310 Introduction to Modern Algebra
Instructor: Petronela Radu
June 29, 2009

Sample Problems for Exam 2 ... and Beyond

1. Solve the following equations:
   (a) $[4]X = [120]$ in $\mathbb{Z}_{28}$;
   (b) $[7]X = [120]$ in $\mathbb{Z}_{20}$;
   (c) $[4]X = [70]$ in $\mathbb{Z}_{33}$;
   (d) $[39]X = -[26]$ in $\mathbb{Z}_{13}$.
2. Is the set $\{m-1, 2m, 3m+1, ..., m^2+m-2\}$ a complete set of representatives in $\mathbb{Z}_m$? If yes, prove it.
3. Assume that if a is a divisor of zero and x_0 is a solution of the equation
   $$ax_0 + b = c$$
   in a commutative ring R. Find other solutions to this equation. Can you write down a formula for the general solution?
4. Show that in a commutative ring if a is a unit with inverse b, then $-a$ is a unit with inverse $-b$.
5. Show that a ring is commutative if and only if $a^2 - b^2 = (a + (-b)) \cdot (a + b)$ for every a and b.
6. Find all units and zero divisors in $\mathbb{Z}/14\mathbb{Z}$. For each unit find its order and its inverse.
7. What is the order of $[5]^{14}$ in $\mathbb{Z}_{13}$? In $\mathbb{Z}_{10}$?
8. In a commutative ring R denote by $U(a) = \{a \cdot u | u \in R, u \text{ is a unit}\}$, where $a \in R$. Show that if $a \in R$ is a unit then $U(a) = U(1)$. Show that $U(a) \cap U(1) = \varnothing$ if a is a zero divisor.
9. Find the order of 5 mod 12, and use this to find the least nonnegative integer in the congruence class $[65^{625}]_{12}$.
10. From Fermat's theorem deduce that, for any integer $n \geq 0, 13 | 11^{12n+6} + 1$.
11. (a) Find the units digit of 3^{100} by the use of Fermat's theorem.
    (b) For any integer a, verify that a^5 and a have the same units digit.
12. Prove that $2n^{11}/11 + n^6/2 + 7n/22$ is an integer for every natural number n.

    MORE PROBLEMS (9C and beyond):
13. Use Euler's Theorem to evaluate 2^{100000} (mod 77).
14. Show that for n an odd integer $\phi(2n) = \phi(n)$.

15. Solve the following system

$$\begin{cases} x \equiv 1 \pmod{9} \\ x \equiv 2 \pmod{11} \\ x \equiv 6 \pmod{13}. \end{cases}$$

16. Solve the equation $x^2 \equiv 1 \pmod{33}$.

17. Find an integer solution x which is simultaneously congruent to 17 mod 35 and to 4 mod 99. Explain how you found x.