MA1312 (Advanced Calculus) Solutions to tutorial sheet 5
[February 19 – 20, 2013]

Let $(x_0, y_0) = (1, -1)$ and

$$f(x, y) = \frac{e^{xy}}{x^4 + y^4}$$

for these problems.

1. Find the directional derivative $D_{\mathbf{u}} f(x_0, y_0)$ for the function $f(x, y)$, the point (x_0, y_0), where $\mathbf{u} = (u_1, u_2)$ is any unit vector.

   Solution:

   We could work out the directional derivative from the definition $D_{\mathbf{u}} f(x_0, y_0) = \frac{d}{dt} |_{t=0} f((x_0, y_0) + t\mathbf{u})$ but it is simpler to rely on the formula

   $$D_{\mathbf{u}} f = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

   or the version evaluated at $(x_0, y_0) = (1, -1)$

   $$D_{\mathbf{u}} f(x_0, y_0) = D_{\mathbf{u}} f(1, -1) = f_x(1, -1)u_1 + f_y(1, -1)u_2$$

   We need the partial derivatives

   $$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{y e^{xy}(x^4 + y^4) - 4x^3 e^{xy}}{(x^4 + y^4)^2} \\ \frac{\partial f}{\partial y} &= \frac{x e^{xy}(x^4 + y^4) - 4y^3 e^{xy}}{(x^4 + y^4)^2} \\ \frac{\partial f}{\partial x} |_{(1,-1)} &= \frac{-2e^{-1} - 4e^{-1}}{4} \\ &= \frac{-3}{2e} \\ \frac{\partial f}{\partial x} |_{(1,-1)} &= \frac{2e^{-1} + 4e^{-1}}{4} \\ &= \frac{3}{2e} \end{aligned}$$

   Thus

   $$D_{\mathbf{u}} f(1, -1) = -\frac{3}{2e} u_1 + \frac{3}{2e} u_2$$

Please turn over

2. For the same $f(x, y)$, the same (x_0, y_0) and the corresponding point (x_0, y_0, z_0) on the graph $z = f(x, y)$, find parametric equations for the line which is tangent to the graph at (x_0, y_0, z_0) and perpendicular to the x-axis. [So it is tangent to the x_0 cross-section curve of the graph.]

   Solution: The line will be tangent to the curve where the graph $z = f(x, y)$ intersects the plane $x = x_0$, and so tangent to the curve

$$\begin{cases} x = x_0 \\ y = y_0 + t \\ z = f(x_0, y_0 + t) \end{cases}$$

   at $t = 0$. The tangent vector to that curve is

$$\left(0, 1, \frac{\partial f}{\partial y} |_{(x_0, y_0)}\right) = (0, 1, \frac{3}{2e}).$$

   So the line is

$$\begin{cases} x = x_0 \\ y = y_0 + t \\ z = z_0 + t(\frac{3}{2e}) \end{cases}$$

   and $z_0 = f(x_0, y_0) = \frac{e^{-1}}{1+1} = 1/(2e)$. Thus the answer is

$$\begin{cases} x = 1 \\ y = -1 + t \\ z = \frac{1}{2e} + \frac{3t}{2e}. \end{cases}$$

3. For the same $f(x, y)$, the same (x_0, y_0) and the corresponding point (x_0, y_0, z_0) on the graph $z = f(x, y)$, find the normal vector to the tangent plane to the graph $z = f(x, y)$ at (x_0, y_0, z_0). Find the equation of the tangent plane at (x_0, y_0, z_0).

   Solution: The normal vector to the tangent plane is

$$\left(\frac{\partial f}{\partial x} |_{(x_0, y)}, \frac{\partial f}{\partial x} |_{(x_0, y)}, -1\right) = (f_x(x_0, y_0), f_y(x_0, y_0), -1)$$

   (or any nonzero multiple of that) and we have those values already. So the normal vector is

$$\left(\frac{-3}{2e}, \frac{3}{2e}, -1\right) = -\frac{3}{2e}\mathbf{i} + \frac{3}{2e}\mathbf{j} - \mathbf{k}$$

   Remark: You may find it easier to remember instead the equation of the tangent plane in the form that arises in the linear approximation formula

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

To read off the normal vector from that you need all the variables on the same side, like

$$-f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + z - f(x_0,y_0) = 0$$

giving normal (components are coefficients of x, y and z)

$$\mathbf{n} = -f_x(x_0,y_0)\mathbf{i} - f_y(x_0,y_0)\mathbf{j} + \mathbf{k}$$

Back to our solution, the equation of the tangent plane must then have the form

$$\alpha x + \beta y + \gamma z = c$$

where $(\alpha, \beta, \gamma) = \left(\frac{-3}{2e}, \frac{3}{2e}, -1\right)$.
So the equation has the form

$$-\frac{3}{2e}x + \frac{3}{2e}y - z = c$$

Also $(x_0, y_0, z_0) = (1, -1, 1/(2e))$ must satisfy the equation and so we have

$$-\frac{3}{2e} - \frac{3}{2e} - \frac{1}{2e} = c$$

or $-7/(2e) = c$. That means the equation is

$$-\frac{3}{2e}x + \frac{3}{2e}y - z = -\frac{7}{2e}$$

or we might prefer

$$3x - 3y + 2ez = 7$$

4. Find the linear approximation formula for $f(x,y)$ centered at $(x,y) = (x_0,y_0)$.

   Solution: The formula we want is

$$f(x,y) \cong f(x_0,y_0) + \frac{\partial f}{\partial x} \mid_{(x_0,y_0)} (x-x_0) + \frac{\partial f}{\partial y} \mid_{(x_0,y_0)} (y-y_0).$$

   We need $f(x_0,y_0) = f(1,-1) = 1/(2e)$ and the values of the partials evaluated at $(1,-1)$ (which we have already).

   So we end up with

$$f(x,y) \cong \frac{1}{2e} - \frac{3}{2e}(x-1) + \frac{3}{2e}(y+1)$$

   (which we expect to be a good approximation for (x,y) close to $(1,-1)$).