221 - Differential Equations
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Substitution Method for an IVP

Consider the DE
$$y' + 2 = \sqrt{y + 3x + 4}. \tag{1}$$
Look at this DE for few seconds and identify the unknown function (you should see it is $y(x)$) and the independent variable (that's x). This DE is neither separable nor linear (check!), but if under the square root there was no $3x + 4$, the equation would be autonomous, hence separable. This is how we get the idea to get rid of this inconvenient term by denoting the entire quantity under the square root, $y + 3x + 4$, by u. Thus, u is a function of x (since y was too), and it becomes our new unknown. Next we will try to rewrite our DE (1) that was given in y, **only** in terms of u

First we need to find the derivative of u in terms of y'. From
$$u(x) = y(x) + 3x + 4$$
we get
$$u' = y' + 3,$$
hence $y' = u' - 3$. We go back in (1) and replace y' and y in terms of u. We get
$$u' - 3 + 2 = \sqrt{u},$$
i.e. $u' = 1 + \sqrt{u}$ which **is** separable! Good job so far. Now we solve this equation by separating and then integrating
$$\int \frac{du}{1 + \sqrt{u}} = \int dx. \tag{2}$$
For the integral on the left hand side (LHS) we will use a substitution (again!) since we do not like $1 + \sqrt{u}$. So let $v := 1 + \sqrt{u}$ (equivalent to $\sqrt{u} = v - 1$) which means $dv = \dfrac{du}{2\sqrt{u}} = \dfrac{du}{2(v-1)}$. Hence $du = 2(v - 1)dv$. We go back with this in our integral on the LHS of (2) and obtain
$$\int \frac{2(v-1)}{v} dv = x + C. \tag{3}$$
and in order to solve the integral on the LHS we write
$$\int \frac{2v - 2}{v} dv = \int \left(2 - \frac{2}{v}\right) dv = 2v - 2 \ln |v| + C = 2v - \ln v^2 + C.$$
Plugging this formula in (3) and remembering that v was $1 + \sqrt{u}$, yields
$$2(1 + \sqrt{u}) - \ln (1 + \sqrt{u})^2 = x + C,$$

and it looks like we are almost done. What is missing is the fact that we were asked to find y and we found u. So let's go back to y by using the fact that $u = y + 3x + 4$. Our solution can be written as

$$2(1 + \sqrt{y + 3x + 4}) - \ln(1 + \sqrt{y + 3x + 4})^2 = x + C.$$

This is a solution given in implicit form, i.e. in the equality above x and y are mixed together with no hope of ever getting y out without having some $x$es stick to it through some nasty function.

Imagine now that we were given an IC $y(0) = 5$. How would we find the unique solution satisfying our IVP ? Well, as in the case of an explicit solution, we plug into our solution $x = 0$ and $y = 5$ to get $C = 2(1 + 3) - \ln 1 + 3 = 8 - \ln 4$. So the unique solution is given implicitly by

$$2(1 + \sqrt{y + 3x + 4}) - \ln(1 + \sqrt{y + 3x + 4})^2 = x + 8 - \ln 4.$$

Now, if it looks like you understood this example, try on your own Problem 6 from the Review Problems for Exam 1 posted on the course website.