1. Consider the level curve $g(x, y) = 0$ for the function $g(x, y) = x^2 + y^3 - 3xy - 13$ and the point $(x_0, y_0) = (2, 3)$.

(a) Show that the point $(x_0, y_0)$ belongs to the level curve.

*Solution:* We have $g(2, 3) = 4 + 27 - 18 - 13 = 0$, hence the point belongs to the level curve.

(b) Use the implicit function theorem to determine if $y$ can be written as a function of $x$ (i.e. there exists $f$ s.t. $y = f(x)$) near $(x_0, y_0)$. If yes, compute $\frac{df}{dx}(x_0) = f'(x_0)$.

*Solution:* We first compute $g_x, g_y$ and note that both are continuous functions

$$g_x(x, y) = 2x - 3y,$$
$$g_y(x, y) = 3y^2 - 3x.$$

For $y$ to be written as a function of $x$ around the given point we must have

$$g_y(2, 3) \neq 0.$$

Indeed, $3 \cdot 3^2 - 3 \cdot 2 = 21 \neq 0$.

We compute

$$f'(3) = -\frac{g_x(2, 3)}{g_y(2, 3)} = \frac{5}{21}.$$

(c) Use the implicit function theorem to determine if $x$ can be written as a function of $y$ (i.e. there exists $h$ s.t. $x = h(y)$) near $(x_0, y_0)$. If yes, compute $\frac{dh}{dy}(y_0) = h'(y_0)$.

*Solution:* We are only left to verify that $g_x(2, 3) \neq 0$. Indeed, $-5 \neq 0$. Hence, $x$ can be written as a function of $y$ near $(2, 3)$. Furthermore,

$$f'(3) = -\frac{g_y(2, 3)}{g_x(2, 3)} = \frac{21}{5}.$$
2. Consider the function 
\[ f(x, y) = 3x^3 + y^2 - 9x + 4y \]

(a) Find all critical points for the function \( f \).

Solution: In order to find the critical points we need to determine the points where the partial derivatives either do not exist, or are zero. We have
\[ \frac{\partial f}{\partial x}(x, y) = 9x^2 - 9, \quad \frac{\partial f}{\partial y}(x, y) = 2y + 4 \]
therefore we have that the critical points must satisfy \( x^2 = 1 \) and \( 2y + 4 = 0 \). Thus we obtain \( x = \pm 1 \) with \( y = -2 \), i.e. two critical points \((-1, -2)\) and \((1, -2)\).

(b) Use the second derivative test to determine which points are local minima, which points are local maxima, and which points are saddle points.

Solution: In order to use the second derivative test, we must compute all second order derivatives. We have 
\[ \frac{\partial^2 f}{\partial x^2}(x, y) = 18x, \quad \frac{\partial^2 f}{\partial x \partial y}(x, y) = 0, \quad \frac{\partial^2 f}{\partial y^2}(x, y) = 2. \]
Hence
\[ D(x, y) = \frac{\partial^2 f}{\partial x^2}(x, y) \cdot \frac{\partial^2 f}{\partial y^2}(x, y) - \left( \frac{\partial^2 f}{\partial x \partial y}(x, y) \right)^2 = 36x. \]
We know that if
- \( D(x, y) > 0 \) and \( \frac{\partial^2 f}{\partial x^2}(x, y) > 0 \) we have a local minimum.
- \( D(x, y) > 0 \) and \( \frac{\partial^2 f}{\partial x^2}(x, y) < 0 \) we have a local maximum.
- \( D(x, y) < 0 \) we have a saddle point.
- \( D(x, y) = 0 \) the test is inconclusive.

At \((-1, -2)\) we have \( D(-1, -2) = -36 < 0 \) and \( \frac{\partial^2 f}{\partial x^2}(-1, -2) = -18 < 0 \) hence this is a saddle point.
At \((1, -2)\) we have \( D(1, -2) = 36 > 0 \) and \( \frac{\partial^2 f}{\partial x^2}(1, -2) = 18 > 0 \) hence this is a local minimum.

(c) Let \( y = 0 \) and compute \( \lim_{x \to \infty} f(x, 0) \), then compute \( \lim_{x \to -\infty} f(x, 0) \). Could you draw a conclusion as to whether any of the critical points found in part a) are global extrema for the function?

(A point of global extremum is a point where the function reaches either its highest or its lowest value among all points in the domain; thus, at a global minimum the function reaches its lowest possible value.)

Solution: We have
\[ \lim_{x \to \infty} f(x, 0) = \lim_{x \to \infty} 3x^3 - 9x = \infty \]
and
\[ \lim_{x \to -\infty} f(x, 0) = \lim_{x \to -\infty} 3x^3 - 9x = -\infty \]

hence, the local minimum point found above is only local, i.e. the function does not have global extrema since it is unbounded (towards both, \(-\infty\) and \(+\infty\)).

Petronela Radu