MA1312 (Advanced Calculus) Solutions to tutorial sheet 7
[March 12 – 13, 2013]

1. Consider the level curve $g(x, y) = 0$ for the function $g(x, y) = x^2 + y^3 - 3xy - 13$ and the point $(x_0, y_0) = (2, 3)$.

   (a) Show that the point (x_0, y_0) belongs to the level curve.
   Solution: We have $g(2, 3) = 4 + 27 - 18 - 13 = 0$, hence the point belongs to the level curve.

   (b) Use the implicit function theorem to determine if y can be written as a function of x (i.e. there exists f s.t. $y = f(x)$) near (x_0, y_0). If yes, compute $\frac{df}{dx}(x_0) = f'(x_0)$.
   Solution: We first compute g_x, g_y and note that both are continuous functions

$$g_x(x, y) = 2x - 3y,$$

$$g_y(x, y) = 3y^2 - 3x.$$

   For y to be written as a function of x around the given point we must have

$$g_y(2, 3) \neq 0.$$

   Indeed, $3 \cdot 3^2 - 3 \cdot 2 = 21 \neq 0$.
   We compute

$$f'(3) = -\frac{g_x(2, 3)}{g_y(2, 3)} = 5/21.$$

   (c) Use the implicit function theorem to determine if x can be written as a function of y (i.e. there exists h s.t. $x = h(y)$) near (x_0, y_0). If yes, compute $\frac{dh}{dy}(y_0) = h'(y_0)$.

Solution: We are only left to verify that $g_x(2, 3) \neq 0$. Indeed, $-5 \neq 0$. Hence, x can be written as a function of y near $(2, 3)$. Furthermore,

$$f'(3) = -\frac{g_y(2, 3)}{g_x(2, 3)} = 21/5.$$

Please turn over

2. Consider the function

$$f(x,y) = 3x^3 + y^2 - 9x + 4y$$

(a) Find all critical points for the function f.

Solution: In order to find the critical points we need to determine the points where the partial derivatives either do not exist, or are zero. We have

$$f_x(x,y) = 9x^2 - 9, \quad f_y(x,y) = 2y + 4$$

therefore we have that the critical points must satisfy $x^2 = 1$ and $2y + 4 = 0$. Thus we obtain $x = \pm 1$ with $y = -2$, i.e. two critical points $(-1,-2)$ and $(1,-2)$.

(b) Use the second derivative test to determine which points are local minima, which points are local maxima, and which points are saddle points.

Solution: In order to use the second derivative test, we must compute all second order derivatives. We have

$$f_{xx}(x,y) = 18x, \quad f_{xy}(x,y) = 0, \quad f_{yy}(x,y) = 2.$$

Hence

$$D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy}(x,y))^2 = 36x.$$

We know that if

- $D(x,y) > 0$ and $f_{xx}(x,y) > 0$ we have a local minimum.
- $D(x,y) > 0$ and $f_{xx}(x,y) < 0$ we have a local maximum.
- $D(x,y) < 0$ we have a saddle point.
- $D(x,y) = 0$ the test is inconclusive.

At $(-1,-2)$ we have $D(-1,-2) = -36 < 0$ and $f_{xx}(-1,-2) = -18 < 0$ hence this is a saddle point.

At $(1,-2)$ we have $D(1,-2) = 36 > 0$ and $f_{xx}(1,-2) = 18 > 0$ hence this is a local minimum.

(c) Let $y = 0$ and compute $\lim_{x\to\infty} f(x,0)$, then compute $\lim_{x\to-\infty} f(x,0)$. Could you draw a conclusion as to whether any of the critical points found in part a) are **global** extrema for the function?

(A point of global extremum is a point where the function reaches either its highest or its lowest value among all points in the domain; thus, at a global minimum the function reaches its lowest possible value.)

Solution: We have

$$\lim_{x\to\infty} f(x,0) = \lim_{x\to\infty} 3x^3 - 9x = \infty$$

and

$$\lim_{x\to-\infty} f(x,0) = \lim_{x\to-\infty} 3x^3 - 9x = -\infty$$

hence, the local minimum point found above is only local, i.e. the function does not have global extrema since it is unbounded (towards both, $-\infty$ and $+\infty$).

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