

## Project - Due April 10, 2006

**Instructions for preparing and submitting the project:** For this project you will work in groups of 2-3 students. On a separate piece of paper each student will summarize the work and the contribution brought to the final product. Attach the piece of paper to the write up of the project.

The report you submit should contain a mix of equations, formulas and prose. The report has to be typed, the graphs should be drawn with a computer system, not by hand. Use complete sentences, proper grammar and correct punctuation. Your report must be self-contained, meaning that it can be understood by anyone that is competent in Differential Equations.

**Note:** You may ask the instructor questions about the project only until April 7, 2006.

### Prices in a Free-Market Economy

In the 18th century, the Scottish economist Adam Smith asserted that in a free market economy governed by the law of demand and supply, prices will eventually tend to an equilibrium. Let us derive a system that models the price change for one commodity (i.e., product or service).

If  $d$  and  $s$  are the time-dependent functions that denote the *demand* and the *supply* of a given commodity, then  $r = d - s$  is called *excess demand*. Let  $p$  denote the price of this commodity (which is also a function of  $t$ ) and assume that  $p$  and  $r$  are differentiable. The law of demand and supply works as follows:

- L1. If at a given time, demand is less than supply, prices decrease, so the demand increases; therefore the excess demand increases.
- L2. If at a given time, demand is greater than supply, prices increase, so the supply increases; therefore the excess demand decreases.
- L3. If at a given time, demand equals supply, prices are constant, so the excess demand remains constant, namely 0.

**Task 1.** Write a mathematical formulation for each of the laws L1, L2, L3 above.

We can thus ignore the change in demand and supply and deal only with the change in excess demand. According to Adam Smith, an economy is a *free market* if the laws L1-L3 are the only ones that rule the variations in price and excess demand.

We can now write the equations that govern a free market. From L1-L3 we see that a possible way to express these laws is that the rate of change in price is proportional to the excess demand, and

similarly, the rate of change in excess demand is proportional to the excess demand, i.e.

$$\begin{cases} p' = \alpha r \\ r' = -\beta r \end{cases}, \quad \alpha, \beta > 0. \quad (\text{S})$$

**Task 2.** Find the solutions of (S) which depend on the parameters  $\alpha$  and  $\beta$ .

**Task 3.** Use the solution obtained in Task 2 and a phase plane analysis to show that the solutions of the system (S) tend to equilibria.

Though the equations in (S) are in agreement with L1-L3, the market may be more complicated. A more accurate description of a free-market economy should involve a system of the type

$$\begin{cases} p' = \alpha f(r) \\ r' = -\beta h(r), \end{cases}$$

where  $f$  and  $h$  are functions with certain properties.

**Task 4.** What would be the natural conditions to impose on  $f$  and  $h$  such that the laws of demand and supply L1-L3 remain true? Add to these conditions the additional condition that

$$\lim_{r \rightarrow 0} f'(r) = 0, \quad (\text{C})$$

which suggests that the variation in price is small whenever the difference between the demand and supply is small.

Can you think of other realistic conditions? For your choice of conditions, can  $f$  and  $h$  be linear functions?

If  $f$  and  $h$  are not linear functions, do prices still tend to equilibria? To explore your model, consider particular cases of functions and values of constants and then use a computer program like Maple to represent the flow of trajectories in the phase plane.

**Task 5.** For one of the nonlinear systems use the improved Euler Method to find the solution of a corresponding initial value problem. (Choose some appropriate initial conditions and some values for  $\alpha$  and  $\beta$  to find the numerical approximation).

**Task 6.** Consider the model of a free-market where other forces occur, i.e. when the system changes to

$$\begin{cases} p' = \alpha f(r) + u_0 \\ r' = -\beta h(r) + v_0. \end{cases} \quad (\text{S1})$$

Show that, in general, for such a system prices do not reach an equilibrium (One counterexample will suffice here, i.e. prove the statement for one particular choice of functions  $f$  and  $h$ ).

**Task 7.** Take in (S1)  $f(r) = g(r) = r$  and  $u_0, v_0$  constant. Find the general solution of the system and explain the dependence of  $p$  and  $r$  on the values of  $\alpha, \beta, u_0$ , and  $v_0$ .