Now we go back in (4.41) and take $\lim\limits_{\delta\to 0}\lim\limits_{\varepsilon\to 0}$ in both sides. First, in (4.42) take $\lim\limits_{\delta\to 0}$ and notice that we can bound the integrand the same way as in (4.43), and since we have the

convergence of the sets $(0,t)\times B_{x_0}(\rho)\times(0,\rho/2)\backslash A\to(0,t)\times B_{x_0}(\rho)\times(0,\rho/2)$ as $\delta\to 0$, by Lebesgue dominated convergence theorem we have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{(0,t)\times B_{x_0}(\rho)\backslash A} f_\varepsilon(u^\varepsilon)u_t^\eta dxds=\int_{(0,t)\times B_{x_0}(\rho)} f(u)u_t^\eta dxds.$$

From (4.43) we have that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_A |f_\varepsilon(u^\varepsilon)u_t^\eta|dxds\le\lim_{\delta\to 0}C|A|^{\frac{1}{\gamma}}\lim_{\varepsilon\to 0}M=0,$$

where M is a bound for the first two factors on the RHS of (4.43). Thus we obtain:

$$\lim_{\varepsilon\to 0}\int_0^t\int_{B_{x_0}(\rho)} f_\varepsilon(u^\varepsilon)u_t^\eta dxds=\int_0^t\int_{B_{x_0}(\rho)} f(u)u_t^\eta dxds.$$

We let $\eta\to 0$ and by using (4.33) and (4.30) we obtain that the limit of this mixed term is $\int_0^t\int_{B_{x_0}(\rho)} f(u)u_t dxds$. The analysis of the second mixed term is similar and so we omit it. Thus we conclude that (4.39) holds, hence (4.38) is valid. Note that (4.37), (4.38),(4.39) also gives us the continuity with respect to time

$$u\in C(0,T;H_0^1(\Omega))\cap C^1(0,T;L^2(\Omega)).$$

The bounds (4.18), (4.19), (4.20) guarantee that our weak solution has the regularity of a strong solution, i.e.

$$(u(\cdot,t),u_t(\cdot,t))\in(H^2(\Omega)\cap H_0^1)(\Omega)\times H^1(\Omega)\text{ for all }t>0.$$

4.4. **The patching argument.** We conclude the proof of existence of solutions in this section, where we construct from small solutions the solution on the entire domain Ω.

4.4.1. *Approximations on small domains.* Let $d>0$. Consider a lattice of points x_k, $k\in\mathbb{N}$ in Ω situated at distance d away from each other, such that in every ball of radius d we find at least one x_k. With ρ that satisfies (4.21) (where ρ depends only on the norms of the initial data), construct the balls of radius ρ centered at x_k. The procedure outlined in subsection 4.2.1 for truncating the initial data around x_0 to obtain a "small piece" denoted by $(u_0^{x_0},u_1^{x_0})$, will be used now to construct around each x_k the truncations $(u_0^{x_k},u_1^{x_k})$ which will satisfy the smallness assumptions

$$\begin{aligned}&|\nabla u_0^{x_k}|_{B_{x_k}(\rho)}<\alpha,\\&\frac{1}{2}|u_1^{x_k}|^2_{B_{x_k}(\rho)}+\frac{1}{2}|\nabla u_0^{x_k}|^2_{B_{x_k}(\rho)}+\int_{B_{x_k}(\rho)}F(u_0^{x_k}(x))\,dx<\Phi(\alpha).\end{aligned}$$

On each of the balls $B_{x_k}(\rho)$ we apply Theorem 3.2 to obtain global existence of solutions u^{ε,x_k} for the problem (SW) with initial data $(u_0^{x_k},u_1^{x_k})$ and with the Lipschitz approximations f_ε for the source term.

4.4.2. *Convergence of approximations on small domains.* At this point, the arguments from section 4.3 for passing to the limit as $\varepsilon \to 0$ in the sequence of approximate problems are applied, where x_0 is successively replaced by different x_k's. First, we apply Sattinger's argument to estimate $|\nabla u^{\varepsilon,x_k}|_2$ on $B_{x_k}(\rho)$ for each k. (Note that we need to make use of the smallness assumptions written in Step 4.1.) The convergence $u^{\varepsilon,x_k} \to u^{x_k}$ takes place on every domain $B_{x_k}(\rho) \times (0, \rho/2)$, so we obtain a global (in time) solution to the boundary value problem (SW) for $B_{x_k}(\rho)$, for every k.

4.4.3. *Patching small domains.* The solutions u^{x_k} found in the above subsection 4.4.2 will now be "patched" together to obtain our general solution. First, we introduce the following notation. For $k \in \mathbb{N}$ let

$$C_k := \{(y, s) \in \mathbb{R}^3 \times [0, \infty); |y - x_k| \leq \rho/2 - s\}$$

be the backward cones which have their vertices at $(x_k, \rho/2)$. For d small enough (i.e. for $0 < d < \rho/2$) any two neighboring cones C_k and C_j will intersect and let

$$I_{k,j} := C_k \cap C_j.$$

The maximum value for the time contained in $I_{k,j}$ is equal to $(\rho - d)/2$ (see Figure 3 below).

For $t < \rho/2$ we define the piecewise function:

$$u(x, t) := u^{x_k}(x, t), \text{ if } (x, t) \in C_k. \tag{4.44}$$

This solution is defined only up to time $(\rho - d)/2$, since this is the height of the intersection set of two cones with their vertices situated at distance d away from each other. By letting $d \to 0$ we can obtain a solution that is well defined up to time $\rho/2$. Thus, we have u defined up to time $\rho/2$, which is the height of all cones C_k. Every pair $(x, t) \in \mathbb{R}^n \times (0, \rho/2)$ belongs to at least one C_k, so in order to show that this function from (4.44) is well defined, we need to check that it is single-valued on the intersection of two cones. Also, we need to show that the above function is the solution generated by the pair of initial data (u_0, u_1). Both proofs will be done in the next step.

4.4.4. *Validity of solutions.* In order to prove the properties that the previously constructed solution is well defined and satisfies the equation, we will go back and look at the solutions u^{x_k} as limits of the approximation solutions u^{ε,x_k}.

Consider first these small solutions on balls that do not intersect the boundary. For each $k \in \mathbb{N}$ we have $(u_0^{x_k}, u_1^{x_k}) = (u_0, u_1)$ for every $x \in B_{x_k}(\rho) = \{y \in \mathbb{R}^n, |y - x_k| < \rho/2\}$ (see the construction of the truncations $(u_0^{x_k}, u_1^{x_k})$ in subsection 4.2.1). Therefore, u^{ε,x_k} (defined in section 4.4.1) is an approximation of the solution generated by the initial data (u_0, u_1) on C_k (from the uniqueness property given by Part 2 of Proposition 3.4). We let $\varepsilon \to 0$ (use

the argument from subsection 4.4.3) to show that the solution u on each C_k is generated by the initial data (u_0, u_1).

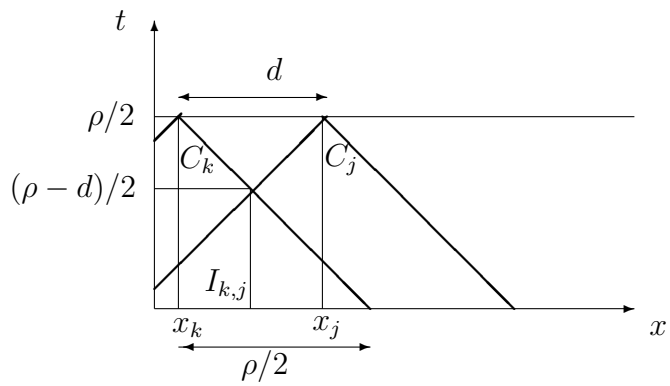

Figure 3: The intersection of the domains $C_k \cap \Omega$ and $C_j \cap \Omega$

Remark 4.3. The above method of using cutoff functions and "patching" solutions based on uniqueness will work the same way in the case when the function f satisfies $F(y) \geq 0, y \in \mathbb{R}$ and there is *no bound on* p. Since we can choose the height of the the cones arbitrarily large, the solutions exist globally in time under the positivity hypothesis for F.

□

5. Final remarks

Remark 5.1. The paper shows that local well-posedness of strong solutions holds without assuming the restriction

$$p + \frac{p}{m} < \frac{2n}{n-2} \qquad \text{for } \frac{n}{n-2} < p < \frac{n+2}{n-2} \tag{R}$$

and a couple of remarks should be made regarding some of the ideas used in the proof. The restriction allows one to transfer a requirement for higher integrability of u to higher integrability for u_t up to this barrier. This fact is illustrated through the key estimates contained in (4.12) and (4.41) where terms of the form:

$$\int_{(0,t)\times\omega} |u|^{\alpha} |u_t|^{p+1-\alpha} dxds$$

had to be estimated. Note that by writing $u = \int_0^t u_t(x,s)dxds$ one can transfer all the powers α onto the exponent of u_t, so it is only important to be able to estimate products of u and u_t raised to exponents which add up to $p+1$.

In this spirit, when looking for strong solutions, one could use the higher integrability for u_t when p is small (instead of using the usual bound for $|u_t|_{m+1}$), find a bound for $|\nabla u_t|_2$ which will eventually lead by Sobolev's embedding theorem to higher integrability for u_t (i.e. $u \in L^{\frac{2n}{n-2}}$) whenever $m+1 < \frac{2n}{n-2}$.

Remark 5.2. These results will hold when one assumes some dependence on x and t as done in [13, 14], but we chose to present a simplified version of the proof here. Also, one may allow variable coefficient elliptic operator instead of the laplacian.

Remark 5.3. The proof can accommodate other growth conditions on f, for example:

$$|f''(u)| \leq f_1|u|^{p-2} + f_2|u|^{q-2} \text{ such that } 1 < q < p < \frac{n+2}{n-2},\ f_1, f_2 > 0,$$

but for simplicity of the presentation we did not write the proof in its entire generality.

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Department of Mathematics

University of Nebraska-Lincoln

203 Avery Hall

Lincoln, NE 68588-0130 USA

e-mail: pradu@math.unl.edu