221 Differential Equations
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April 17, 2006

Review Problems for Laplace Transform

1. Apply Duhamel's Principle to write an integral solution for the solution of the following IVP:

$$x'' + 6x' + 9x = f(t), \quad x(0) = x'(0) = 0$$

Solution: Consider the problem

$$y'' + 6y' + 9y = \delta(t), \quad y(0) = y'(0) = 0. \tag{1}$$

Duhamel's principle gives us that a particular solution x_p for the given IVP can be computed as

$$x_p = y \star f.$$

We use the Laplace transform in order to compute y. From (1) we have:

$$L[y'' + 6y' + 9y] = L[\delta(t)]$$

which implies

$$(s^2 + 6s + 9)L[y] = 1.$$

Hence $L[y] = \dfrac{1}{s^2 + 6s + 9} = \dfrac{1}{(s+3)^2}$. By using the shifted formula in s (# 7 in the table from the front cover of the textbook) we get that $y = e^{-3t}t$. Hence,

$$x(t) = e^{-3t} \star f(t).$$

2. Find the Laplace transform of

$$f(t) = \begin{cases} \sin \pi t, & 2 \leq t \leq 3 \\ 0, & \text{otherwise.} \end{cases}$$

Solution: Write

$$f(t) = \sin \pi t(H(t-2) - H(t-3)) = H(t-2)\sin \pi t - H(t-3)\sin \pi t.$$

By one of the shifted formula (#8 from the table) we get that

$$L[f] = e^{-2s}L[\sin \pi(t+2)] - e^{-3s}L[\sin \pi(t+3)] = e^{-2s}L[\sin \pi t] + e^{-3s}L[\sin \pi t] = e^{-2s}\frac{\pi}{s^2+\pi^2} + e^{-3s}\frac{\pi}{s^2+\pi^2}.$$

Above we used that

$$\sin \pi(t+2) = \sin(\pi t + 2\pi) = \sin \pi t$$

and

$$\sin \pi(t+3) = \sin(\pi t + 3\pi) = -\sin \pi t$$