221 Differential Equations
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# Solutions to Practice Problems for Test 2

1. Verify that the given functions are solutions of the DE.

$$y''' + 2y'' - y' - 2y = 0, \quad e^t, e^{-t}, e^{-2t}.$$

Determine the Wronskian of each pair of two functions.

**Solution:** Plug each of the functions $e^t, e^{-t}, e^{-2t}$ into the DE and check the equality. Compute the Wronskians

$$W[e^t, e^{-t}] = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2 \neq 0$$

$$W[e^t, e^{-2t}] = \begin{vmatrix} e^t & e^{-2t} \\ e^t & -2e^{-2t} \end{vmatrix} = -3e^{-t} \neq 0$$

$$W[e^{-t}, e^{-2t}] = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = -e^{-3t} \neq 0$$

This implies that each pair of two functions is linearly independent. (Note: From Linear Algebra, it follows that the three functions are linearly independent).

2. Find the general solution of the given DE

$$2y''' - 4y'' - 2y' + 4y = 0.$$

**Solution:** The characteristic equation is $2r^3 - 4r^2 - 2r + 4 = 2(r^2 - 1)(r - 2) = 0$ whose roots are $-1, 1$ and 2. The fundamnetal solutions of the DE are $e^t, e^{-t}, e^{2t}$, so the general solution is given by

$$y(t) = c_1e^t + c_2e^{-t} + c_3e^{2t}.$$

3. Let the linear differential operator $L$ be defined by

$$L[y] = Ay^{(4)} - 5y^{(3)} + By'' + y,$$

where $A, B$ are real constants.

(a) Find $L[t^4]$.

(b) Find $L[e^{rt}]$.

(c) Write the characteristic equation for $L[y] = 0$.

**Solution:**

$$L[t^4] = 4 \cdot 3 \cdot 2 \cdot A - 5 \cdot 4 \cdot 3 \cdot 2t + 4 \cdot 3 \cdot Bt^2 + t^4 = 24A - 120t + 12Bt^2 + t^4$$

$$L[e^{rt}] = r^4Ae^{rt} - 5r^3e^{rt} + Br^2e^{rt} + e^{rt} = (r^4A - 5r^3 + Br^2 + 1)e^{rt}.$$

The characteristic equation will be

$$r^4A - 5r^3 + Br^2 + 1.$$

4. Use the method of reduction of order to solve the DE:

$$(t - 1)y'' - ty' + y = 0, \quad t > 1,$$

knowing that a particular solution is $y_1(t) = e^t$. (*Hint:* Use the substitution $y = y_1(t)v(t)$ and derive a DE for $v$.

**Solution:** We compute $y' = e^t(v + v'),\ y'' = e^t(v + 2v' + v'')$. Since $y$ is a solution of the DE, we have after simplification by $e^t$:

$$(t-1)(v + 2v' + v'') - t(v + v') + v = 0.$$

We simplify to obtain:

$$(t-1)v'' + (t-2)v' = 0.$$

Take $w = v'$, so we obtain $(t-1)w' + (t-2)w = 0$. This is a separable equation, that has a solution $w = e^{-t}(1-t)$, so $v = \int w dt = te^{-t}$. This means that $y_1 = t$.

5. Find the solution of the initial value problem

$$u'' + u = F(t), \quad u(0) = 0, \quad u'(0) = 0,$$

where

$$F(t) = \begin{cases} F_0 t, & 0 \leq t \leq \pi, \\ F_0(2\pi - t), & \pi < t \leq 2\pi, \\ 0, & 2\pi < t. \end{cases}$$

*Hint:* Treat each time interval separately, and match the solutions in the different intervals by requiring that $u$ and $u'$ be continuous functions of $t$.
**Solution:** Solve each of the problems (by the method of undetermined coefficients):

$$u_1'' + u_1 = F_0 t$$

$$u_2'' + u_2 = F_0(2\pi - t)$$

$$u_3'' + u_3 = 0$$

$u_1$ is be the solution on the interval $[0, \pi)$, $u_2$ is the solution on $[\pi, 2\pi)$, and $u_3$ satisfies satisfies the DE on $[2\pi, \infty)$. We impose $u_1(0) = 0, u_1'(0) = 0$. This will uniquely determine $u_1$. Compute $u_1(\pi), u_1'(\pi)$ and with these values solve the IVP:

$$u_2'' + u_2 = F_0(2\pi - t), \quad u_2(\pi) = u_1(\pi), \quad u_2'(\pi) = u_1'(\pi)$$

Now $u_2$ is uniquely determined, so we can solve the IVP for $u_3$:

$$u_3'' + u_3 = 0, \quad u_3(2\pi) = u_2(2\pi), \quad u_3'(2\pi) = u_2'(2\pi).$$

The solution will be:

$$u = \begin{cases} F_0(t - \sin t), & 0 \leq t \leq \pi, \\ F_0[(2\pi - t) - 3\sin t], & \pi < t \leq 2\pi, \\ -4F_0 \sin t, & 2\pi < t. \end{cases}$$

6. Use the method of variation of parameters to determine the solution of the given IVP:

$$y'' + y = \sec t; \quad y(0) = 2, \quad y'(0) = 1.$$

**Solution** The solution of the homogeneous equation is a linear combination of $y_1 = \sin t$ and $y_2 = \cos t$. The solution of the nonhomogeneous equation is $y = Ay_1 + By_2 + u_1 y_1 + u_2 y_2$, where $u_1, u_2$ satisfy:

$$u_1' = \frac{-y_2 \sec t}{W[y_1, y_2]}, \quad u_2' = \frac{y_1 \sec t}{W[y_1, y_2]}$$

The Wronskian is equal to 1, so $u_1 = -t, u_2 = -\ln|\cos t|$. By plugging in the initial conditions we obtain:

$$y = -t\sin t - \cos t \ln|\cos t| + 2(\sin t + \cos t).$$

7. Use the method of undetermined coefficients to solve the following DEs:

(a) $2y'' + 3y' + y = t^2 + 3\sin t$
**Solution** The complementary solution is $y_c = Ae^{-t} + Be^{-t/2}$. A trial solution for

$$(1) \quad 2y'' + 3y' + y = t^2$$

is of the form $Y_1 = Ct^2 + Dt + E$, and for

$$(2) \quad 2y'' + 3y' + y = 3\sin t$$

we take the trial solution $Y_2 = F\sin t + G\cos t$. For (1) and (2) we find the particular solutions $y_{p_1}, y_{p_2}$, so the solution to our problem will be

$$y = y_c + y_{p_1} + y_{p_2}.$$

(b) $y'' + 2y' + 5y = 4e^{-t}\cos 2t$
**Solution** The complementary solution is $y_c = e^{-t}(A\cos 2t + B\sin 2t)$. Take the trial solution $Y = te^{-t}(C\cos 2t + D\sin 2t)$.

8. If an undamped spring-mass system with a mass that weighs 6lb and a spring constant 1lb/in is suddenly set in motion at $t = 0$ by an external force of $4\cos t$ lb, determine the position of the mass at any time.

   **Solution:** The equation is $6y'' + y = 4\cos t$. with initial conditions $y(0) = 0, y'(0) = 0$. A particular solution is $y_p = -4/5\cos t$, so the solution is $y = A\sin(t/\sqrt{6}) + B\cos(t/\sqrt{6}) + y_p$. From the initial conditions we get: $A = 0, B = 4/5$.

9. In the absence of damping the motion of a spring-mass system satisfies the initial value problem

$$mu'' + ku = 0, \quad u(0) = a, \quad u'(0) = b.$$

   (a) Show that the kinetic energy initially imparted to the mass is $mb^2/2$ and that the potential energy initially stored in the spring is $ka^2/2$, so that initially the total energy in the system is $(ka^2 + mb^2)/2$.
   (b) Solve the given initial value problem.
   (c) Using the solution in part b), determine the total energy in the system at any time. Your result should confirm the principle of conservation of energy for this system.

**Solution:**
a)The kinetic energy for a body is $mv^2/2$, where $v$ is the velocity. The potential energy for a spring is $kx^2/2$, where $k$ is the elasticity constant and $x$ is the displacement. These facts imply that the initial kinetic energy is $mu'(0)^2/2 = mb^2/2$, and the initial potential energy is $ku(0)^2/2 = ka^2/2$.
b) $u = a\cos(\sqrt{k/mt}) + b\sqrt{m/k}\sin(\sqrt{k/mt})$ .
c) The kinetic energy is $mu'(t)^2/2$, the potential energy is $ku(t)^2/2$. By using the solution from part b) we get that the total energy (kinetic +potential) is conserved in time.