MA1132 (Advanced Calculus) - Sample Exam 2
[Tuesday, April 2, 2013]

Note: Students who turn in **complete** solutions to the problems below by the due date will be able to drop a low score from one of their tutorials.

1. Let
$$f(x,y,z) = \frac{x^2e^xy + 4xe^yz}{x+2y-z}.$$

   (a) Find the direction of largest increase for f at $(1, 1/2, 0)$.
   Solution: The direction of the largest increase is given by the gradient
$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)(x,y,z)$$
$$\frac{\partial f}{\partial x} = \frac{(2xe^xy + x^2e^xy + 4e^yz)(x+2y-z) - (x^2e^xy + 4xe^yz)}{(x+2y-z)^2}$$
$$\frac{\partial f}{\partial y} = \frac{(x^2e^x + 4xe^yz)(x+2y-z) - 2(x^2e^xy + 4xe^yz)}{(x+2y-z)^2}$$
$$\frac{\partial f}{\partial z} = \frac{4xe^y(x+2y-z) - (-1)(x^2e^xy + 4xe^yz)}{(x+2y-z)^2}$$
   Hence,
$$\nabla f\,(1, 1/2, 0) = (\frac{5e}{8}, \frac{e}{4}, \frac{e}{8} + 2\sqrt{e}).$$

   (b) Find the equation of the tangent plane to the surface $f(x,y,z) = 2$ at $(1, 0, -1)$.
   Solution: Using the formula for the gradient derived in the first part, we find
$$\nabla f(1, 0, -1) = (-1, \frac{e}{2}, 1).$$
   Since the gradient vector is normal to the tangent plane, the equation of the tangent plane at the given point is given by
$$(-1)(x-1) + \frac{e}{2}y + (z+1) = 0,$$
   or $x - ye/2 - z - 2 = 0$

2. Use the implicit function theorem to determine if y can be written as a function of x near the point $(3, 0)$ when x and y satisfy $9e^{2y} = xy + x^2$. If yes, find $\frac{\partial y}{\partial x}$ at $x = 3$ and $y = 0$. Similarly, determine if x can be written as a function of y near the same point; if yes, find $\frac{\partial x}{\partial y}$ at (3,0).

   Solution: Let $f(x, y) = 9e^{2y} - xy - x^2$. Since this is a continuous function with continuous partial derivatives, we are left to check that $f_y(3, 0) \neq 0$. We have that

$$f_x(x, y) = -y - 2x, \quad f_y(x, y) = 18e^{2y} - x$$

   hence $f_y(3, 0) = 18 - 3 = 15 \neq 0$. The derivative $\frac{\partial y}{\partial x}$ can be computed use the formula

$$\frac{\partial y}{\partial x}|_{x=3} = -\frac{f_x}{f_y}|_{(3,0)} = \frac{6}{15} = \frac{2}{5}.$$

   Similarly, we check that $f_x(3, 0) = -6$, hence x can also be written as a function of y. We compute

$$\frac{\partial x}{\partial y}|_{y=0} = -\frac{f_y}{f_x}|_{(3,0)} = \frac{5}{2}.$$

3. Sketch the region of integration and evaluate the integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2}\ dydx$$

   Solution: The region of integration is described by $0 \leq x \leq 2$ and $0 \leq y \leq \sqrt{x^2 + y^2}$, hence it describes the upper half of the disk

$$y^2 + (x - 1)^2 \leq 1,$$

   which is centered at $x = 1$ and $y = 0$ with radius 1. Note that since the circle is contained in the strip $0 \leq x \leq 2$, the entire circle is contained in the region of integration.

   The domain and the form of the function prompts us to use polar coordinates. Note that for a given point on the circle $y^2 + (x - 1)^2 = 1$, if its polar angle is θ, then the polar radius r (a.k.a. the distance from the origin) is given by $2R \cos \theta$ with $R = 1$, i.e. by $r = 2 \cos \theta$. By changing the integral to polar coordinates we obtain

$$\int_0^{\pi/2} \int_0^{2\cos\theta} r^2 dr d\theta = \int_0^{\pi/2} \frac{r^3}{3}|_{r=0}^{r=2\cos\theta} d\theta = \frac{8}{3} \int_0^{\pi/2} (\cos\theta)^3 d\theta = \frac{8}{3} \int_0^{\pi/2} \cos\theta(1 - \sin^2\theta) d\theta.$$

   The last integral is the sum of the two integrals; for the integrand $\cos\theta \sin^2\theta$ we use the change of variable $u = \sin\theta$, so our initial integral is equal to

$$\frac{8}{3}(\sin\theta|_0^{\pi/2} - \int_0^1 u^2 du) = \frac{8}{3}(1 - \frac{1}{3}) = \frac{16}{9}.$$

4. Express the volume of the part of the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{25} \leq 1$$

where $y \geq 0$ and $z \geq 4$, as an iterated triple integral.

Solution: We start by noticing that the ellipsoid is between the planes $z = -5$ to $z = 5$ (the minimum and maximum values that z can have when satisfying the given inequality). The range of z allowed though will be $4 \leq z \leq 5$. With z fixed we find the range of values for y given by

$$0 \leq y \leq 4\sqrt{1 - \frac{z^2}{25}}.$$

Finally, the bounds for x in terms of y and z (corresponding to the integral in x inside) are

$$-\sqrt{4 - \frac{y^2}{4} - \frac{4z^2}{25}} \leq x \leq \sqrt{4 - \frac{y^2}{4} - \frac{4z^2}{25}}.$$

Therefore the volume can be expressed as

$$\int_4^5 \int_0^{4\sqrt{1-\frac{z^2}{25}}} \int_{-\sqrt{4-\frac{y^2}{4}-\frac{4z^2}{25}}}^{\sqrt{4-\frac{y^2}{4}-\frac{4z^2}{25}}} 1 \; dxdydz.$$

Petronela Radu