Substitution Method for an IVP

Consider the DE
\[ y' + 2 = \sqrt{y + 3x + 4}. \]  
(1)

Look at this DE for few seconds and identify the unknown function (you should see it is \( y(x) \)) and the independent variable (that’s \( x \)). This DE is neither separable nor linear (check!), but if under the square root there was no \( 3x + 4 \), the equation would be autonomous, hence separable. This is how we get the idea to get rid of this inconvenient term by denoting the entire quantity under the square root, \( y + 3x + 4 \), by \( u \). Thus, \( u \) is a function of \( x \) (since \( y \) was too), and it becomes our new unknown. Next we will try to rewrite our DE (1) that was given in \( y \), only in terms of \( u \).

First we need to find the derivative of \( u \) in terms of \( y' \). From
\[ u(x) = y(x) + 3x + 4 \]
we get
\[ u' = y' + 3, \]
hence \( y' = u' - 3 \). We go back in (1) and replace \( y' \) and \( y \) in terms of \( u \). We get
\[ u' - 3 + 2 = \sqrt{u}, \]
i.e. \( u' = 1 + \sqrt{u} \) which is separable! Good job so far. Now we solve this equation by separating and then integrating
\[ \int \frac{du}{1 + \sqrt{u}} = \int dx. \]  
(2)

For the integral on the left hand side (LHS) we will use a substitution (again!) since we do not like \( 1 + \sqrt{u} \). So let \( v := 1 + \sqrt{u} \) (equivalent to \( \sqrt{u} = v - 1 \)) which means \( dv = \frac{du}{2\sqrt{u}} = \frac{du}{2(v - 1)} \). Hence \( du = 2(v - 1)dv \). We go back with this in our integral on the LHS of (2) and obtain
\[ \int \frac{2(v - 1)}{v}dv = x + C. \]  
(3)

and in order to solve the integral on the LHS we write
\[ \int \frac{2v - 2}{v}dv = \int \left( 2 - \frac{2}{v} \right)dv = 2v - 2\ln|v| + C = 2v - \ln v^2 + C. \]

Plugging this formula in (3) and remembering that \( v \) was \( 1 + \sqrt{u} \), yields
\[ 2(1 + \sqrt{u}) - \ln (1 + \sqrt{u})^2 = x + C, \]
and it looks like we are almost done. What is missing is the fact that we were asked to find \( y \) and we found \( u \). So let’s go back to \( y \) by using the fact that \( u = y + 3x + 4 \). Our solution can be written as

\[
2(1 + \sqrt{y + 3x + 4}) - \ln(1 + \sqrt{y + 3x + 4})^2 = x + C.
\]

This is a solution given in implicit form, i.e. in the equality above \( x \) and \( y \) are mixed together with no hope of ever getting \( y \) out without having some \( x \)es stick to it through some nasty function.

Imagine now that we were given an IC \( y(0) = 5 \). How would we find the unique solution satisfying our IVP? Well, as in the case of an explicit solution, we plug into our solution \( x = 0 \) and \( y = 5 \) to get \( C = 2(1 + 3) - \ln 1 + 3 = 8 - \ln 4 \). So the unique solution is given implicitly by

\[
2(1 + \sqrt{y + 3x + 4}) - \ln(1 + \sqrt{y + 3x + 4})^2 = x + 8 - \ln 4.
\]

Now, if it looks like you understood this example, try on your own Problem 6 from the Review Problems for Exam 1 posted on the course website.