MA 2326 Differential Equations
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# Solutions to Assignment 1

1. Find the general solutions of the following ODEs:

   (a) $\sqrt{2t}\ x' = \tan x$
   **Solution:** It is a separable equation as we can write it as
   $$\frac{dx}{\tan x} = \frac{dt}{\sqrt{2t}}.$$
   Integrate it to obtain
   $$\ln(|\sin x|) = \sqrt{2}t^{1/2} + C$$
   which gives after exponentiation and relabeling the constant
   $$\sin x = Ce^{\sqrt{2}t^{1/2}}.$$
   We could solve it explicitly with the inverse sine function so
   $$x(t) = \sin^{-1}(Ce^{\sqrt{2}t^{1/2}}).$$

   (b) $y'(t) = \dfrac{y}{2t} + \dfrac{t+1}{2t}$
   **Solution:** This is the linear equation $y'(t) - \dfrac{y}{2t} = \dfrac{t+1}{2t}$ which has the integrating factor
   $$\mu(t) = e^{-\int 1/(2t)dt} = \frac{1}{\sqrt{t}}.$$
   We multiply the equation by the integrating factor and obtain
   $$\left(y(t)\frac{1}{\sqrt{t}}\right)' = \frac{t+1}{2t\sqrt{t}}.$$
   After integration we obtain that
   $$y(t) = \sqrt{(}t)\left(\sqrt{t} - \frac{1}{\sqrt{t}} + C\right) = t - 1 + Ct^{1/2}.$$

   (c) $x't = x + t$
   **Solution:** This is the same as the linear equation $x' - \frac{x}{t} = 1$ with the integrating factor
   $$\mu(t) = e^{-\int 1/tdt} = \frac{1}{t}.$$
   After multiplying the equation by the integrating factor we can write it as
   $$\left(\frac{x}{t}\right)' = \frac{1}{t}$$
   so $x(t) = t\ln t + Ct$ for $t > 0$.

   (d) $(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$
   **Solution:** This is an exact equation as
   $$\frac{\partial}{\partial y}(6xy - y^3) = 6x - 3y^2 = \frac{\partial}{\partial x}(4y + 3x^2 - 3xy^2).$$

We will then to try to find $F(x,y)$ such that $\frac{\partial F}{\partial x} = 6xy - y^3$ and $\frac{\partial F}{\partial y} = 4y + 3x^2 - 3xy^2$. Integrating both equations we obtain that

$$F(x,y) = 3x^2y - y^3x + C_1(y) = 2y^2 + 3x^2y - xy^3 + C_2(x).$$

This implies that the solutions are given (implicitly) by

$$2y^2 + 3x^2y - xy^3 = C,$$

where $C$ is determined by the initial conditions.

2. An accident at a nuclear power plant has left the surrounding area polluted with radioactive material that decays naturally. The initial amount of radioactive material present is 15 su (safe units), and 5 months later is still 10 su.

   (a) Write a formula giving the amount $A(t)$ of radioactive material (in su) remaining after $t$ months.

   (b) What amount of radioactive material will remain after 8 months?

   (c) How long - total number of months or fraction thereof - will it be until $A = 1$ su, so it is safe for people to return to the area?

   **Solution:** This is a model of natural decay, so $A(t)$ satisfies

   $$A'(t) = kA(t), \quad A(0) = 15$$

   which has the solution $A(t) = 15e^{kt}$. To obtain the constant $k$ we use the fact that $A(5) = 10 = 15e^{5k}$. Hence

   $$k = \frac{1}{5}\ln\frac{2}{3}.$$

   We compute then that

   $$A(t) = 15\left(\frac{2}{3}\right)^{t/5}.$$

   After 8 months $A(t) = 15\left(\frac{2}{3}\right)^{8/5} = 7.8405$.

   The time $t_0$ when $A(t_0) = 1$ is given by $15\left(\frac{2}{3}\right)^{t_0/5} = 1$ or, $t_0 = 5\frac{\ln(1/15)}{\ln(2/3)} = 33.394367$.

3. Solve $y' - \frac{1}{3}y = e^{-t}$, with $y(0) = a$, and discuss how the behavior of $y$ as $t \to \infty$ depends on the initial value $a$.

   **Solution:** This is a linear equation with integrating factor $\mu(x) = e^{-t/3}$ so the solution is given by

   $$y(t) = e^{t/3}\left(\frac{-3}{4}e^{-4t/3} + C\right) = Ce^{t/3} - \frac{3}{4}e^{-t}.$$

   Imposing the initial condition gives $C - \frac{3}{4} = a$, hence $C = a + \frac{3}{4}$. The solution is then

   $$y(t) = (a + \frac{3}{4})e^{t/3} - \frac{3}{4}e^{-t}.$$

   We distinguish the following cases:

   (a) if $a = -\frac{3}{4}$ then $\lim_{t\to\infty} y(t) = 0$

   (b) if $a > -\frac{3}{4}$ then $\lim_{t\to\infty} y(t) = \infty$

   (c) if $a < -\frac{3}{4}$ then $\lim_{t\to\infty} y(t) = -\infty$.

4. Solve

$$y'(x) = \frac{1+3x^2}{3y^2-6y}, \quad y(0) = 1$$

and identify the interval where the solution is valid.

**Solution:** This is a separable equation, so after separating we obtain

$$(3y^2 - 6y)dy = (1 + 3x^2)dx$$

which after integrating becomes

$$y^3 - 3y^2 = x + x^3 + C$$

The initial condition yields $C = -2$. The solution is then defined implicitly by the equation

$$y^3 - 3y^2 = x + x^3 - 2$$

The solution can not exist when the derivative does not exist, hence we impose that $3y^2 \neq 6y$ which yields $y \neq 0$ or $y \neq 2$. From the equality $y^3 - 3y^2 = x + x^3 - 2$ we obtain that $y = 0$ implies $x + x^3 - 2 = 0$ which means $(x-1)(x^2+x+2) = 0$ with the only solution $x = 1$. When $y = 2$ we obtain $x + x^3 + 2 = 0$ equivalent to $(x+1)(x^2 - x + 2) = 0$ with the only real solution $x = 1$. In order to avoid the points $x = \pm 1$, but to have an interval that includes $x = 0$ (where the IC is given) we must have that the interval of existence is $(-1, 1)$.

5. An investment is modeled by the ODE $y' = y(6-y)$, where $y$ is the amount (in thousands) of dollars at time $t$ (in months).

   (a) What kind of growth does the investment follow?

   (b) Assume that the investment starts losing \$5,000 per month. For the new equation, discuss the stability of the critical points with a phase line analysis. Find the long term outcome of an initial investment of \$500, \$3,000, \$5,000, and respectively \$6,000.

   **Solution:** The initial investment follows logistic growth with growth constant $k = 1$ and carrying capacity $M = 6$ (this is \$6,000 for the model). Once money starts being taken out, the new model is given by

$$y' = y(6-y) - 5 = -(y-1)(y-5)$$

   which has critical points $y = 1$ and $y = 5$. By studying the sign of $y' = -(y-1)(y-5)$ we obtain that $y$ is decreasing for $y \in (-\infty, 1) \cup (5, \infty)$ and increasing for $y \in (1, 5)$. Thus, $y = 1$ is a source (hence, unstable) and $y = 5$ is a sink (hence, stable). The long time behavior for an initial investment of

   - $y(0) = 0.5$ (we convert in units, which are thousands) goes to $-\infty$ (assuming that no investor would take an infinite debt, we will stop the investment at $y = 0$)
   - $y(0) = 3$ will grow to $y = 5$ (i.e. \$5,000)
   - $y(0) = 5$ will stay constant.
   - $y(0) = 6$ will decrease to $y = 5$.

6. A tank contains 50 Kg of salt dissolved in 100 l of water. The tank capacity is 400 l. From t = 0, 1/4 kg of salt/l is entering at a rate of 4 l/min, and the well-mixed mixture is drained at 2 l/min. Find:

   (a) the time t when the tank overflows;

   (b) amount of salt in the tank before overflow;

   (c) the concentration of salt in the tank at overflow.

   **Solution:** First, note that there are 2l of liquid gained per minute, and since the tank had $400 - 100 = 200l$ available in volume, it will take 150 minutes to reach the point of overflow.

   Set up the model and denote

$$Q(t) = \text{amount (kg) of salt in the tank at time } t \text{ (min)}$$

so $Q(0) = 50$. Next we compute

$$\text{In-rate:} = 4 \text{ l/min} \times \ 1/4 \text{ kg/l} = 1 \text{ kg /min}$$

$$\text{Out-rate:} = 2 \text{ l/min} \times \frac{Q(t)}{100+2t} \text{ kg /l } = \frac{Q(t)}{50+t} \text{ kg/min}$$

We formulate the IVP:

$$\begin{cases} \frac{dQ}{dt} = [\text{In-rate}] - \ [\text{Out-rate}] = 1 - \frac{1}{50+t}Q \\ Q(0) = 50. \end{cases}$$

Solve the linear equation $Q' + \frac{1}{50+t}Q = 1$ by computing the integrating factor $\mu(t) = e^{\ln(50+t)} = 50+t$. After integrating and using the IC we obtain that the solution is

$$Q(t) = \frac{1}{50+t}\left(50t + \frac{t^2}{2} + 2500\right)$$

Note that as $t \to \infty$ (assuming an infinitely large tank) we have

$$\lim_{t\to\infty} Q(t) = \infty.$$

7. Consider the DE

$$y' = \frac{(t-1)^3}{y^4+1}.$$

   (a) Compute the slopes of the minitangents in the slope field at the points (0, $a$) and ($b$,0) in the $ty$ coordinate system.

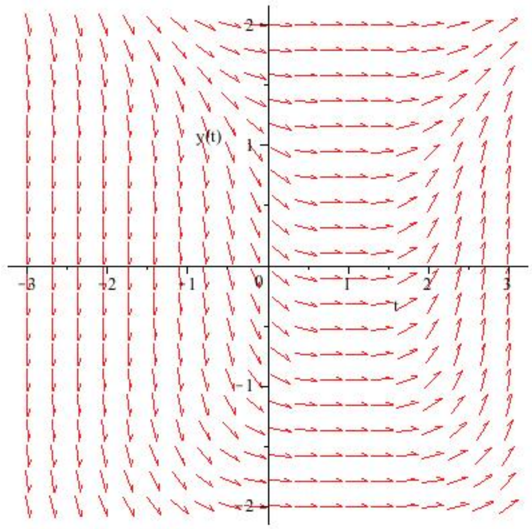
   **Solution:** The slope at $(0, a)$ is given by $\frac{-1}{a^4+1}$ which is negative and it increases as $a$ gets large. On the $Ot$ axis the slope at $(b, 0)$ is $(b-1)^3$ which increases as $b \to \infty$.

   (b) Draw the minitangents to the solution curves at the points (-1,1), (2,1), (1,1), and (0,0).

   **Solution:** The minitangent at (-1,1) has slope $-4$, at (2,1) has slope $\frac{1}{2}$, at (1,1) the slope is 0, and at (0,0) is $-1$.

   (c) Draw the curve of points $(x, t)$ for which $y' = 0$ and indicate on the diagram the regions where the solutions of the given DE are increasing, respectively decreasing.

   **Solution:** The derivative $y' = 0$ whenever $t = 1$ (see the horizontal slopes below). The solutions are increasing for all $t \geq 1$ and decreasing for $t < 1$.




8. Solve the following IVP and find the interval of validity for the solution

$$xy' = y(\ln x - \ln y), \quad y(1) = 4, \quad x > 0.$$

This differential equation becomes homogeneous after using a quick logarithm property

$$y' = \frac{y}{x} \ln\left(\frac{x}{y}\right)$$

Applying the substitution $v(x) = \dfrac{y}{x}$ with $y'(x) = xv'(x) + v(x)$ gives

$$xv' = v(-\ln v - 1)$$

which is separable, so

$$\int \frac{dv}{v(-\ln v - 1)} = \int \frac{dx}{x}.$$

For the integral on the LHS use the substitution $u = -\ln v - 1$ with $du = -\dfrac{1}{v}dv$. We obtain

$$\ln(\ln(-v) - 1) = C - \ln x.$$

Solve for v and note that well need to exponentiate both sides a couple of times and play fast and loose with the constants

$$\ln(-v) - 1 = e^{-\ln x + C} = \frac{C}{x},$$

from which we have

$$v(x) = e^{-\frac{C}{x} - 1}.$$

Therefore

$$y(x) = xe^{-\frac{C}{x} - 1}$$

where $C$ is determined from the initial condition: $4 = e^{-C-1}$, i.e. $C = -(1 + \ln 4)$.

The solution is then

$$y(x) = xe^{\frac{1 + \ln 4}{x} - 1}$$

which in order to exist needs to avoid $x = 0$.Therefore, the interval of existence is $(0, \infty)$.

9. Find the interval of existence for solutions to the following IVP:

$$(\tan t)\, v' = \sqrt{t-2}\, v - \ln(7-t), \quad v(4) = 2.$$

**Solution:** This equation is linear and it can be written as

$$v' = \frac{\sqrt{t-2}}{\tan t}\, v - \frac{\ln(7-t)}{\tan t}.$$

From the existence theorem for linear ODEs we know that the solution exists as long as the coefficients are continuous around the initial time. Therefore, we need to impose

$$t \geq 2, \quad t \neq k\frac{\pi}{2} \text{ for all } k \in \mathbb{Z}, \quad t < 7.$$

Note that we had to impose that $\tan t$ exists, but also that it is nonzero. The largest interval around $t = 4$ which avoids all these discontinuities is $(\pi, \frac{3\pi}{2})$.

10. Solve the following Bernoulli equation:

$$y' + xy = xy^n, \quad n \neq 0, 1$$

by transforming it into a linear equation through the following steps:

(a) multiply the equation by $y^{-n}$

(b) use the substitution $w(x) := y^{1-n}(x)$.

(*Hint:* The resulting linear equation should be $\frac{1}{1-n}w' + xw = x$.)

**Solution:** Using the suggested substitution with $w' = (1-n)y^{-n}y'$ we obtain

$$\frac{1}{1-n}w' + xw = x.$$

This is a linear equation with integrating factor $\phi(x) = e^{(1-n)x^2/2}$ so we can rewrite it as

$$\left(e^{(1-n)x^2/2}w\right)' = (1-n)xe^{(1-n)x^2/2}$$

which after integration yields

$$w(x) = Ce^{(1-n)x^2/2} + 2.$$

Therefore

$$y(x) = w(x)^{1/(1-n)} = \left(Ce^{(1-n)x^2/2} + 2\right)^{1/(1-n)}.$$