Weak Solutions to the Initial Boundary Value Problem of a Semilinear Wave Equation with Damping and Source Terms

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Abstract

In this paper we show local existence of solutions to the initial boundary value problem corresponding to a semilinear wave equation with interior damping and source terms. The difficulty in dealing with these two competitive forces comes from the fact that the source term is not a locally Lipschitz function from $H^1(\Omega)$ into $L^2(\Omega)$ as typically assumed in the literature. The strategy behind the proof is based on the physics of the problem, so it does not use the damping present in the equation. The arguments are natural and adaptable to other settings/other PDEs.
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1 Introduction

Consider the boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + f(x,t,u) + g(x,t,u_t) = 0 \text{ a.e. } (x,t) \in \Omega \times [0,\infty); \\ u(x,t) = 0, \text{ a.e. } (x,t) \in \partial\Omega \times [0,\infty); \\ (u,u_t)|_{t=0} = (u_0,u_1), \text{ a.e. } x \in \Omega. \end{cases} \tag{SW}$$

where Ω is a connected bounded domain with C^2 boundary. In [13] we proved local existence of solutions to the Cauchy problem ($\Omega = \mathbb{R}^n$) under a set of assumptions A_f, A_g concerning the nonlinearities f and g that we will list below. In this work we are interested in solving the problem on a

bounded domain, issue which is of great interest in control theory ([9, 10]). This paper complements the work [13] as we will prove that the techniques of [13] apply to bounded domains as well, however, some nontrivial arguments have to be used in order to deal with the boundary conditions.

1.1 Assumptions

At this point we present the list of assumptions that govern our result. Note that the set (A0) – (A8) refers to the source and the damping terms, while (A9) is a geometric assumption on the boundary.

(A0) f is measurable in x, differentiable in t a.e., differentiable in u a.e., and there exists a continuous function k such that for a.e. x, t

$$|f_u(x,t,u)| \leq k(r) \text{ for a.e.} |u| \leq r;$$

(A1) Growth conditions on the source term f:

(i) $f(x,t,0) = 0$;

(ii) $|f(x,t,u)| \leq m_1|u|^p + m_2|u|^q$ such that $1 < q < p < 2^* - 1$, $m_1, m_2 > 0$, where $2^* = \dfrac{2n}{n-2}$;

(A2) $|f_t(x,t,u)| \leq K$ for some $K > 0$;

(A2)* f does not depend on t and $F(x,u) = \displaystyle\int_0^u f(x,v)dv \geq 0$;

(A3) $g = g(x,t,v)$ is measurable in t, differentiable in x, and continuous in v;

(A4) For every x, t the function $v \to g(x,t,v)$ is increasing and $g(x,t,0) = 0$;

(A5) $vg(x,t,v) \geq C_1|v|^{m+1}$ and $|g(x,t,v)| \leq C_2|v|^m$ for some $m \geq 0$;

(A6) $|\nabla_x g(x,t,v)| \leq C|v|$;

(A7) $|g_t(x,t,v)| \leq C|v|$.

(A8) Either (a) $1 < p < \dfrac{2^*}{2}$, $m > 0$, (b) $p + \dfrac{p}{m} < 2^*$, $m > 0$, or (c) $1 < p < 2^* - 1$, where p is given by (A1), and g is either linear in v or it does not depend on v. Note that the range of exponents (p, m) given by (A8)(c) is **strictly** included in the set $(1, 2^* - 1) \times \{0, 1\}$.

(A9) Ω is convex such that $\Omega \cap B(z,r)$ are open, bounded, connected domains with Lipschitz boundary for any $z \in \Omega$, $r > 0$ (regular domains in the sense of Sobolev's embedding theorem.)

The constants C, C_1, C_2 denote nonnegative numbers which may change from line to line.

The above assumptions are generalizations of the much studied differential equation:

$$u_{tt} - \Delta u \pm u|u|^{p-1} \pm u|u|^{q-1} + a(x,t)u_t|u_t|^{m-1} = 0,$$

where $1 < q < p < 2^* - 1$, $m \geq 0$, and $a(x,t) \geq 0$. The "$-$" sign in the above equation corresponds to a situation when energy is introduced into the system, whereas the "$+$" sign gives us a model whose energy is dissipating. Special cases of this equation arise in quantum field theory (e.g., the Klein-Gordon equation) and in mechanical applications (the dynamics of a membrane in presence of friction when energy is introduced in the system).

1.2 Relationship to previous literature

The interaction between the damping and the source terms has been extensively studied such that many results and conjectures abound in the literature. Thus, it is well-known that the source term $f(x,t,u)$ alone can drive the solution to blow-up in finite time if it corresponds to an accretion of energy [7, 8, 12, 15]. In contrast, the damping term $g(x,t,u_t)$ leads to global existence of weak solutions. The decisive factor in establishing long time behavior of solutions under the interaction of these two effects appears to be the power of the nonlinearity, such that a higher power of damping would yield long time existence, while a dominant source term will cause the solutions to go to infinity in finite time. This relationship has been studied on both bounded and unbounded domains [2, 3, 4, 6, 9, 11, 13, 16, 19].

The largest set of exponents p and m for which local existence of solutions was proven for $\mathbb{R}^n$ was found in [13] to be given by (A8). This result extended the set of exponents found by J. Serrin, G. Todorova and E. Vitillaro in [16] while using a natural approach with finite energy spaces which are adapted to the equation. For bounded domains we prove that the local existence holds for the same range of exponents as for the entire space, thus generalizing the results of [6, 11] where the source term is locally Lipschitz from H^1 into L^2. When the mapping f is locally Lipschitz (in other words, f has subcritical exponent $p \leq 3$ in $n = 3$) local existence of solutions can be proved through monotone semigroup theory (see the Appendix of [4]); for global existence the source term has to additionally satisfy a coercivity-type condition involving the first eigenvalue of the Laplacian. Under these Lipschitz assumptions for the interior source terms the authors of [3, 9, 4] study existence of solutions and stability issues, where nonlinear source and damping terms act on the boundary of the domain. This set up is different from ours since problems with nonlinear damping on the boundary in general do not satisfy the Lopatinski condition.

We would like to point out that in contrast with the existing literature ([2, 6, 16]), our arguments *do not make use of the smoothing effect of the*

damping. This allows us to handle exponents p that go all the way up to 2^*-1 in the case of **linear** damping and in the case of **no damping**. Several

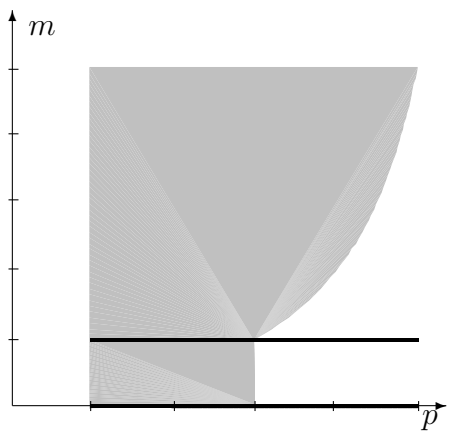

Figure 1: The range of exponents p and m in $n = 3$ where the thick lines mark only the case of linear damping and no damping, so they do not cover *nonlinear* terms $g(x, t, v)$ whose growth exponent is $m = 0$ or $m = 1$.

works mention that the damping is essential in dealing with source terms, our results, however, prove the contrary. For local existence of solutions, at least for subcritical exponents for the source term, the extra regularity given by the damping is not needed. This fact strongly suggests that one should be able to extend the allowable range of exponents to the full box $1 < p < 2^*-1$, $m \geq 0$, as one naturally expects from the Sobolev embedding theorem. In other words, we conjecture that the exponents p and m do not have to satisfy the restriction

$$p + \frac{p}{m} < 2^* \text{ for } \frac{2^*}{2} < p < 2^* - 1. \tag{R}$$

The methods used in the study of nonlinear wave equations are usually suited to either bounded *or* unbounded domains and very few methods are applicable in both setups. We show here that our arguments are some of these fortunate situations. The proof relies on two main ingredients: the potential well method due to Sattinger ([12, 14]) and a "patching" argument which was originally used by Crandall and Tartar (see [17, 18]) to show global existence of arbitrarily large solutions for the Broadwell model. In both problems ([13] and [17]) the boundary conditions were absent since the PDEs were considered on the entire space, but we show

here that the methods work in the presence of boundary conditions with appropriate modifications. We will often make reference to various results and arguments of [13] without including the proof due to space limitations.

Another fact that distinguishes this paper from other works is that the existence of solutions is proved by passing to the limit for a sequence of approximating solutions where *only the source term is approximated with Lipschitz functions and not the damping term.* This allows us to stay within the natural framework of Lebesgue spaces. Note that the usual approach of approximating the damping term with Lipschitz functions is not feasible in Lebesgue spaces (in [16] this obstacle was overcome through a recourse to Orlicz spaces). This is due to the incompatibility between the Lipschitz assumptions and the polynomial growth conditions $vg(x,t,v) \geq \frac{1}{C}|v|^m$, $|g(x,t,v)| \leq C|v|^{m-1}$ for some $m \geq 1$, and $g(x,t,0) = 0$, which makes it impossible to work in a Lebesgue space (see [16] for more details).

In conclusion, we summarize the main contributions of this manuscript as follows:

- We provide an extension of the allowed range of exponents p and m through a stronger and more substantial use of the monotonicity of g. Prior to this work local existence results for wave equations with interior source and damping terms on **bounded** domains were proved only for the range of exponents described by (A8)(a) which corresponds to the case f Lipschitz from H^1 into L^2. Our results extend the prior range in two directions, by allowing (A8)(b) ($p+\frac{p}{m} < 2^*$) and (A8)(c) ($\frac{p^*}{2} \leq p < 2^* - 1$ for linear or no damping).

- In the presence of absorption terms (when f satisfies (A2)*) and damping terms whose range of exponents satisfy (A8) we prove global existence of solutions.

- These results hold for finite energy initial data, not necessarily with compact support as it is assumed in other works (e.g. [16, 19]).

- Our work has the advantage of allowing some dependence on t in the function f.

- The results apply for wave equations with variable coefficients as pointed out in Remark 1 of [13].

- This work together with [13] illustrate the great applicability of these methods since they work for problems which are set on bounded and unbounded domains.

2 Main Result

We are concerned in this paper with existence of weak solutions, whose definition we present below.

Definition 2.1. *Let $\Omega_T := \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is an open connected set with smooth boundary $\partial\Omega$. Suppose the functions f and g satisfy the assumptions* (A1) *and* (A5)*, and further suppose that $u_0 \in H_0^1(\Omega) \cap L^{p+1}(\Omega)$ and $u_1 \in L^2(\Omega) \cap L^{m+1}(\Omega)$.*

A weak solution on Ω_T of the boundary value problem

$$\begin{cases} u_{tt} - \Delta u + f(x,t,u) + g(x,t,u_t) = 0 \ \text{in } \Omega_T; \\ (u,u_t)|_{t=0} = (u_0,u_1); \\ u = 0 \ \text{on } \partial\Omega \times (0,T). \end{cases} \tag{SW}$$

is any function u satisfying

$$u \in C(0,T;H_0^1(\Omega)) \cap L^{p+1}(\Omega_T), \quad u_t \in L^2(\Omega_T) \cap L^{m+1}(\Omega_T),$$

and

$$\int_{\Omega_T} \Big(u(x,s)\phi_{tt}(x,s) + \nabla u(x,s) \cdot \nabla\phi(x,s) + f(x,s,u)\phi(x,s) + g(x,s,u_t)\phi(x,s) \Big)\, dxds = \int_\Omega \Big(u_1(x)\phi(x,0) - u_0(x)\phi_t(x,0) \Big) dx$$

for every $\phi \in C_c^\infty(\Omega \times (-\infty,T))$.

In the sequel we use the following notation: $|\cdot|_q$ is the norm in $L^q(\Omega)$. If the domain over which the norm is considered is not clear from the context, then we use $|\cdot|_{q,\Omega}$.

Below we present our main result and also the main steps of the proof. The steps which are identical with the steps for the proof done on the entire space $\mathbb{R}^n$ (as presented in [13]) are sketched so that we cam focus instead on the patching argument where we have to deal with boundary conditions.

Theorem 2.2. (Main Theorem) *Let* $(u_0, u_1) \in H_0^1(\Omega) \cap L^{p+1}(\Omega) \times L^2(\Omega) \cap L^{m+1}(\Omega)$ *and consider the Cauchy problem*

$$\begin{cases} u_{tt} - \Delta u + f(x,t,u) + g(x,t,u_t) = 0 \ \ a.e. \ \ in \ \Omega \times [0,\infty); \\ u(x,t) = 0, \ \ a.e. \ \ (x,t) \in \partial\Omega \times [0,\infty); \\ (u,u_t)|_{t=0} = (u_0,u_1), \ \ a.e. \ \ x \in \Omega. \end{cases} \tag{SW}$$

where f *and* g *satisfy (A0)-(A8). Additionally, assume that* $G(x,0,u_1) \in L^1(\Omega)$*, where* $G(x,t,v) = \int_0^v g(x,t,u)du$*. Then, there exists a time* $0 < T < 1$ *such that (SW) admits a weak solution on* $[0,T]$ *in the sense of Definition* 2.1. *In addition, if (A2)* is satisfied, then the solution is global, so* T *can be taken arbitrary. The solution is also continuous in time in the topology of the finite energy space, i.e.*

$$u \in C(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega)).$$

Remark: The restriction $T < 1$ is arbitrary, the time of existence is naturally limited by the physics of the problem, since one expects blow-up of solutions whenever the source term dominates the damping.

Proof. In order to facilitate the exposition we present a brief outline of the proof. As we mentioned in the introduction we approximate the problem with problems that have Lipschitz source terms. We will then use a compactness and a monotonicity argument in order to pass to the limit in the sequence of approximating solutions. This approach will actually give us existence of "small" solutions, so we will "patch" these small solutions in order to obtain a solution on the entire domain Ω.

Step 1. We begin by considering the problem with a globally Lipschitz source term in the u argument and arbitrary damping. It is known that for the boundary value problem with f is Lipschitz in the third argument and g taken as general as possible under the assumptions A_g, we have the following theorem on global existence, uniqueness and regularity of strong solutions for (SW) (see [1] or [13]).

Theorem 2.3. (Existence and uniqueness of solutions for dissipative wave equations with Lipschitz source terms) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with smooth boundary* $\partial\Omega$*, and the functions* $f(x,t,u)$ *and* $g(x,t,v)$ *satisfy assumptions* (A0), (A2), (A3)-(A7), *and* f *is globally Lipschitz in the last argument with Lipschitz constant* L*, i.e.*

$$|f(x,t,u) - f(x,t,v)| \leq L|u-v|,$$

for every $u, v \in \mathbb{R}$. Let $u_0, u_1 \in H_0^1(\Omega) \times L^2(\Omega)$ with $G(x, 0, u_1) \in L^1(\Omega)$, where G is defined by the formula

$$G(x,t,v) = \int_0^v g(x,t,y)dy.$$

Then (SW) admits a unique solution u on the time interval [0, T] such that

$$u \in C(0,T; H_0^1(\Omega)) \cap L^{p+1}(\Omega_T), \quad u_t \in L^2(\Omega_T) \cap L^{m+1}(\Omega_T).$$

The solutions to the wave equation with Lipschitz source terms and polynomial-like damping satisfy have finite speed of propagation. In [13] two statements are proven, one that looks at the growth of the support of solutions, and the second, which regards uniqueness of solutions which start with the same initial data. Due to the presence of the nonlinearities f and g, these properties are not equivalent, even though the proofs are very similar. The statement concerning the size of the support was proven prior to [13] only for source terms of "good" sign (when their contribution to the energy was a nonnegative quantity). The complete statement of the theorem follows below with the only modification that the balls are permitted to intersect the boundary. The proof remains the same as in [13] bur we include it here for the sake of completeness.

Theorem 2.4. (Finite speed of propagation) *Consider the problem (SW) under the hypothesis of Theorem* 2.3 *(i.e. we additionally impose that the source terms are globally Lipschitz). Then*

(1) *if the initial data u_0, u_1 is compactly supported inside the domain $B(x_0, R) \cap \Omega$, then $u(x,t) = 0$ outside $B(x_0,\ R+t) \cap \Omega$;*

(2) *if $(u_0, u_1), (v_0, v_1)$ are two pairs of initial data with compact support, with the corresponding solutions $u(x,t)$, respectively $v(x,t)$, with*

$$(u_0(x), u_1(x)) = (v_0(x), v_1(x)) \text{ for } x \in B(x_0, R) \cap \Omega,$$

then $u(x,t) = v(x,t)$ inside $B(x_0,\ R-t) \cap \Omega$ for any $t < R$.

Proof. Part (1) The proof extends an argument used for the linear wave equation by L. Tartar [18].

Assume for now that $f(x,t,u) = 0$ for $|x - x_0| \geq R + t$. Since the equation is invariant by translations, without loss of generality we can take $x_0 = 0$. First we approximate the initial data uniformly by smooth functions (u_{0_η}, u_{1_η}) with compact support inside $B(R_\eta)$, with $R_\eta \nearrow R$ as $\eta \to 0$. By

Theorem 2.3, for any $T > 0$, the solution of:

$$\begin{cases} u_{\eta_{tt}} - \Delta u_\eta + f(x,t,u_\eta) + g(x,t,u_{\eta_t}) = 0 \text{ in } \Omega \times (0,T); \\ (u_\eta, u_{\eta_t})|_{t=0} = (u_{\eta_0}, u_{\eta_1}); \\ u_\eta = 0 \text{ on } \partial\Omega \times (0,T) \end{cases} \tag{SW$_\eta$}$$

exists on $[0,T]$ and it has the additional regularity

$$u \in C^1([0,T]; L^2(\Omega)) \text{ with } u_{tt},\, \Delta u \in L^2(0,T; L^2(\Omega)).$$

Consider a function ϕ_η with $\phi_\eta(r) = 0$ on $(-\infty, R_\eta]$, $\phi_\eta(r) > 0$ on (R_η, ∞), such that $\phi'(r) \geq 0$ on $\mathbb{R}$. Since $u_{\eta_t} \in L^\infty(0,T; H_0^1(B(R_\eta)))$, we are allowed to multiply (SW$_\eta$) by $u_{\eta_t}(t,x)\phi_\eta(|x|-t)$, for any $0 < t < T$. The quantity:

$$I_\eta(t) := \int_\Omega (|u_{\eta_t}(x,t)|^2 + |\nabla u_\eta(x,t)|^2)\phi_\eta(|x|-t)\, dx$$

is well defined and assume for now that $\frac{dI_\eta}{dt} \leq 0$. It can be easily seen that $I_\eta(0) = 0$, since

$$I_\eta(0) = \int_{B(R_\eta)\cap\Omega} (|u_{1_\eta}(x)|^2 + |\nabla u_{0_\eta}(x)|^2)\phi_\eta(|x|)\, dx \\ + \int_{\Omega\setminus B(R_\eta)} (|u_{1_\eta}(x)|^2 + |\nabla u_{0_\eta}(x)|^2)\phi_\eta(|x|)\, dx.$$

The first integral is 0 since $\phi_\eta(|x|) = 0$ for $|x| < R_\eta$. The initial data has support inside the ball $|x| < R_\eta$, so the second integral is zero.

The assumption that the mapping $t \to I_\eta(t)$ is decreasing leads us to $I_\eta(t) \leq I_\eta(0) = 0$, which means that $u_\eta(x,t) = 0$ if $|x| - t > R_\eta$. We pass to the limit in η (see Theorem 2.2 in [13] on the convergence of approximations of solutions) to obtain $u(x,t) = 0$ for $|x| - t > R$, and this is what we set out to prove in Part 1.

It suffices then to prove that $\dfrac{dI_\eta}{dt} \leq 0$. For the regularized initial data we have $u_\eta \in L^\infty(0,T; H_0^1(B(R_\eta)))$, $u_{\eta_t} \in L^\infty(0,T; H_0^1(B(R_\eta)))$, $u_{\eta_{tt}} \in L^1(0,T; L^2(B(R_\eta)))$, which enables us to compute (we drop the subscript η

in the remainder of the proof)

$$\frac{dI}{dt}(t) = \int_\Omega 2\phi(|x|-t)(u_t u_{tt} + \sum_{i=1}^n u_{x_i} u_{tx_i})(x,t)\,dx$$
$$- \int_\Omega \phi'(|x|-t)(u_t^2 + |\nabla u|^2)(x,t)\,dx = \int_\Omega 2\phi(|x|-t)(u_t u_{tt})(x,t)\,dx$$
$$- \int_\Omega \sum_{i=1}^n \left((2\phi(|x|-t)u_{x_i})_{x_i} u_t\right)(x,t)\,dx - \int_\Omega \phi'(|x|-t)(u_t^2 + |\nabla u|^2)(x,t)\,dx$$
$$= \int_\Omega 2\phi(|x|-t)((u_{tt} - \Delta u)u_t)(x,t)\,dx - \int_\Omega \sum_{i=1}^n 2\phi'(|x|-t)\frac{x_i}{r}(u_{x_i}u_t)(x,t)\,dx$$
$$- \int_\Omega \phi'(|x|-t)(u_t^2 + |\nabla u|^2)(x,t)\,dx$$

By (SW)

$$u_{tt} - \Delta u = -f(x,t,u) - g(x,t,u_t),$$

hence the assumptions on the support of ϕ and f, together with the fact that g is nondecreasing, make the first term of the last equality negative. We factor out $\phi'(|x|-t)$ in the other two terms, and since $\phi'(r) \geq 0$ for every r, it is enough to show that

$$u_t^2 + |\nabla u|^2 + 2\sum_{i=1}^n \frac{x_i}{|x|} u_{x_i} u_t \geq 0, \tag{2.1}$$

which is obtained by summing the inequalities:

$$\left(\frac{x_i}{|x|}u_t + u_{x_i}\right)^2 \geq 0$$

for all $i = 1, ..., n$.

It remains to show that the function f vanishes for $|x| \geq R + t$. A fixed point argument will establish this fact now. Consider the iterative equation:

$$\begin{cases} u_{tt}^{k+1} - \Delta u^{k+1} + f(x,t,u^k) + g(x,t,u_t^{k+1}) = 0 \\ (u^{k+1}, u_t^{k+1})|_{t=0} = (u_0, u_1) \\ u^{k+1} = 0 \text{ on } \partial\Omega \times (0,T), \end{cases}$$

for every $k \in \mathbb{N}$, with $(u^0, u_t^0) = (u_0, u_1)$. The existence of a unique weak solution is guaranteed by Theorem 2.3. An induction argument, together with the first part of the proof, will show that $u^k(x,t) = 0$ for $|x| > R + t$,

for every $k \in \mathbb{N}$. It is enough then to show that $u^k(x,t) \to u(x,t)$ a.e. as $k \to \infty$. Since f is Lipschitz we obtain that $f(x,t,u^k(x,t))$, which is zero for $|x| \geq R+t$, converges a.e. to $f(x,t,u(x,t))$, hence f vanishes outside the cone $|x| < R+t$. The sequence of difference functions $v^k(x,t) := u^k(x,t) - u(x,t)$ satisfies:

$$\begin{cases} v_{tt}^{k+1} - \Delta v^{k+1} + f(x,t,v^k+u) - f(x,t,u) \\ \qquad\qquad + g(x,t,v_t^{k+1}+u_t) - g(x,t,u_t) = 0 \\ (v^{k+1}, v_t^{k+1})|_{t=0} = (0,0) \\ v^{k+1} = 0 \text{ on } \partial\Omega \times (0,T). \end{cases}$$

Upon multiplication by v_t^{k+1} and integration over $(0,t) \times \Omega$, we use the monotonicity of g to derive the following inequality:

$$\begin{aligned} &\int_\Omega (v_t^{k+1}(x,t))^2 + |\nabla v^{k+1}(x,t)|^2 \, dx \\ &\leq \int_0^t \int_\Omega 2|f(x,t,v^k+u) - f(x,t,u)||v_t^{k+1}(x,s)| dx \, ds, \end{aligned}$$

which by the Lipschitz assumption on f is

$$\leq \int_0^t 2L|v^k(s)|_2 |v_t^{k+1}(s)|_2 \, ds \leq L \int_0^t |v^k(s)|_2^2 + |v_t^{k+1}(s)|_2^2 \, ds.$$

We now need a bound for $\displaystyle\int_0^t |v^k(s)|_2^2 ds$, which we obtain by writing:

$$\begin{aligned} |v^k(t)|_2^2 &= 2\int_0^t \int_\Omega v^k(x,s) v_t^k(x,s) dx \, ds \\ &\leq \int_0^t \int_\Omega (v^k(x,s))^2 + (v_t^k(x,s))^2 dx \, ds. \end{aligned}$$

Gronwall's inequality for the function $|v^k(s)|_2^2$ will give us for any $t < T$ the bound:

$$|v^k(t)|_2^2 \leq e^T \int_0^t |v_t^k(s)|_2^2 ds.$$

At this point, to simplify the writing let

$$\phi^k(t) := \int_0^t |v_t^k(s)|_2^2 + |\nabla v^k(t)|_2^2 ds.$$

By summarizing the estimates above, we have that ϕ^{k+1} satisfies the inequality:

$$\phi_t^{k+1}(t) \leq L\phi^{k+1}(t) + C\phi^k(t),$$

which after integration becomes:

$$\phi^{k+1}(t) \leq C\int_0^t e^{L(t-s)}\phi^k(s)ds \leq Ce^{LT}\int_0^t \phi^k(s)ds.$$

A simple induction argument will show that:

$$\phi^{k+1}(t) \leq K\frac{C^{k+1}e^{LT(k+1)}t^{k+1}}{(k+1)!},$$

where K is a bound on $|\phi^1(t)|$ for all t in $[0,T]$. Thus we proved the convergence for $u^k(x,t)$ a.e. (x,t), so $u(x,t) = 0$ outside the domain of dependence, i.e. for $x \in \Omega$, $|x| \geq R+t$. Recall that this actually is proven for the sequence of approximated solutions u_η. By Theorem 2.2 in [13] we have the convergence $u_\eta(t) \to u(t) \in H_0^1(\Omega)$, hence $u_\eta(t) \to u(t)$ a.e. This concludes the proof of Part (1).

Part (2). We follow here a similar argument as in Part (1). Initially, we work under the assumption that

$$f(x,t,u(x,t)) = f(x,t,v(x,t)) \tag{2.2}$$

for $|x| < R-t$ (again, take $x_0 = 0$). The difference $u-v$ satisfies:

$$\begin{cases} (u-v)_{tt} - \Delta(u-v) + f(x,t,u) - f(x,t,v) + g(x,t,u_t) - g(x,t,v_t) = 0; \\ ((u-v),(u-v)_t)\,|_{t=0} = (0,0) \\ u-v = 0 \text{ on } \partial\Omega \times (0,T). \end{cases}$$

Consider a function ψ strictly positive on $(-\infty, R)$, such that $\psi(r) = 0$ on $[R,\infty)$ and $\psi'(r) \leq 0$ everywhere. Define the function $J(t)$ by

$$J(t) := \int_{\mathbb{R}^n} \Big((u_t(x,t) - v_t(x,t))^2 + |\nabla(u(x,t) - v(x,t))|^2\Big)\psi(|x|-t)dx.$$

We will show that $\dfrac{dJ}{dt} \leq 0$. As before, this will show that $u(x,t) = v(x,t)$ on the support of $\psi(|x|+t)$, i.e. if $|x| < R-t$. The proof follows along the same lines as in part (1). □

The energy identity allows us to obtain bounds for solutions by using a natural approach based on the physics of the problem (a complete proof can be found in [13]).

Proposition 2.5. (The energy identity) *If u is a weak solution of (SW), under the assumptions of Theorem 2.3 we have the following equality:*

$$E(t) + \int_\Omega F(x,t,u(x,t))\,dx - \int_0^t \int_\Omega f_t(x,s,u(x,s))\,dx\,ds + \int_0^t \int_\Omega g(x,s,u_t(x,s))u_t(x,s)\,dx\,ds = E(0), \quad (2.3)$$

where $E(t) := \frac{1}{2}|u_t(t)|_2^2 + \frac{1}{2}|\nabla u(t)|_2^2$.

At this point we employ the well known Sattinger potential well method to obtain bounds for $|\nabla u(t)|_2$ if the initial data satisfies the smallness assumption

$$|\nabla u_0|_2 < \alpha, \quad \frac{1}{2}|u_1|_2^2 + \frac{1}{2}|\nabla u_0|_2^2 + \int_\Omega F(x,0,u_0(x))\,dx + K|\Omega| < \Phi(\alpha), \quad (2.4)$$

where $|\Omega|$ is the Lebesgue measure of Ω, K is the constant from (A2), and Φ is the potential well function with α being its global maximum point. More precisely,

$$\Phi(x) = \frac{x^2}{2} - Ax^p - Bx^q, \quad x \geq 0. \quad (2.5)$$

where A and B are given by

$$A = m_1 C^p (R+1)^{n\frac{2^*-p}{2^*}}, \qquad B = m_2 C^q (R+1)^{n\frac{2^*-q}{2^*}}.$$

and α is the only positive root of the equation $pA\alpha^{p-2} + qB\alpha^{q-2} = 1$. (see (3.2) in [13] for more details). As a consequence of Sattinger's well potential argument we obtain

$$|\nabla u(t)|_2 \leq \alpha \quad (2.6)$$

for all $t > 0$ if the smallness assumptions 2.4 are satisfied. If (A2)* is satisfied then the positivity of the $F(u) = \int_0^u f(v)dv$ immediately gives us the desired bounds and we also have global in time estimates for the gradient of the solution. It is important to remark that once the Lipschitz approximations f_ε of f are used in the above problem these bounds remain independent of ε so one can use a compactness argument in order to pass to the limit.

Step 2. At this stage we show how one can construct the small solutions which will be later "patched" to obtain the solution on the entire domain. Consider a pair of initial data (u_0, u_1) such that $u_0 \in H^2(\Omega), u_1 \in H^1(\Omega)$,

and $G(x,0,u_1) \in L^1(\Omega)$ (recall that $G(x,t,v) = \int_0^v g(x,t,y)\,dy$). The higher differentiability assumptions on the initial data are removed with a standard approximation argument, exactly as in [13]. For now, we keep the Lipschitz assumptions for f.

For each point $x_0 \in \Omega$ we will find a domain ω_{x_0} around it such that the smallness assumptions are satisfied on ω_{x_0}. We will distinguish between (a) interior points that are far away from the boundary and (b) points which are close to the boundary.

(a) First fix $x_0 \in \Omega$ far away from the boundary (this statement will be made more precise below). As in [13] we find a domain ω_{x_0} small enough, and construct a new pair of initial data $(u_0^{x_0}, u_1^{x_0})$ such that $(u_0^{x_0}, u_1^{x_0})$ satisfy (2.4) inside ω_{x_0}. More precisely, $\omega_{x_0} = B(x_0,\rho)$ where the radius $\rho < 1$ is chosen independent of x_0 and small enough such that

$$\begin{aligned}
&\rho < \Big(\frac{\Phi(\alpha)}{4K\omega_n}\Big)^{1/n}, \quad |\nabla u_0|_{2,B(x_0,\rho)} < \frac{\alpha}{2}, \quad \text{and} \quad |\nabla u_0|^2_{2,B(x_0,\rho)} \le \frac{\Phi(\alpha)}{8},\\
&2(C^*\omega_n)^{\frac{1}{n}}\left(|\nabla u_0|_{2,B(x_0,\rho)} + |u_0|_{2,B(x_0,\rho)}\right) \le \min\Big\{\frac{\alpha}{2}, \frac{\sqrt{\Phi(\alpha)}}{8}\Big\} \text{ and}\\
&4{C^*}^2\omega_n^{\frac{2^*-2}{2^*}}|u_0|^2_{2,B(x_0,\rho)} \le \frac{\Phi(\alpha)}{8}, \quad \frac{1}{2}|u_1|^2_{2,B(x_0,\rho)} \le \frac{\Phi(\alpha)}{4},\\
&m_1(C^*)^p\Big(|\nabla u_0|_{2,B(x_0,\rho)} + |u_0|_{2,B(x_0,\rho)}\big(\frac{2}{\rho}+1\big)\Big)^p \le \frac{\Phi(\alpha)}{8},\\
&m_2(C^*)^q\Big(|\nabla u_0|_{2,B(x_0,\rho)} + |u_0|_{2,B(x_0,\rho)}\big(\frac{2}{\rho}+1\big)\Big)^q \le \frac{\Phi(\alpha)}{8},
\end{aligned} \tag{2.7}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and C^* is the constant from the Sobolev inequality. It can be easily shown that the above inequalities are satisfied by $\rho < (\frac{\Phi(\alpha)}{4K\omega_n})^{1/n}$ such that

$$|\nabla u_0|_{2,B(x_0,\rho)} < \min\left\{\frac{\alpha}{2}, \sqrt{\frac{\Phi(\alpha)}{8}}, \frac{\alpha}{4(C^*\omega_n)^{\frac{1}{n}}}, \frac{1}{2C^*}\left(\frac{\Phi(\alpha)}{8m_1}\right)^{1/p}, \frac{1}{2C^*}\left(\frac{\Phi(\alpha)}{8m_2}\right)^{1/q}\right\}, \tag{2.8}$$

$$|u_0|_{2,B(x_0,\rho)} < \min\left\{\frac{\alpha}{4(C^*\omega_n)^{\frac{1}{n}}}, \frac{1}{\omega_n C^*}\sqrt{\frac{\Phi(\alpha)}{32\omega_n^{\frac{2^*-2}{2^*}}}}, \frac{1}{8C^*}\left(\frac{\Phi(\alpha)}{8m_1}\right)^{1/p}, \frac{1}{8C^*}\left(\frac{\Phi(\alpha)}{8m_2}\right)^{1/q}\right\}, \tag{2.9}$$

and

$$|u_1|^2_{2,B(x_0,\rho)} \leq \sqrt{\frac{\Phi(\alpha)}{8}}. \tag{2.10}$$

The fact that ρ can be chosen independently of x_0 is motivated by the equi-integrability of the functions $u_0, \nabla u_0, u_1$. More precisely, for each of the functions $u_0, \nabla u_0, u_1$ we apply the following result of classical analysis:

If $f \in L^1(A)$, with A a measurable set, then for every given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\int_E |f(x)|dx < \varepsilon$, for every measurable set $E \subset A$ of measure less than δ (see [5]).

Note that δ in the above result does not depend on E, hence ρ does not vary with x_0.

In order to apply the results of Step 1 we need to have that u_0 has zero trace on $\partial\omega_{x_0}$, so we multiply it by a smooth cut-off function θ such that

$$\theta(x) = \begin{cases} 1, & |x - x_0| \leq \rho/2 \\ 0, & |x - x_0| \geq \rho \end{cases}$$

and

$$|\theta|_{\infty,B(x_0,\rho)} \leq 1, \quad |\nabla\theta|_{\infty,B(x_0,\rho)} \leq \frac{2}{\rho}. \tag{2.11}$$

On ω_{x_0} we denote

$$u_0^{x_0} = \theta u_0, \quad u_1^{x_0} = u_1,$$

and by u^{x_0} the solution generated by $(u_0^{x_0}, u_1^{x_0})$. In order to show that the smallness conditions are satisfied we start with the following estimate

$$|\nabla u_0^{x_0}|_{2,B(x_0,\rho)} \leq |\theta|_{\infty,B(x_0,\rho)}|\nabla u_0|_{2,B(x_0,\rho)} + |\nabla\theta|_{\infty,B(x_0,\rho)}|u_0|_{2,B(x_0,\rho)}.$$

By (2.11), (2.7), Hölder's inequality, followed by Sobolev's inequality we conclude that:

$$\begin{aligned} |\nabla u_0^{x_0}|_{2,B(x_0,\rho)} &< \frac{\alpha}{2} + |B(x_0,\rho)|^{\frac{1}{n}}\frac{2}{\rho}|u_0|_{2^*,B(x_0,\rho)} \\ &\leq \frac{\alpha}{2} + 2(C^*\omega_n)^{\frac{1}{n}}\left(|\nabla u_0|_{2,B(x_0,\rho)} + |u_0|_{2,B(x_0,\rho)}\right) \overset{(2.7)}{\leq} \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha. \end{aligned}$$

so the first part of (2.4) is satisfied. For the second part, first note that by (2.7) we have

$$\frac{1}{2}|u_1^{x_0}|^2_{2,B(x_0,\rho)} < \frac{\Phi(\alpha)}{4}$$

since $u_1 = u_1^{x_0}$ on $B(x_0,\rho)$. Also with a similar argument as above we have the estimates

$$|\nabla u_0^{x_0}|^2_{2,B(x_0,\rho)} \leq |\theta|^2_{\infty,B(x_0,\rho)}|\nabla u_0|^2_{2,B(x_0,\rho)} + |\nabla\theta|^2_{\infty,B(x_0,\rho)}|u_0|^2_{2,B(x_0,\rho)}$$
$$< \frac{\Phi(\alpha)}{8} + 4(C^*\omega_n)^{\frac{2}{n}}\left(|\nabla u_0|_{2,B(x_0,\rho)} + |u_0|_{2,B(x_0,\rho)}\right)^2 \overset{(2.7)}{\leq} \frac{\Phi(\alpha)}{4}. \quad (2.12)$$

For the third term in the second inequality of (2.4), we use assumption (A1)(a), the Sobolev embedding theorem and the restrictions for ρ in (2.7) to obtain:

$$\int_{B(x_0,\rho)} F(x,0,u_0^{x_0}(x))\,dx \leq \int_{B(x_0,\rho)} (m_1|u_0^{x_0}(x)|^p + m_2|u_0^{x_0}(x)|^q)\,dx$$
$$\leq m_1(C^*)^p\left(|\nabla u_0^{x_0}|_{B(x_0,\rho)} + |u_0^{x_0}|_{B(x_0,\rho)}\right)^p + m_2(C^*)^q\left(|\nabla u_0^{x_0}|_{B(x_0,\rho)} + |u_0^{x_0}|_{B(x_0,\rho)}\right)^q$$
$$\leq m_1(C^*)^p\left(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)}\left(\frac{2}{\rho}+1\right)\right)^p$$
$$+ m_2(C^*)^q\left(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)}\left(\frac{2}{\rho}+1\right)\right)^q < \frac{\Phi(\alpha)}{8} + \frac{\Phi(\alpha)}{8} = \frac{\Phi(\alpha)}{4}.$$

We also have that $K|B(x_0,\rho)| < \dfrac{\Phi(\alpha)}{4}$, so by summing the above inequalities we have that $(u_0^{x_0}, u_1^{x_0})$ satisfies the second inequality of (2.4) i.e.

$$\frac{1}{2}|u_1^{x_0}|^2_{2,B(x_0,\rho)} + \frac{1}{2}|\nabla u_0^{x_0}|^2_{B(x_0,\rho)} + \int_{B(x_0,\rho)} F(x,0,u_0^{x_0}(x))\,dx + K|B(x_0,\rho)| < \Phi(\alpha). \quad (2.13)$$

In order to eliminate the higher regularity restrictions for the initial data we approximate $u_0 \in H^1(\mathbb{R}^n)$, $u_1 \in L^2(\mathbb{R}^n)$ by smooth functions and pass to the limit in (2.13) to obtain the conclusions of this step for finite energy initial data.

Since we showed that the pair $(u_0^{x_0}, u_1^{x_0})$ satisfies (2.4) we apply the results of Step 1 to obtain

$$|\nabla u^{x_0}(t)|_2 < \alpha, \quad t > 0. \quad (2.14)$$

(b) Assume now that $x_0 \in \Omega$ such that $\mathrm{dist}(x_0, \partial\Omega) < \rho$, where ρ is the number which satisfies (2.7). Let

$$\omega_{x_0} = B(x_0,\rho) \cap \Omega.$$

Note that $\partial\omega_{x_0} = \gamma_0 \cup \gamma_1$, where $\gamma_0 \subset \partial\Omega$ and $\gamma_1 \subset \Omega$. We need to construct $u_0^{x_0} \in H_0^1(\omega_{x_0})$ so to this end we use again the cutoff function θ which was introduced in part (a). Note that we already have $u_0^{x_0}|_{\gamma_0} = 0$ so multiplication by θ will ensure that the trace on γ_1 is also zero. The smallness assumptions are satisfied on ω_{x_0} (the proof is identical with the proof in part (a)).

A key observation here is that the geometrical assumption (A9) we imposed for $\partial\Omega$ implies that ω_{x_0} is a nice domain with Lipschitz boundary on which the embedding inequalities (Sobolev and Rellich-Kondrachov) and the Green identities hold (the integration by parts formula is needed to establish the energy identity).

In both cases (a) and (b) we have that $u_0^{x_0}(x) = u_0(x)$ and $u_1^{x_0}(x) = u_1(x)$, for all $x \in B(x_0, \rho/2) \cap \Omega$.

Step 3. At this point we construct Lipschitz approximations f_ε for f such that f_ε satisfy the assumptions (A0) with a new function k_ε instead of k, and (A1), (A2) with constants $m_1, m_2, p, q, 1+K$ (see [13] for details). It is easy to see that the coefficients A, B from (2.5), corresponding to f_ε, as well as the root α, and the radius ρ chosen above, will not depend on ε. By solving the problem with initial data $(u_0^{x_0}, u_1^{x_0})$ on ω_{x_0} we obtain a solution u^{x_0} which satisfies the estimate:

$$|\nabla u_\varepsilon^{x_0}(t)|_{2,\omega_{x_0}} < \alpha, \tag{2.15}$$

for all $t > 0$. The maximum time of existence will be restricted through the patching argument by $\rho/2$ so in the sequel all our estimates will be considered for $0 \le t \le \rho/2$.

The energy identity (2.3), (A2), and the fact that g is increasing imply that:

$$|u_{\varepsilon_t}^{x_0}(t)|_{2,\omega_{x_0}}^2 + |\nabla u_\varepsilon^{x_0}(t)|_{2,\omega_{x_0}}^2 + \int_{\omega_{x_0}} F_\varepsilon(x,t,u_\varepsilon^{x_0}(x,t))dx \le 2K|\omega_{x_0}| + E(0),$$

so, with the growth condition (A1)(a) on F^ε, Sobolev's inequality and (2.15) we obtain the bound:

$$\begin{aligned} &|u_{\varepsilon_t}^{x_0}(t)|_{2,\omega_{x_0}}^2 + |\nabla u_\varepsilon^{x_0}(t)|_{2,\omega_{x_0}}^2 \\ &\le 2K|\omega_{x_0}| + E(0) + \int_{\omega_{x_0}} (m_1|u_\varepsilon^{x_0}(x,s)|^p + m_2|u_\varepsilon^{x_0}(x,s)|^q)dx \\ &\le 2K|\omega_{x_0}| + E(0) + C(\rho,m_1)|\nabla u_\varepsilon^{x_0}(t)|_{2,\omega_{x_0}}^p + C(\rho,m_2)|\nabla u_\varepsilon^{x_0}(t)|_{2,\omega_{x_0}}^q < C, \end{aligned} \tag{2.16}$$

if $0 \le t \le \rho/2$. By integrating in time (2.16) up to $\rho/2$, we deduce from Alaoglu's Theorem the existence of a subsequence, denoted also by $u_\varepsilon^{x_0}$, for which we have the convergences:

$$\begin{aligned} &u_\varepsilon^{x_0} \to u^{x_0} \text{ weakly star in } L^\infty(0,\rho/2; H_0^1(\omega_{x_0})) \\ &u_{\varepsilon_t}^{x_0} \to u_t^{x_0} \text{ weakly star in } L^\infty(0,\rho/2; L^2(\omega_{x_0})). \end{aligned} \tag{2.17}$$

Also, by Aubin's Theorem and Rellich-Kondrachov compactness embedding theorem we have the convergence

$$u_\varepsilon^{x_0} \to u^{x_0} \text{ strongly in } L^2((0,\rho/2)\times\omega_{x_0})$$

so for a subsequence we have

$$u_\varepsilon^{x_0}(x,t) \to u^{x_0}(x,t) \text{ a.e. } (x,t) \in \omega_{x_0} \times (0,\rho/2). \tag{2.18}$$

It is shown in [13] that this implies

$$f_\varepsilon(x,t,u_\varepsilon^{x_0}(x,t)) \to f(x,t,u^{x_0}(x,t)) \text{ a.e. } (x,t) \in \omega_{x_0} \times (0,\rho/2), \tag{2.19}$$

so we finally have (after using the subcritical growth of f)

$$f_\varepsilon(x,t,u_\varepsilon^{x_0}(x,t)) \to f(x,t,u^{x_0}(x,t)) \text{ in } L^1(\omega_{x_0} \times (0,\rho/2)), \tag{2.20}$$

hence, also in the sense of distributions.

A monotonicity argument will be applied in order to pass to the limit in the nonlinear dissipative term $g(x,t,u_{\varepsilon_t}^{x_0})$. From (A4), the energy identity and the bounds on F_ε obtained above, we have:

$$\begin{aligned} &|u_{\varepsilon_t}^{x_0}(t)|_{2,\omega_{x_0}}^2 + |\nabla u_\varepsilon^{x_0}(t)|_{2,\omega_{x_0}}^2 + \int_0^t \int_{\omega_{x_0}} |u_{\varepsilon_t}^{x_0}(x,s)|^{m+1} dxds \\ &\le |u_{\varepsilon_t}^{x_0}(t)|_{2,\omega_{x_0}}^2 + |\nabla u_\varepsilon^{x_0}(t)|_{2,\omega_{x_0}}^2 + \int_0^t \int_{\omega_{x_0}} g(x,s,u_{\varepsilon_t}^{x_0}(x,s)) u_{\varepsilon_t}^{x_0}(x,s) dxds \le C. \end{aligned} \tag{2.21}$$

Therefore, we can again extract a subsequence $u_\varepsilon^{x_0}$ such that:

$$\begin{aligned} &u_{\varepsilon_t}^{x_0} \rightharpoonup u_t^{x_0} \text{ in } L^{m+1}((0,\rho/2)\times\omega_{x_0}) \\ &g(x,t,u_{\varepsilon_t}^{x_0}) \rightharpoonup \xi \text{ in } L^{(m+1)'}((0,\rho/2)\times\omega_{x_0}). \end{aligned} \tag{2.22}$$

Passing to the limit in ε we obtain (we drop the ^{x_0} superscript for u in the sequel)

$$u_{tt} - \Delta u + f(x,t,u) + \xi = 0 \text{ in the sense of distributions.} \tag{2.23}$$

We need to verify that $\xi = g(x,t,u_t)$.

Let u_ε, u_η be two terms in the sequence $(u_\varepsilon)_{\varepsilon>0}$. We subtract the equations satisfied by u_ε, u_η and we obtain

$$\int_{\omega_{x_0}} |u_{\varepsilon_t}(x,t) - u_{\eta_t}(x,t)|^2 + |\nabla u_\varepsilon(x,t) - \nabla u_\eta(x,t)|^2 dx \tag{2.24}$$
$$+ \int_0^t \int_{\omega_{x_0}} (g(x,s,u_{\varepsilon_t}(x,s)) - g(x,s,u_{\eta_t}(x,s)))(u_{\varepsilon_t}(x,s) - u_{\eta_t}(x,s))dxds$$
$$= -\int_0^t \int_{\omega_{x_0}} (f_\varepsilon(x,s,u_\varepsilon(x,s)) - f_\eta(x,s,u_\eta(x,s)))(u_{\varepsilon_t}(x,s) - u_{\eta_t}(x,s))dxds.$$

The standard process through which we obtained the above equality (multiplication by $u_{\varepsilon_t} - u_{\eta_t}$, integration by parts etc.) is motivated exactly the same way as in obtaining the energy identity for equations with Lipschitz source terms with nonlinear damping (Proposition 2.5).

In the spirit of the monotonicity argument found in [9] on pg. 518 (although we are here in a different setup with approximations on the source terms, not on the damping) we obtain

$$\lim_{\eta,\varepsilon\to 0} \int_0^t \int_{\omega_{x_0}} (g(x,s,u_{\varepsilon_t}) - g(x,s,u_{\eta_t}))(u_{\varepsilon_t} - u_{\eta_t})dxds = 0. \tag{2.25}$$

This equality is obtained by passing to the limit as $\eta, \varepsilon \to 0$ in (2.24). The convergence to zero of the term on the RHS of (2.24) is shown below. After using the positivity of all of the terms on the LHS the conclusion of (2.25) follows. We repeat the argument used in [13] to show the convergence to zero of the difference of source terms, i.e.

$$\lim_{\eta,\varepsilon\to 0} \int_0^t \int_{\omega_{x_0}} (f_\varepsilon(x,s,u_\varepsilon) - f_\eta(x,s,u_\eta))(u_{\varepsilon_t} - u_{\eta_t})dxds = 0. \tag{2.26}$$

First we multiply out the quantities in the integrand and show convergence for each of them. We start with the study of the "non-mixed" product $f_\varepsilon(u_\varepsilon)u_{\varepsilon_t}$ (identical analysis for $f_\eta(u_\eta)u_{\eta_t}$). We have the equality

$$\int_0^t \int_{\omega_{x_0})} f_\varepsilon(x,s,u_\varepsilon(x,s))u_{\varepsilon_t}(x,s)dxds$$
$$= \int_{\omega_{x_0}} F_\varepsilon(x,s,u_\varepsilon(x,s))dx|_{s=0}^{s=t} - \int_0^t \int_{\omega_{x_0}} F_{\varepsilon_t}(x,s,u_\varepsilon(x,s))dxds,$$

where we notice that we can pass to the limit in the first term of the RHS by (2.16) combined with the condition of subcritical growth for F. For the second term by (A2), we have $|F_{\varepsilon_t}(u_\varepsilon)| \leq K|u_\varepsilon|$, and since u_ε is bounded in L^1 (as a consequence Sobolev embedding theorem) by the Lebesgue dominated convergence we get $f_\varepsilon(x,t,u_\varepsilon) \to f(x,t,u)$ in $L^1((0,t)\times B(x_0,\rho^*))$.

The analysis of the "mixed" terms (which are $f_\varepsilon(u_\varepsilon)u_{\eta_t}$ and $f_\eta(u_\eta)u_{\varepsilon_t}$) will, however, impose some restrictions on the exponents p and m. We first analyze $f_\varepsilon(u_\varepsilon)u_{\eta_t}$ which converges a.e. to $f(u)u_{\eta_t}$ as $\varepsilon \to 0$ by (2.19). By Egoroff's Theorem for every $\delta > 0$ there exists a set $A \subset (0,t)\times\omega_{x_0}$ with $|A| < \delta$ such that $f_\varepsilon(u_\varepsilon)u_{\eta_t} \to f(u)u_{\eta_t}$ uniformly (hence, in L^1) on $(0,t)\times\omega_{x_0}\backslash A$.

We write

$$\int_0^t \int_{\omega_{x_0}} f_\varepsilon(u_\varepsilon)u_{\eta_t}dxds = \int_{(0,t)\times\omega_{x_0}\backslash A} f_\varepsilon(u_\varepsilon)u_{\eta_t}dxds + \int_A f_\varepsilon(u_\varepsilon)u_{\eta_t}dxds. \tag{2.27}$$

Due to the uniform convergence of $f_\varepsilon(u_\varepsilon)u_{\eta_t} \to f(u)u_{\eta_t}$ (as $\varepsilon \to 0$) on $(0,t)\times\omega_{x_0}\backslash A$, we have

$$\lim_{\varepsilon\to 0}\int_{(0,t)\times\omega_{x_0}\backslash A} f_\varepsilon(u_\varepsilon)u_{\eta_t}dxds = \int_{(0,t)\times\omega_{x_0}\backslash A} f(u)u_{\eta_t}dxds. \tag{2.28}$$

In order to analyze the integral on A from (2.27) we apply Hölder's inequality with conjugate exponents α, β, and γ:

$$\int_A |f_\varepsilon(u_\varepsilon)u_{\eta_t}|dxds \leq C\left(\int_A |u_\varepsilon|^{\alpha p}dxds\right)^{\frac{1}{\alpha}}\left(\int_A |u_{\eta_t}|^{\beta}dxds\right)^{\frac{1}{\beta}}|A|^{\frac{1}{\gamma}}. \tag{2.29}$$

Our goal is to bound the first two factors on the RHS above, and to this end we have two options for choosing α, β, and γ. First we take:

$$\alpha = \frac{2^*}{p}, \quad \beta = 2, \quad \gamma = \frac{2\cdot 2^*}{2^* - 2p},$$

and by Sobolev embedding theorem and by (2.15) we have the desired bounds in (2.29) if $\gamma > 0$. The positivity of γ amounts to $p < \dfrac{2^*}{2}$ which is condition (a) in (A8). The second choice is

$$\alpha = \frac{m+1}{m-(m+1)\nu}, \quad \beta = m+1, \quad \gamma = \frac{1}{\nu},$$

for some $0 < \nu < 1$. We need to impose that $\alpha p \leq 2^*$ and by letting $\nu \to 0\,(\nu \neq 0)$, we get the restriction $p + \dfrac{p}{m} < 2^*$ (condition (b) in (A8)).

Now we go back in (2.27) and take $\lim_{\delta\to 0}\lim_{\varepsilon\to 0}$ in both sides. First, in (2.28) take $\lim_{\delta\to 0}$ and notice that we can bound the integrand the same way as in (2.29), and since we have the convergence of the sets $(0,t)\times\omega_{x_0}\times(0,\rho/2)\backslash A \to (0,t)\times\omega_{x_0}\times(0,\rho/2)$ as $\delta\to 0$, by Lebesgue dominated convergence theorem we have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{(0,t)\times\omega_{x_0}\backslash A} f_\varepsilon(u_\varepsilon)u_{\eta_t}dxds = \int_{(0,t)\times\omega_{x_0}} f(u)u_{\eta_t}dxds.$$

From (2.29) we have that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_A |f_\varepsilon(u_\varepsilon)u_{\eta_t}|dxds \le \lim_{\delta\to 0} C|A|^{\frac{1}{\gamma}}\lim_{\varepsilon\to 0} M = 0,$$

where M is a bound for the first two factors on the RHS of (2.29). Thus we obtain:

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\omega_{x_0}} f_\varepsilon(u_\varepsilon)u_{\eta_t}dxds = \int_0^t\int_{\omega_{x_0}} f(u)u_{\eta_t}dxds.$$

We let $\eta\to 0$ and by using (2.20) and (2.17) we obtain that the limit of this mixed term is $\int_0^t\int_{\omega_{x_0}} f(u)u_t dxds$. The analysis of the second mixed term is similar and so we omit it. Thus we conclude that (2.26) holds, hence (2.25) is valid. Note that (2.24), (2.25),(2.26) also gives us the continuity with respect to time

$$u\in C(0,T;H_0^1(\Omega))\cap C^1(0,T;L^2(\Omega)).$$

We continue the monotonicity argument by multiplying out the quantities in (2.25). By using (2.17) and (2.22) we have

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\omega_{x_0}}(g(x,s,u_{\varepsilon_t})u_{\varepsilon_t}-\xi(x,s)u_{\varepsilon_t}-g(x,s,u_{\varepsilon_t})u_t)dxds$$
$$+\lim_{\eta\to 0}\int_0^t\int_{\omega_{x_0}} g(x,s,u_{\eta_t})u_{\eta_t}dxds = 0.$$

After using again (2.17) and (2.22) and changing η to ε in the last term above we get

$$2\lim_{\varepsilon\to 0}\int_0^t\int_{\omega_{x_0}} g(x,s,u_{\varepsilon_t})u_{\varepsilon_t}dxds = 2\int_0^t\int_{\omega_{x_0}}\xi(x,s)u_t dxds. \tag{2.30}$$

By the monotonicity of g we have that:

$$\int_0^t \int_{\omega_{x_0}} (g(x,s,u_{\varepsilon_t}(x,s)) - g(x,s,\phi(x,s)))(u_{\varepsilon_t}(x,s) - \phi(x,s))dxds \geq 0. \tag{2.31}$$

for every $\phi \in L^{m+1}((0,t) \times \omega_{x_0})$. In order to close the argument we use (2.30) to obtain that:

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_0^t \int_{\omega_{x_0}} &(g(x,s,u_{\varepsilon_t}(x,s)) - g(x,s,\phi(x,s)))(u_{\varepsilon_t} - \phi(x,s))dxds \\ &\leq \int_0^t \int_{\omega_{x_0}} (\xi(x,s) - g(x,s,\phi(x,s)))(u_t(x,s) - \phi(x,s))dxds, \end{aligned} \tag{2.32}$$

By combining (2.31) and (2.30) we obtain:

$$\int_0^t \int_{\omega_{x_0}} (\xi(x,s) - g(x,s,\phi(x,s)))(u_t(x,s) - \phi(x,s))dxds \geq 0,$$

for all $t < \rho/2$, so by passing to limit as $t \to \rho/2$, it holds also for $t = \rho/2$. We choose ϕ appropriately ($\phi_\pm := u_t \pm \lambda v$ for $\lambda > 0$) and take v arbitrary in $C_c^\infty(\omega_{x_0})$. Let $\lambda \to 0$ for both choices, ϕ_+, respectively ϕ_-, to obtain the desired equality $\xi = g(u_t)$.

If $m = 0$ and g does not depend on u_t or $m = 1$ and g is linear in u_t then one does not need the monotonicity argument in order to obtain $g(u_t) = \xi$, only $(2.17)_2$ and $(2.22)_1$. Since there is no other restriction imposed on p, these values for p and m cover the case (c) in (A8).

At this stage we have global existence of the small solutions u^{x_0} on the domains ω_{x_0}. Through the patching argument described in Step 4 we will construct the arbitrarily large solution u on the entire domain Ω.

Step 4. In this last step will put together the previous arguments to construct the solution u from an arbitrarily large pair of initial data (u_0, u_1) considered on the domain Ω. The order in which we assemble all the pieces is very important, so for a better understanding of this intricate procedure we identify the following substeps:

Step 4.1. Cut the initial data in small pieces on bounded domains ω_{x_0} and for each piece obtain global existence of solutions for the approximate problems with Lipschitz source terms f_ε.

Step 4.2. For each bounded domain, obtain bounds for $|\nabla u_\varepsilon^{x_0}(t)|_2$ and pass to the limit in the approximate solutions; hence, we obtain existence for the problem with a *general* source term.

Step 4.3. Up to some time $T < 1$, "patch all solutions" obtained in Step 4.2 to obtain a solution for the problem with a *general* source term with initial data on Ω.

Step 4.4. Show that the solution defined in Step 4.3 is well defined function and it is the solution generated by the initial data (u_0, u_1).

Below is a detailed discussion of the above construction.

Step 4.1. Let $d > 0$. Consider a lattice of points x_k, $k \in \mathbb{N}$ in Ω situated at distance d away from each other, such that in every ball of radius d we find at least one x_k. With ρ that satisfies (2.7) (where ρ depends only on the norms of the initial data), construct the balls of radius ρ centered at x_k. The procedure outlined in Step 2 for truncating the initial data around x_0 to obtain a "small piece" denoted by $(u_0^{x_0}, u_1^{x_0})$, will be used now to construct around each x_k the truncations $(u_0^{x_k}, u_1^{x_k})$ which will satisfy the smallness assumptions

$$|\nabla u_0^{x_k}|_{\omega_{x_k}} < \alpha, \quad \frac{1}{2}|u_1^{x_k}|^2_{\omega_{x_k}} + \frac{1}{2}|\nabla u_0^{x_k}|^2_{\omega_{x_k}} + \int_{\omega_{x_k}} F(x, 0, u_0^{x_k}(x))\, dx + K|\omega_{x_k}| < \Phi(\alpha).$$

On each of the balls ω_{x_k} we apply Theorem 2.3 to obtain global existence of solutions $u_\varepsilon^{x_k}$ for the problem (SW) with initial data $(u_0^{x_k}, u_1^{x_k})$ and with the Lipschitz approximations f_ε for the source term.

Step 4.2. At this point, the arguments of Step 3 for passing to the limit as $\varepsilon \to 0$ in the sequence of approximate problems are applied, where x_0 is successively replaced by different x_k's. First, we apply Sattinger's argument to estimate $|\nabla u_\varepsilon^{x_k}|_2$ on ω_{x_k} for each k. (Note that we need to make use of the smallness assumptions written in Step 4.1.) The convergence $u_\varepsilon^{x_k} \to u^{x_k}$ takes place on every domain $\omega_{x_k} \times (0, \rho/2)$, so we obtain a global (in time) solution to the boundary value problem (SW) for ω_{x_k}, for every k.

Step 4.3. The solutions u^{x_k} found in Step 4.2 will now be "patched" together to obtain our general solution. First, we need to introduce the following notation. For $k \in \mathbb{N}$ let :

$$C_k := \{(y, s) \in \mathbb{R}^3 \times [0, \infty); |y - x_k| \leq \rho/2 - s\}$$

be the backward cones which have their vertices at $(x_k, \rho/2)$. For d small enough (i.e. for $0 < d < \rho/2$) any two neighboring cones C_k and C_j will intersect. For every intersection set $I_{k,j}$ defined by

$$I_{k,j} := C_k \cap C_j$$

the maximum value for the time contained in it is equal to $(\rho - d)/2$ (see Figure 3 below).

For $t < \rho/2$ we define the piecewise function:

$$u(x,t) := u^{x_k}(x,t), \text{ if } (x,t) \in C_k. \tag{2.33}$$

This solution is defined only up to time $(\rho - d)/2$, since this is the height of the intersection set of two cones with their vertices situated at distance d away from each other. By letting $d \to 0$ we can obtain a solution well defined up to time $\rho/2$. Thus, we have u defined up to time $\rho/2$, which is the height of all cones C_k. Every pair $(x,t) \in \mathbb{R}^n \times (0, \rho/2)$ belongs to at least one C_k, so in order to show that this function from (2.33) is well defined, we need to check that it is single-valued on the intersection of two cones. Also, we need to show that the above function is the solution generated by the pair of initial data (u_0, u_1). Both proofs will be done in the next step.

Step 4.4. In order to prove the properties that we set out to do in this step, we will go back and look at the solutions u^{x_k} as limits of the approximation solutions $u_\varepsilon^{x_k}$.

Consider first balls which do not intersect the boundary.

For each $k \in \mathbb{N}$ we have $(u_0^{x_k}, u_1^{x_k}) = (u_0, u_1)$ for every $x \in \omega_{x_k} = \{y \in \mathbb{R}^n, |y - x_k| < \rho/2\}$ (see the construction of the truncations $(u_0^{x_k}, u_1^{x_k})$ in Step 2). Therefore, $u_\varepsilon^{x_k}$ (defined in Step 4.1) is an approximation of the solution generated by the initial data (u_0, u_1) on C_k (from the uniqueness property given by Part 2 of Proposition 2.4). We let $\varepsilon \to 0$ (use the argument from Step 3) to show that the solution u on each C_k is generated by the initial data (u_0, u_1).

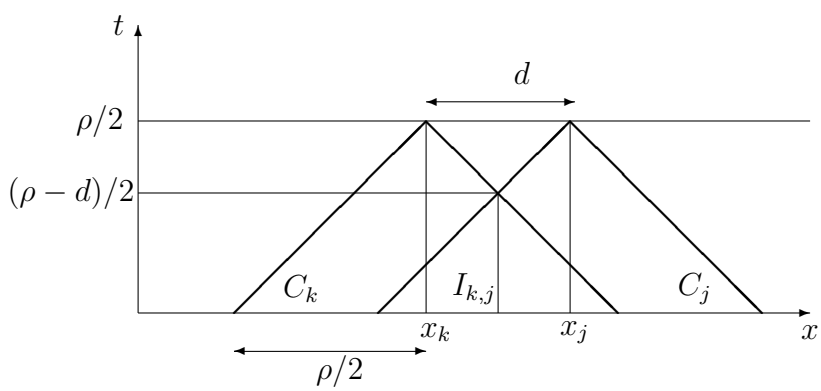

Figure 2: The intersection of the cones C_k and C_j

To show that u defined by (2.33) is a proper function, we use the same result of uniqueness given by the finite speed of propagation. First note that

for $n \geq 3$ the intersection $I_{k,j}$ is not a cone, but it is contained by the cone $C_{k,j}$ with the vertex at $((x_k+x_j)/2, (\rho-d)/2)$ of height $(\rho-d)/2$. In this cone we use the uniqueness asserted by the finite speed of propagation as follows. First note that the cones $C_{k,j}$ contain the sets $I_{k,j}$, but $C_{k,j} \subset C_k \cup C_j$. In C_k we have the solution $u_\varepsilon^{x_k}$, while in C_j the solution is given by $u_\varepsilon^{x_j}$ (see construction in 4.1); hence, in $C_{k,j}$ we now have defined two solutions which we need to show that are equal. Since $u_\varepsilon^{x_k}$ and $u_\varepsilon^{x_j}$ start with the same initial data $((u_0^{x_k}, u_1^{x_k}) = (u_0, u_1) = (u_0^{x_j}, u_1^{x_j})$ on $\omega_{x_k} \cap B_j)$, they are equal on $C_{k,j}$ by part 2 of the finite speed of propagation property. We let $\varepsilon \to 0$ to obtain $u_k^{x_0} = u_j^{x_0}$ in $C_{k,j}$, and since $I_{k,j} \subset C_{k,j}$ we proved $u_k^{x_0} = u_j^{x_0}$ on $I_{k,j}$. Therefore, u is a single-valued (proper) function.

Two issues arise when we treat the case of the balls that intersect of the boundary. The first deals with the geometry of the intersection sets, so that we can apply the uniqueness from part II of the finite speed of propagation property. The second issue that needs to be addressed is to show that the Dirichlet boundary conditions are still satisfied.

Assume that $B(x_j, \rho)$ and $B(x_k, \rho)$ are such balls that intersect the boundary $\partial\Omega$ and let $\omega_{x_j} := B(x_j, \rho) \cap \Omega$ and $\omega_{x_k} := B(x_k, \rho) \cap \Omega$. On $\omega_{x_k}$and ω_{x_j} we have again two solutions which by the same finite speed of propagation argument can be shown that they coincide on the intersection. In the picture below consider that the line $x = 0$ represents the boundary $\partial\Omega$. We see that intersecting the boundary $\partial\Omega$ with the cones C_k and C_j continues to allow us to show that on the intersection $I_{k,j}$ we have a unique solution.

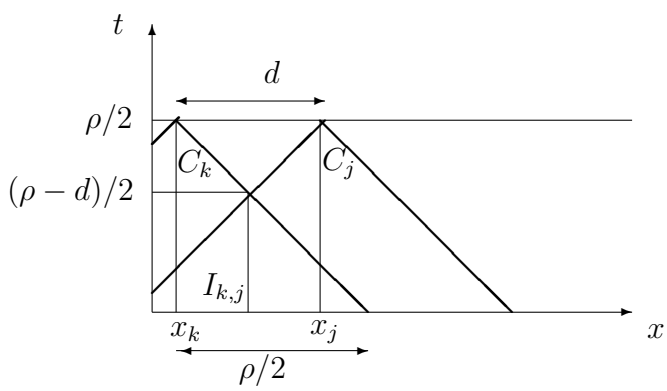

Figure 3: The intersection of the domains $C_k \cap \Omega$ and $C_j \cap \Omega$

The above method of using cutoff functions and "patching" solutions based on uniqueness will work the same way in the case when we additionally assume (A2)*. Since we can choose the height of the the cones as large as

we wish, the solutions exist globally in time under the positivity hypothesis for F.

The boundary conditions are satisfied since we have $u^{x_k} = 0$ on each $\partial\omega_{x_k} \cap \partial\Omega$. Since the solution u is obtained by patching all the solutions u^{x_k} we obtain $u = 0$ on $\partial\Omega$.

□

Acknowledgement and Errata to [13] The author would like to warmly thank the referee for pointing out an error in the original draft of the manuscript and for making other suggestions which helped improve the paper. Th error (which unfortunately also appeared in [13]), regards the range of exponents (p, m) which **does not** include the full set $(1, 2^* - 1) \times \{0, 1\}$, as previously thought. In the case $(p, m) \in (1, 2^* - 1) \times \{0, 1\}$ we can allow only g linear in the velocity argument (subcase of $m = 1$), or g independent of the velocity (subcase of $m = 0$). The correct assumptions and arguments for dealing with this case are outlined in this paper.

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