310 Introduction to Modern Algebra
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Sample Problems for Exam 2

1. Find all units in $\mathbb{Z}/12\mathbb{Z}$. For each unit find its order and its inverse.
2. In a commutative ring R denote by $U(a) = \{a \cdot u | u \in R, u \text{ is a unit}\}$, where $a \in R$. Show that if $a \in R$ is a unit then $U(a) = U(1)$. Show that $U(a) \cap U(1) = \varnothing$ if a is a zero divisor.
3. Show that in a commutative ring if a is a unit with inverse b, then $-a$ is a unit with inverse $-b$.
4. Find the order of 5 mod 12, and use this to find the least nonnegative integer in the congruence class $[65^{625}]_{12}$.
5. From Fermat's theorem deduce that, for any integer $n \geq 0, 13 | 11^{12n+6} + 1$.
6. (a) Find the units digit of 3^{100} by the use of Fermat's theorem.
   (b) For any integer a, verify that a^5 and a have the same units digit.
7. Prove that $2n^{11}/11 + n^6/2 + 7n/22$ is an integer for every natural number n.
8. Find an integer solution x which is simultaneously congruent to 17 mod 35 and to 4 mod 99. Explain how you found x.
9. Use Euler's Theorem to evaluate $2^{100000} \pmod{77}$.
10. Show that for n an odd integer $\phi(2n) = \phi(n)$.
11. Solve the following system
$$\begin{cases} x \equiv 1 \pmod{9} \\ x \equiv 2 \pmod{11} \\ x \equiv 6 \pmod{13}. \end{cases}$$
12. Solve the equation $x^2 \equiv 1 \pmod{33}$.