MA1132 (Advanced Calculus) Solutions to tutorial sheet 9

1. Evaluate

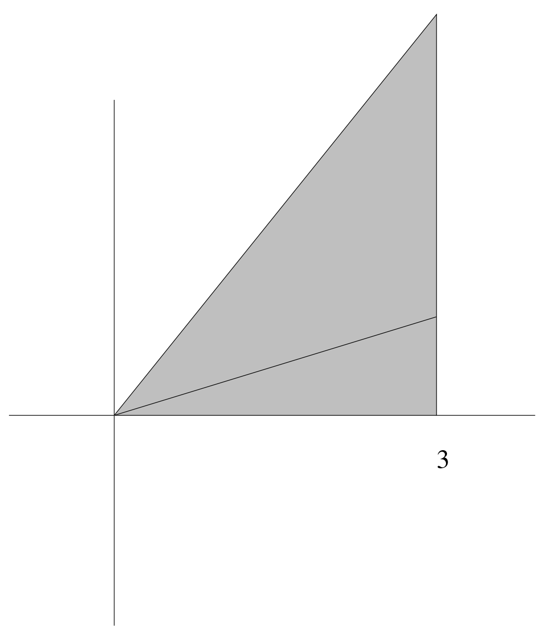
$$\int_0^3 \int_{y=0}^{y=\sqrt{3}x} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx$$

by making a change of variables to polar coordinates. [Hint: Sketch the region first. Then do the dr integral first, before $d\theta$; for fixed θ find the limits for r in terms of θ. Recall that $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$].

Solution: This iterated integral is the same (by Fubini's theorem) as the double integral

$$\iint_R \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$$

where R is the triangle in the plane bounded by the x-axis, the line $x = 3$ and the line $y = \sqrt{3}x$.



The limit $x = 3$ is $r\cos\theta = 3$ or $r = 3/\cos\theta$ and the angle is $\pi/3$ (because $\tan(\pi/3) = \sqrt{3}$).

Changing this integral to polar coordinates, and remembering that $dx\,dy = r\,dr\,d\theta$, we get

$$\begin{aligned}
\int_{\theta=0}^{\theta=\pi/3}\left(\int_{r=0}^{r=3/\cos\theta}\frac{1}{r}r\,dr\right)d\theta &= \int_{\theta=0}^{\theta=\pi/3}[r]_{r=0}^{r=3/\cos\theta}\,d\theta \\
&= \int_{\theta=0}^{\theta=\pi/3}\frac{3}{\cos\theta}\,d\theta \\
&= \int_{\theta=0}^{\theta=\pi/3}3\sec\theta\,d\theta \\
&= [3\ln|\sec\theta+\tan\theta|]_0^{\pi/3} \\
&= 3\ln(2+\sqrt{3})
\end{aligned}$$

2. Consider the function

$$f(x,y,z) = 1 + \frac{x^2e^xy + xe^yz}{x+y+z}$$

(a) Find ∇f (evaluated at an unspecified point (x,y,z)).
Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{(2xe^xy + x^2e^xy + e^yz)(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2} \\
\frac{\partial f}{\partial y} &= \frac{(x^2e^x + xe^yz)(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2} \\
\frac{\partial f}{\partial z} &= \frac{xe^y(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2}
\end{aligned}$$

$$\begin{aligned}
\nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
&= \left(\frac{(2xe^xy + x^2e^xy + e^yz)(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2},\right. \\
&\qquad \frac{(x^2e^x + xe^yz)(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2}, \\
&\qquad \left.\frac{xe^y(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2}\right)
\end{aligned}$$

(b) Find the equation of the tangent plane to the surface $f(x,y,z) = 1 + \frac{2e}{3}$ at the point $(1,1,1)$.
Solution: We are getting the tangent plane to a level surface $f(x,y,z) = 2e/3$. So the normal vector to the tangent plane is the gradient of f evaluated at the point $(1,1,1)$.

$$\nabla f\,|_{(1,1,1)} = \left(\frac{(4e)(3) - 2e}{3^2}, \frac{(2e)(3) - 2e}{3^2}, \frac{3e - 2e}{3^2}\right) = \left(\frac{10e}{9}, \frac{4e}{9}, \frac{e}{9}\right).$$

For the tangent plane we get

$$\frac{10e}{9}(x-1) + \frac{4e}{9}(y-1) + \frac{e}{9}(z-1) = 0$$

and we could simplify this by dividing by $e/9$ to get the equivalent equation

$$10(x-1) + 4(y-1) + (z-1) = 0.$$

(c) Find the direction $\mathbf{u} = (u_1, u_2, u_3)$ for which the directional derivative $D_{\mathbf{u}} f(1,1,-1)$ is as small as possible. Also, what is that smallest possible value of $D_{\mathbf{u}} f(1,1,-1)$?

Solution: What we need is $\mathbf{u}$ to be the unit vector in the direction opposite to $\nabla f \mid_{(1,1,-1)}$ and the smallest possible value is $-\|\nabla f \mid_{(1,1,-1)} \|$.

$$\nabla f \mid_{(1,1,-1)} = \left(\frac{(2e + e - e)(1) - 0}{1^2}, 0 - 0, \frac{e(1) - 0}{1^2} \right) = (2e, 0, e)$$

So

$$\mathbf{u} = -\frac{1}{\|\nabla f \mid_{(1,1,-1)} \|} \nabla f \mid_{(1,1,-1)} = \frac{-1}{e\sqrt{5}}(2e, 0, e) = \left(-\frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}} \right)$$

and the corresponing smallest value of the directional derivative is

$$D_{\mathbf{u}} f(1,1,-1) = -\|\nabla f \mid_{(1,1,-1)} \| = -\sqrt{5}e$$