221 Differential Equations
Instructor: Petronela Radu
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Name: ____________________

Final Exam

Show all your work; no credit will be given for unsupported answers.
No communication devices, notes nor books may be used.

DE is abbreviation for a differential equation.
IVP is abbreviation for an initial value problem.

Problem	**Maximum**	
1	15	
2	10	
3	10	
4	10	
5	10	
6	10	
7	15	
8	10	
9	10	
Total	100	

You only need 100 points for full credit!

1. (15 points) Find the general solutions of the following DEs:

   (a) $\sqrt{2t}\; x' = \tan x$

   (b) $x't = x + t$ (homogeneous equation)

(c) $(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$ (exact equation)

(d) $y^{(4)} + 4y'' = 0$.

2. (10 points) An accident at a nuclear power plant has left the surrounding area polluted with radioactive material that decays naturally. The initial amount of radioactive material present is 15 su (safe units), and 5 months later is still 10 su.

   (a) Write a formula giving the amount $A(t)$ of radioactive material (in su) remaining after t months.

   (b) What amount of radioactive material will remain after 8 months?

   (c) How long - total number of months or fraction thereof - will it be until $A = 1$ su, so it is safe for people to return to the area?

3. (10 points) An investment is modelled by the ODE $y' = y(6-y)$, where y is the amount (in thousands) of dollars at time t (in months).

   (a) What kind of growth does the investment follow?

   (b) Assume that the investment starts losing \$5,000 per month. For the new equation, discuss the stability of the critical points with a phase line analysis. Find the long term outcome of an initial investment of \$500, \$3,000, \$5,000, and respectively \$6,000.

4. (10 points) Given the IVP:

$$x' = x - t, \quad x(0) = 2.$$

(a) Use Euler's method with step size $h = 0.2$ to estimate $x(1)$.

(b) Solve the equation and compute the difference between the correct value and the approximate one at $t = 1$.

For full credit, write the formulas that you are using and show all your work.

5. (10 points) Use the method of undetermined coefficients to find the unique solution of the IVP:

$$y'' = -y + 2t + 6\cos t, \quad y(0) = 1, \quad y'(0) = 1.$$

6. (10 points) Use variation of parameters to find the solution of the following DE:

$$x'' = -3x' - 2x + e^t.$$

7. (15 points) Find the exact solution of the system

$$\begin{cases} x' = -ax - y \\ y' = x - ay, \end{cases}$$

for a an arbitrary real number and discuss the stability of the origin. For **one** value of a (your choice!) perform a phase plane analysis and draw some typical trajectories.

8. (10 points) Determine the solution of the IVP:

$$y'' + y = \begin{cases} 5, & \text{if } t < 1 \\ 2, & \text{if } t > 1, \end{cases} \qquad y(0) = 1, \quad y'(0) = 0.$$

9. (10 points) A pendulum inside a rigid frame starts swinging with small amplitude. Five seconds later a karate master hits the frame from above. Since frictions are negligible for short time intervals, we can model the pendulum's motion with the second-order equation

$$x'' + x = \delta(t-5),$$

where δ is the Dirac delta function. Assuming that the blow does not break the frame, determine the motion of the pendulum for the initial conditions $x(0) = 1$, $x'(0) = 0$.

Formula Sheet

The method of variation of parameters

For a second order linear equation $y'' + py' + qy = g$, let $y_c = c_1 y_1 + c_2 y_2$ be the general solution of the homogeneous equation $y'' + py' + qy = 0$ and $W[y_1, y_2] = y_1 y_2' - y_1' y_2$ be its Wronskian. Then, a **particular** solution for $y'' + py' + qy = g$ is of the form $u_1 y_1 + u_2 y_2$, where

$$u_1' = \frac{-y_2 g}{W[y_1, y_2]}, \quad u_2' = \frac{y_1 g}{W[y_1, y_2]},$$

with y_1, y_2 being the fundamental solutions of the homogeneous equation and W is the Wronskian of y_1, y_2.

Laplace Transform Formulas

With $\quad F(s) = \mathcal{L}[f(t)], \quad G(s) = \mathcal{L}[g(t)]$

$$1. \quad \mathcal{L}[e^{at} g(t)] = G(s-a)$$

$$2. \quad \mathcal{L}[g(t)H(t-a)] = e^{-as}\mathcal{L}[g(t+a)] \quad (a > 0)$$

$$3. \quad \mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s) \quad (a > 0)$$

$$4. \quad \mathcal{L}[\delta(t-a)] = e^{-as}, a \geq 0$$

$$5. \quad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad n = 0, 1, ..$$

$$6. \quad \mathcal{L}[e^{at}] = \frac{1}{s-a}$$

$$7. \quad \mathcal{L}[\sin kt] = \frac{k}{s^2 + k^2}$$

$$8. \quad \mathcal{L}[\cos kt] = \frac{s}{s^2 + k^2}$$

$$9. \quad \mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}$$