Review Problems for Laplace Transform

1. Apply Duhamel’s Principle to write an integral solution for the solution of the following IVP:

\[ x'' + 6x' + 9x = f(t), \quad x(0) = x'(0) = 0 \]

**Solution:** Consider the problem

\[ y'' + 6y' + 9y = \delta(t), \quad y(0) = y'(0) = 0. \]  \hspace{1cm} (1)

Duhamel’s principle gives us that a particular solution \( x_p \) for the given IVP can be computed as

\[ x_p = y \ast f. \]

We use the Laplace transform in order to compute \( y \). From (1) we have:

\[ L[y'' + 6y' + 9y] = L[\delta(t)] \]

which implies

\[ (s^2 + 6s + 9)L[y] = 1. \]

Hence

\[ L[y] = \frac{1}{s^2 + 6s + 9} = \frac{1}{(s + 3)^2}. \]

By using the shifted formula in \( s \) (#7 in the table from the front cover of the textbook) we get that \( y = e^{-3t} \). Hence,

\[ x(t) = e^{-3t} \ast f(t). \]

2. Find the Laplace transform of

\[ f(t) = \begin{cases} 
\sin \pi t, & 2 \leq t \leq 3 \\
0, & \text{otherwise.} 
\end{cases} \]

**Solution:** Write

\[ f(t) = \sin \pi t(H(t - 2) - H(t - 3)) = H(t - 2) \sin \pi t - H(t - 3) \sin \pi t. \]

By one of the shifted formula (#8 from the table) we get that

\[ L[f] = e^{-2s}L[\sin \pi (t+2)] - e^{-3s}L[\sin \pi (t+3)] = e^{-2s}L[\sin \pi t] + e^{-3s}L[\sin \pi t] = e^{-2s} \frac{\pi}{s^2 + \pi^2} + e^{-3s} \frac{\pi}{s^2 + \pi^2}. \]

Above we used that

\[ \sin \pi (t + 2) = \sin(\pi t + 2\pi) = \sin \pi t \]

and

\[ \sin \pi (t + 3) = \sin(\pi t + 3\pi) = -\sin \pi t \]