

### Sample Exam 1

1. Prove by induction the following statements:
  - (a)  $n! > n^3$ , for every  $n \geq 6$ .
  - (b)  $\frac{(2n)!}{2^n n!}$  is an integer for every  $n \geq 1$ .
2. Find  $(320, 112)$  and  $[320, 112]$ . Solve the equation  $320x + 112y = a$  in the following situations: (i)  $a = 32$  and (ii)  $a = 10$ .
3. Prove that  $(a, bc) = 1$  if and only if  $(a, b) = 1$  and  $(a, c) = 1$ . (Note: for this problem you may use ANY of the results that we proved in class or in the homework.)
4. Show that if  $p$  is prime such that  $p|a^n$  then  $p^n|a^n$ .
5. Is  $24^{1/4}$  rational or irrational? Prove it.
6. Find all solutions for the equation:

$$x \equiv 1 \pmod{15}$$

#### More Problems...

Before attempting these problems carefully read the material from your notes and the textbook. **Make sure you fully understand and know how to solve the problems from the homework assignments and quizzes.**

7. Let  $a$  and  $b$  natural numbers. Is the following relation an equivalence relation  
$$a \sim b \text{ iff } a + b \equiv 0 \pmod{2}?$$
8. Prove or disprove the following statements:
  - (a)  $(a, b) = [a, b]$  if and only if  $a = b$ .
  - (b)  $[a, b, 1] = [a, b]$ .
  - (c) For every  $a$  and  $b$  natural numbers there exist unique integers  $q$  and  $r$  such that  $a = b \cdot q + r$  where  $r$  satisfies  $a \leq r < 2a$ .
9. Show that if  $b$  and  $c$  are positive integers such that  $bc$  is a perfect square and  $(b, c) = 1$ , then both  $b$  and  $c$  are perfect squares.
10. Prove that every integer of the form  $n^4 + 4$ , with  $n > 1$  is composite. [*Hint*: Write  $n^4 + 4$  as a product of two quadratic factors].
11. Find all numbers  $p$  such that  $[2^p]_3 = [1]_3$ . Find all numbers  $r$  such that  $[2^r]_4 = [0]_4$ .
12. A man has \$4.55 in change composed entirely of dimes and quarters. What are the maximum and the minimum number of coins that he can have? Is it possible for the number of dimes to equal the number of quarters?

#### Even More Problems ...

13. Let  $a, b$  and  $m$  be natural numbers such that  $b|a$ . Prove that the operation of division  $\frac{[a]_m}{[b]_m} = \left[ \frac{a}{b} \right]_m$  is not well defined. (i.e. Rationals over  $\mathbb{Z}/m\mathbb{Z}$  do not make sense.)
14. Find sufficient conditions such that
  - (a)  $[a]_m \subset [b]_m$  for every  $m \in \mathbb{N}$  (find a condition on  $a$  and  $b$ ). Is it possible to have  $[a]_m \subset [b]_m$ , but  $[a]_m \neq [b]_m$ ?
  - (b)  $[a]_m \subset [a]_n$ , where  $n$  is a multiple of  $m$  (find a condition on  $a$ ).