MA1132 (Advanced Calculus) Tutorial sheet 4

[February 12 – 13 , 2013]

1. Find parametric equations for the tangent line to the parametric curve

$$\mathbf{x}(t) = (t\cosh t, t\sinh t, t)$$

at the point where $t = 0$.

Solution: We need $\mathbf{x}'(0)$ (which is a vector parallel to the tangent line) and $\mathbf{x}(0) = (0, 0, 0)$ (which is a point on the line).

$$\mathbf{x}'(t) = \frac{d\mathbf{x}}{dt} = (\cosh t + t\sinh t, \sinh t + t\cosh t, 1)$$

and so

$$\mathbf{x}'(0) = (1 + 0, 0 + 0, 1) = (1, 0, 1).$$

The line has parametric equations

$$\begin{cases} x &= 0 + 1t \\ y &= 0 + 0t \\ z &= 0 + 1t \end{cases}$$

or

$$\begin{cases} x &= t \\ y &= 0 \\ z &= t \end{cases}$$

2. Find the partial derivatives with respect to x and y evaluated at $(x_0, y_0) = (2, -2)$ for

$$f(x, y) = \frac{x\cos(\pi y)}{x^2 + y^2}$$

Solution: We could do this

$$\begin{aligned}
f(x, y_0) &= f(x, -2) \\
&= \frac{x\cos(-2\pi)}{x^2 + (-2)^2} \\
&= \frac{x}{x^2 + 4} \\
\frac{\partial f}{\partial x}\,|_{(x,-2)} &= \frac{d}{dx} f(x, y_0) = \frac{d}{dx}\frac{x}{x^2 + 4} \\
&= \frac{(x^2 + 4) - x(2x)}{(x^2 + 4)^2} \\
&= \frac{4 - x^2}{(x^2 + 4)^2} \\
\frac{\partial f}{\partial x}\,|_{(2,-2)} &= 0
\end{aligned}$$

and this

$$\begin{aligned}
f(x_0, y) &= f(2, y) \\
&= \frac{2\cos(\pi y)}{4 + y^2} \\
\frac{\partial f}{\partial y}\,|_{(2,y)} &= \frac{d}{dy} f(x_0, y) = \frac{d}{dy}\frac{2\cos(\pi y)}{4 + y^2} \\
&= \frac{-2\pi\sin(\pi y)(4 + y^2) - 2\cos(\pi y)(2y)}{(x^2 + 4)^2} \\
\frac{\partial f}{\partial x}\,|_{(2,-2)} &= \frac{0 + 8}{8^2} = \frac{1}{8}
\end{aligned}$$

But this is how we would usually do it:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\cos(\pi y)(x^2 + y^2) - x\cos(\pi y)(2x)}{(x^2 + y^2)^2} \\
&= \frac{(x^2 + y^2 - 2x^2)\cos(\pi y)}{(x^2 + y^2)^2} \\
&= \frac{(y^2 - x^2)\cos(\pi y)}{(x^2 + y^2)^2} \\
\frac{\partial f}{\partial y} &= \frac{-\pi x\sin(\pi y)(x^2 + y^2) - x\cos(\pi y)(2y)}{(x^2 + y^2)^2} \\
&= \frac{-\pi x(x^2 + y^2)\sin(\pi y) - 2xy\cos(\pi y)}{(x^2 + y^2)^2} \\
\frac{\partial f}{\partial x}\,|_{(2,-2)} &= 0 \\
\frac{\partial f}{\partial y}\,|_{(2,-2)} &= \frac{0 + 8}{8^2} = \frac{1}{8}
\end{aligned}$$

3. Find the partials $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ evaluated at (x_0, y_0) (which means $\frac{\partial f}{\partial x}\,|_{(x_0,y_0)}= f_x(x_0, y_0)$ and $\frac{\partial f}{\partial y}\,|_{(x_0,y_0)}= f_y(x_0, y_0)$) for the following cases.

   (a) $f(x, y) = e^{(x+2y-3)}$, $(x_0, y_0) = (-1, 1)$

Solution:

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= e^{(x+2y-3)} \frac{\partial}{\partial x}(x+2y-3) \\
&= e^{(x+2y-3)} \\
\frac{\partial f}{\partial y} &= e^{(x+2y-3)} \frac{\partial}{\partial y}(x+2y-3) \\
&= 2e^{(x+2y-3)} \\
\frac{\partial f}{\partial x} |_{(-1,1)} &= e^{-2} \\
\frac{\partial f}{\partial y} |_{(-1,1)} &= 2e^{-2}
\end{aligned}
$$

(b) $f(x, y) = \sqrt{x^2 + y^2}$, $(x_0, y_0) = (-3, 4)$

Solution:

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{1}{2\sqrt{x^2+y^2}}(2x) \\
&= \frac{x}{\sqrt{x^2+y^2}} \\
\frac{\partial f}{\partial y} &= \frac{1}{2\sqrt{x^2+y^2}}(2y) \\
&= \frac{y}{\sqrt{x^2+y^2}} \\
\frac{\partial f}{\partial x} |_{(-3,4)} &= \frac{-3}{5} \\
\frac{\partial f}{\partial y} |_{(-3,4)} &= \frac{4}{5}
\end{aligned}
$$