MA 2326 Differential Equations
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Friday, March 15, 2013

Solutions to Assignment 3

1. (3 points) Determine if the systems below could be interpreted as models for predatory-prey, competing, or cooperating species. Motivate your answer.

   (a) $x' = x(1-x+y), \quad y' = y(4-3y-x)$.
   (b) $x' = x(1-x-y), \quad y' = y(4-3y-x)$.
   (c) $x' = x(1-x+y), \quad y' = y(4-3y+x)$.

   Solution:

   (a) The system can be rewritten as $x' = x(1-x)+xy, \quad y' = y(4-3y)-xy$ which shows that both populations undergo logistic growth and the interaction with y affects x positively, while y is affected negatively from the presence of x. Therefore, this is a predator-prey system with x predator and y prey.
   (b) For this system, both populations lose from the interaction with each other, hence it is a competitive system.
   (c) Both populations benefit from the presence of each other, hence it is a cooperative system.

2. (7 points) Solve the following initial value problem:

$$\mathbf{X}'(t) = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix} \mathbf{X}(t), \quad \mathbf{X}(0) = \begin{bmatrix} -5 \\ 2 \end{bmatrix}.$$

   Solution: We begin by finding the characteristic roots from the equality

$$(5-\lambda)(1-\lambda)+5=0$$

   which can be rewritten as $\lambda^2 - 6\lambda + 10 = 0$ with roots $\lambda_{1,2} = 3 \pm i$. For the eigenvalue $\lambda_2 = 3 - i$ we find that the corresponding eigenvector $\mathbf{V}_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ must satisfy $(2+i)v_1 - 5v_2 = 0$ (the second equation is just a multiple of the first; think about why that is.) With the choice $v_1 = 5$ we obtain the eigenvector $\begin{bmatrix} 5 \\ 2+i \end{bmatrix}$. The solution can be written as a linear combination of the real and respectively the imaginary parts of $e^{\lambda_2 t}\mathbf{V}_2 = e^{3t}\begin{bmatrix} 5 \\ 2+i \end{bmatrix} = e^{3t}(\cos t - i\sin t)\begin{bmatrix} 5 \\ 2+i \end{bmatrix} = e^{3t}\left(\begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} + i\begin{bmatrix} 5\sin t \\ -2\sin t + \cos t \end{bmatrix}\right)$. Therefore,

$$\mathbf{X}(t) = e^{3t}\left(c_1\begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2\begin{bmatrix} 5\sin t \\ -2\sin t + \cos t \end{bmatrix}\right)$$

   By plugging in the IC we obtain that $c_1 = -1$ and $c_2 = -4$.

3. (10 points) Compute the matrix exponential e^{tA} in the case when $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$

   Solution: The eigenvalues of the matrix are $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 3$. For λ_1 the eigenvector $\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ would satisfy $2u_1 = 0, \quad u_1 + u_2 + 2u_3 = 0$, hence we can choose $\mathbf{U} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

The eigenvectors $\mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ corresponding to $\lambda_{2,3}$ would satisfy

$$v_1 - 2v_2 = 0, \quad v_1 + v_2 = 0$$

hence, the eigenspace has dimension 1 and it is spanned by the vector $\mathbf{V} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The generalized eigenvector $\mathbf{W} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ would have to satisfy $(A - 3I)\mathbf{W} = \mathbf{V}$, hence be a solution of the system

$$w_1 - 2w_2 = 0, \quad w_1 + w_2 = 1,$$

so we could take it $\mathbf{W} = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix}$.

We know that A can be written as $A = BJB^{-1}$, where $B = \begin{bmatrix} 0 & 0 & 2/3 \\ 2 & 0 & 1/3 \\ -1 & 1 & 0 \end{bmatrix}$, the Jordan form is $J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, and we compute $B^{-1} = \begin{bmatrix} -1/4 & 1/2 & 0 \\ -1/4 & 1/2 & 1 \\ 3/2 & 0 & 0 \end{bmatrix}$.

We then compute the matrix exponential

$$e^{tA} = Be^{tJ}B^{-1} = Be^{tD+tE}B^{-1}$$

where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $E = \begin{bmatrix} o & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Note that $E^2 = 0$.

We obtain

$$e^{tA} = \begin{bmatrix} e^{3t} & 0 & 0 \\ (e^{3t} - e^t)/2 & e^t & 0 \\ (e^t - 3e^{3t})/4 + 3te^{3t}/2 & (e^{3t} - e^t)/2 & e^{3t} \end{bmatrix}$$

4. (45 points) For the differential system $\mathbf{x}$'(t)=A$\mathbf{x}$(t):

   (a) (6 points) Draw the nullclines and perform a nullcline analysis;

   (b) (6 points) Find the general solution;

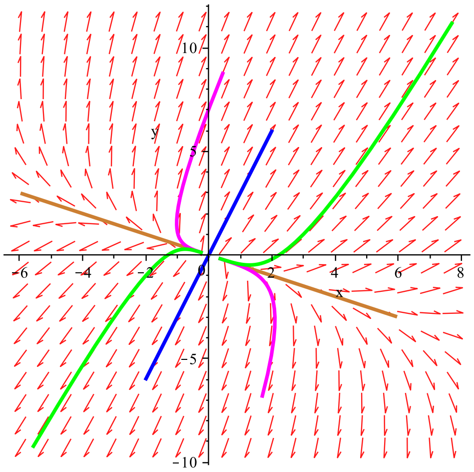
   (c) (1 points) Discuss the stability of the origin based on parts (a) and (b);

   (d) (2 points) Draw some trajectories to illustrate what type of a critical point the origin is.

   where

   (a) (15 points) $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$

   Solution: The $x-$nullcline is given by the line $3x + 2y = 0$ or $y = -3/2x$. Above this line, x is increasing, below it is decreasing. The $y-$nullcline is $3x + 8y = 0$ or $y = -3/8x$. Above the line y is increasing, below it is decreasing.
   The characteristic equation is $(3 - \lambda)(8 - \lambda) - 6 = \lambda^2 - 11\lambda + 18 = 0$ with roots $\lambda_1 = 2$ and $\lambda_2 = 9$. An eigenvector corresponding to the first eigenvalue is $\mathbf{V}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. The eigenspace corresponding to the second eigenvalue is generated by $\mathbf{V}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Therefore, the solution is given by



$\mathbf{X}(t) = c_1 e^{2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{9t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. These are all unbounded solutions, hence the origin is unstable.

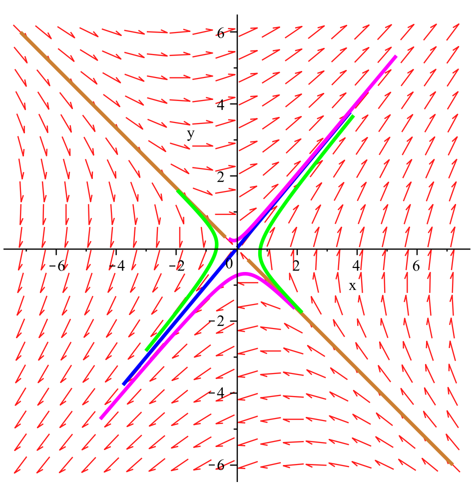
A plot of the phase portrait with the linear trajectories in blue, respective brown and sample trajectories in green and purple is below.

(b) (10 points) $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

Solution. The $x-$ nullcline is given by the line $x + 6y = 0$,; above it $x' > 0$, below $x' < 0$. The $y-$nullcline is $5x + 2y = 0$; above the line $y' > 0$, below the line $y' < 0$.

The characteristic equation is $\lambda^2 - 3\lambda - 28 = 0$ with roots $\lambda_1 = -4, \lambda_2 = 7$. The first eigenspace is generated by $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$, the second one is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution is given by $\mathbf{X}(t) = c_1 e^{-4t} \begin{bmatrix} 6 \\ -5 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. By looking at the eigenvalues we see the the origin is a saddle point (some trajectories approach it, while others move away).

A plot of the phase portrait with the linear trajectories in blue, respective brown and sample trajectories in green and purple is below.



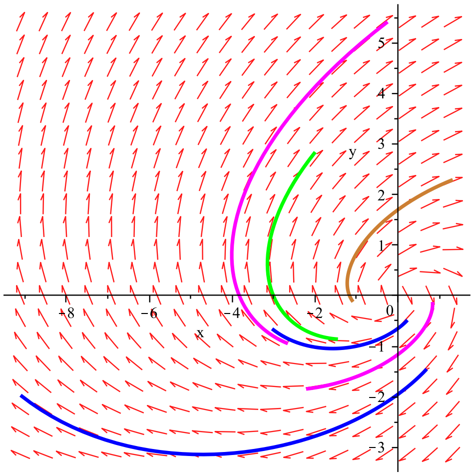
(c) (10 points) $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$

Solution: The $x-$nullcline is given by $y = -x/5$ and the $y-$nullcline is $y = 2/3x$. The sign analysis proceeds as before.

The characteristic equation $\lambda^2 - 4\lambda + 13 = 0$ has the complex roots $\lambda = 2 \pm 3i$, hence there are no linear trajectories (in fact, the origin will be a spiral source). The general solution is obtained as a linear combination of the real and imaginary parts of $e^{\lambda t}\mathbf{V}$ where λ is an eigenvalue and $\mathbf{V}$ is the corresponding eigenvector. We obtain

$$\mathbf{X}(t) = c_1 e^{2t} \begin{bmatrix} 5\cos 3t \\ \cos 3t - 3\sin 3t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 5\sin 3t \\ \sin 3t + 3\cos 3t \end{bmatrix}.$$

A plot of the phase portrait with sample trajectories in blue, brown, green and purple is below. Note that there are **no linear trajectories**.



5. (21 points) Solve the following initial value problems:

   (a) (7 points) $x' = 5x - y, \quad y' = 3x + y$ with $x(0) = 2, \quad y(0) = -1$.
   Solution: The characteristic equation $\lambda^2 - 6\lambda + 8 = 0$ has the root $\lambda_1 = 2$ with eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and the second root $\lambda_2 = 4$ with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution is given by $\mathbf{X}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. From the initial conditions we get $c_1 = -3/2$, $c_2 = 7/2$.

   (b) (7 points) $x' = x - 5y, \quad y' = x - 3y$ with $x(0) = 1, \quad y(0) = 1$.
   Solution: The characteristic equation $\lambda^2 + 2\lambda + 2 = 0$ has complex valued roots $\lambda_{1,2} = -1 \pm i$. The eigenvector corresponding to $\lambda_1 = -1 + i$ is chosen to be $\mathbf{V} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$.
   We obtain the solution as a linear combination of the real, respective imaginary parts of $e^{\lambda t}\mathbf{V}$. $\mathbf{X}(t) = c_1 e^{-t} \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$.
   The initial conditions give $c_1 = 1, c_2 = -1$.

   (c) (7 points) $x' = 3x + 9y, \quad y' = -x - 3y$ with $x(0) = 2, \quad y(0) = 4$.
   Solution: The characteristic equation $\lambda^2 = 0$ gives the root zero with multiplicity 2. Take a corresponding eigenvector to be $\mathbf{V} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and the generalized eigenvector $\mathbf{W} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. The solution will be

$$\mathbf{X}(t) = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3t+2 \\ -t-1 \end{bmatrix}.$$

   We find $c_1 = 10, c_2 = 14$.

6. (4 points) Two interconnected tanks contain a solution of salt and water. Water containing 1lb salt per gallon flows into Tank 1 at a rate of 5 gal/min and into Tank 2 at a rate of 7 gal/min. Some of the

mixture in Tank 1 flows out carrying every minute 2/7 of the amount of salt present in Tank 1 at time t. Half of this amount flows into Tank 2 and the other half leaves the system. Some of the mixture in Tank 2 flows out carrying 1/4 of the amount of salt present in Tank 2 at time t. Denote by $x(t)$ and $y(t)$ the amounts of salt at time t in Tank 1, respectively, Tank 2. Write a differential system of equations that model the flow process.

Solution: Per the problem's suggestion, let us denote by $x(t)$ the amount of salt (in pounds) in the first tank at time t (in minutes) and by $y(t)$ the amount of salt in the second tank, in the same units.

To formulate both differential equations we start with the conservation of mass, which is valid for the rates of change in mass

$$\text{Rate\_change(t) =Rate\_in(t) - Rate\_out(t).}$$

Rates of change are the derivatives for x, respectively for y. The rates in/out are computed as a product between the rates of flow in/out and the concentration of the fluid in/out.

$$\begin{cases} x'(\text{lb/min}) = 5\text{gal/min} \cdot 1\text{lb/gal} - (2/7x)\text{lb/min} \\ y'(\text{lb/min}) = 7\text{gal/min} \cdot 1\text{lb/gal} + (1/7x)\text{lb/min} - (1/4y)\text{lb/min} \end{cases}$$

After canceling units we obtain the system

$$\begin{cases} x' = 5 - 2/7x \\ y' = 7 + 1/7x - 1/4y. \end{cases}$$