MA1132 (Advanced Calculus)
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Solutions to Sample Exam 1

1. (a) Write the Maclaurin series of $f(x) = \ln(1-x)$. Write the series with the $\sum$ notation, as well as an enumeration of (the first few) terms of the series.

   Solution:

   The derivatives of the function f (that is infinitely times differentiable away from $x = 1$) are

$$f'(x) = \frac{1}{x-1}, \quad f''(x) = -\frac{1}{(x-1)^2}, \quad f'''(x) = \frac{2!}{(x-1)^3}, \dots,$$

$$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{(x-1)^n}$$

   Therefore, the Maclaurin series is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{(-1)(n-1)!}{n!}x^n = \sum_{n=0}^{\infty}(-1)\frac{1}{n}x^n = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^n}{n} - \dots.$$

   It can be shown that the series converges for $x \in [-1, 1)$.

   (b) Write the fifth order Taylor polynomial for $f(x) = \sin(2-x)$ at $x = 1$.

   Solution: We start by computing the derivatives

$$f'(x) = -\cos(2-x), \;\; f''(x) = -\sin(2-x), \;\; f'''(x) = \cos(2-x),$$

$$f^{(4)}(x) = \sin(2-x), \; f^{(5)}(x) = -\cos(2-x).$$

   The fifth order Taylor polynomial at $x = 1$ is then

$$P_5(x) = \sin 1 - (\cos 1)x - \frac{\sin 1}{2}x^2 + \frac{\cos 1}{3!}x^3 + \frac{\sin 1}{4!}x^4 - \frac{\cos 1}{5!}x^5.$$

   (c) Find the radius and interval of convergence for the following power series:

      i. $\displaystyle\sum_{k=1}^{\infty} \frac{(-2)^{k-1}}{(2k-1)}x^{2k-1}$

      Solution: We use the ratio test so we compute

$$\lim_{k\to\infty}\left|\frac{(-2)^k x^{2k+1}}{(2k+1)} \cdot \frac{(2k-1)}{(-2)^{k-1}x^{2k-1}}\right| = \lim_{k\to\infty}\frac{2(2k+1)|x|^2}{2k-1} = 2|x|^2$$

In order to guarantee convergence of the series we must have $2|x|^2 < 1$, i.e. $|x| < \sqrt{2}/2$. We check the endpoints of the interval:

$$x = \frac{\sqrt{2}}{2}: \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)} \frac{\sqrt{2}}{2} \text{ converges as an alternating series.}$$

$$x = -\frac{\sqrt{2}}{2}: \quad \sum_{k=1}^{\infty} \frac{(-1)^{k}}{(2k-1)} \frac{\sqrt{2}}{2} \text{ again, converges as an alternating series.}$$

Therefore the radius of convergence is $R = \sqrt{2}/2$, while the interval of convergence is $[-\sqrt{2}/2, \sqrt{2}/2]$.

ii. $\displaystyle\sum_{k=1}^{\infty} \frac{2k+3}{k \cdot k!}(x+2)^k$

Solution: Again, the ratio test will be used, so compute

$$\lim_{k\to\infty} \left| \frac{(2k+5)(x+2)^{k+1}}{(k+1)\cdot(k+1)!} \cdot \frac{k \cdot k!}{(2k+3)(x+2)^k} \right| = \lim_{k\to\infty} \frac{k(2k+5)|x+2|}{(k+1)^2(2k+3)} = 0.$$

Since this limit is always less than 1, we have that the series converges for all x, hence the radius of convergence $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.

2. Find the partials $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ evaluated at $(x_0, y_0) = (1, 1)$ for

$$f(x,y) = e^{(x+2y-3)} \sin(\pi x)$$

Solution:

We compute

$$\frac{\partial f}{\partial x}(x,y) = e^{(x+2y-3)}(\sin(\pi x) + \pi \cos \pi x),$$
$$\frac{\partial f}{\partial y}(x,y) = 2e^{(x+2y-3)} \sin(\pi x).$$

Evaluated at the point (1,1) these functions give us

$$\frac{\partial f}{\partial x}(1,1) = e^0(0-\pi) = -\pi, \quad \frac{\partial f}{\partial y}(1,1) = 2e^0 \sin(\pi) = 0.$$

3. Find parametric equations for the tangent line to the parametric curve

$$\mathbf{x}(t) = ((1+t)\cos(\pi t), (1+t)\sin(\pi t), 2t-1)$$

at the point where $t = 1/2$.

Solution: The tangent vector to the curve is given by the vector

$$\mathbf{x}'(t) = (\cos(\pi t) - \pi(1+t)\sin(\pi t), \sin(\pi t) + \pi(1+t)\cos(\pi t), 2).$$

At $t = 1/2$ the tangent vector will be given by $\mathbf{v} = \mathbf{x}'(1/2) = (-3\pi/2, 1, 2)$. The parametric equations of a line passing through a point $\mathbf{x}_0$ of direction $\mathbf{v}$ are given by

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}, \quad t \in \mathbb{R}.$$

At $t = 1/2$ we obtain the point $\mathbf{x}_0 = (0, 3/2, 0)$ so the parametric equations are

$$\begin{cases} x_1 = -3t\pi/2 \\ x_2 = 3/2 + t \\ x_3 = 2. \end{cases}, \quad t \in \mathbb{R}.$$

4. For $f(x, y) = (x + y)^5 \cos(\pi x)$ and $(x_0, y_0) = (1/2, 1/2)$ find the equation of the tangent plane to the graph $z = f(x, y)$ at the point on the graph where $(x, y) = (x_0, y_0)$.

   Solution: The equation of the tangent plane to the graph gives the linear approximation of a surface:
$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

   We have

$$f_x(1/2, 1/2) = [5(x+y)^4 \cos(\pi x) - \pi(x+y)^5 \sin(\pi x)]|_{(x=1/2, y=1/2)} = -\pi$$

$$f_y(1/2, 1/2) = [5(x+y)^4 \cos(\pi x)]|_{(x=1/2, y=1/2)} = 0$$

   The equation of the tangent plane is then given by

$$z = f(1/2, 1/2) - \pi(x - 1/2)$$

   or $2z + 2\pi x = \pi$, since $f(1/2, 1/2) = 0$.