Assignment 5 - Due Friday, April 5

1. Consider the unforced van der Pol equation
   \[ u''(t) + (u^2 - 1)u' + u = 0. \]
   (a) Write the equation as a system of 2 ODEs;
   (b) Find the equilibrium points and study their stability by linearization.

2. Determine the stability of the equilibrium solutions of the system
   \[
   \begin{bmatrix}
   x'_1 \\
   x'_2
   \end{bmatrix}
   =
   \begin{bmatrix}
   -4x_1 - 2x_2 + 4 \\
   x_1 x_2
   \end{bmatrix}.
   \]

3. Find the equilibria of the nonlinear system below and study their stability
   \[
   \begin{bmatrix}
   u' \\
   v'
   \end{bmatrix}
   =
   \begin{bmatrix}
   u(v - 1) \\
   4 - u^2 - v^2
   \end{bmatrix}.
   \]

4. Consider the mass-spring nonlinear model with the equation
   \[ x''(t) + kx'(t) + g(x) = 0, \]
   where \( g \) satisfies \( xg(x) > 0 \) for \( x \neq 0 \) and \( k > 0 \) is the friction constant.
   (a) Rewrite the equation as a nonlinear system of differential equations.
   (b) Show that the function
   \[
   V(x, y) := \frac{1}{2} + \int_0^x g(s)ds
   \]
   is a Lyapunov function. Is it a strict Lyapunov function?
   (c) Find an example that shows that the condition \( xg(x) > 0 \) for \( x \neq 0 \) is essential for stability.

5. Use the Lyapunov function \( V(x) = \frac{1}{2}(x^2 + 3y^2) \) to show that the origin of the system
   \[
   \begin{cases}
   x' = -3y \\
y' = x - \alpha(2y^3 - y)
   \end{cases}
   \]
   is asymptotically stable for \( \alpha < 0 \).

6. Consider the non-dimensionlized harmonic oscillator equation:
   \[ x''(t) + bx' + \sin x = 0. \]
   Study the stability of the origin for all \( b \geq 0 \) by using the Lyapunov function given by the energy of the system, i.e. \( V(x, y) = 1 - \cos x + \frac{y^2}{2} \), where \( y = x' \).