221 Differential Equations
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Solutions to Practice Problems for Test 2

1. Verify that the given functions are solutions of the DE, and determine their Wronskian.

$$y''' + 2y'' - y' - 2y = 0, \quad e^t, e^{-t}, e^{-2t}.$$

Solution: Plug each of the functions e^t, e^{-t}, e^{-2t} into the DE and check the equality. Compute the Wronskian

$$W[e^t, e^{-t}, e^{-2t}] = \begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e^{-t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{vmatrix} = -6e^{-2t}$$

by factoring out e^t, e^{-t}, and respectively e^{-2t} from each column of the determinant.

2. Find the general solution of the given DE

$$2y''' - 4y'' - 2y' + 4y = 0.$$

Solution: The characteristic equation is $2r^3 - 4r^2 - 2r + 4 = 2(r^2 - 1)(r - 2) = 0$ whose roots are $-1, 1$ and 2. The solutions of the DE are e^t, e^{-t}, e^{2t}.

3. Let the linear differential operator L be defined by

$$L[y] = a_4 y^{(4)} + a_3 y^{(3)} + a_2 y'' + a_1 y' + a_0 y,$$

where a_4, a_3, a_2, a_1, a_0 are real constants.

   (a) Find $L[t^4]$.

   (b) Find $L[e^{rt}]$.

   (c) Use a) and b) to determine four solutions of the equation $y^{(4)} - 5y'' + 4y = 0$ and two solutions of the equation $y'' - 6y' + 9 = 0$.

Partial solution was given in class.

4. Use the method of reduction of order to solve the DE:

$$(2 - t)y''' + (2t - 3)y'' - ty' + y = 0, \quad t < 2,$$

knowing that a particular solution is $y_1(t) = e^t$. (*Hint:* Use the substitution $y = y_1(t)v(t)$ and derive a DE for v.
Partial solution was given in class.

5. Find the solution of the initial value problem

$$u'' + u = F(t), \quad u(0) = 0, \quad u'(0) = 0,$$

where

$$F(t) = \begin{cases} F_0 t, & 0 \le t \le \pi, \\ F_0(2\pi - t), & \pi < t \le 2\pi, \\ 0, & 2\pi < t. \end{cases}$$

Hint: Treat each time interval separately, and match the solutions in the different intervals by requiring that u and u' be continuous functions of t.
Solution: Solve each of the problems (by the method of undetermined coefficients):

$$u_1'' + u_1 = F_0 t$$

$$u_2'' + u_2 = F_0(2\pi - t)$$
$$u_3'' + u_3 = 0$$

u_1 is be the solution on the interval $[0, \pi)$, u_2 is the solution on $[\pi, 2\pi)$, and u_3 satisfies satisfies the DE on $[2\pi, \infty)$. We impose $u_1(0) = 0, u_1'(0) = 0$. This will uniquely determine u_1. Compute $u_1(\pi), u_1'(\pi)$ and with these values solve the IVP:

$$u_2'' + u_2 = F_0(2\pi - t), \quad u_2(\pi) = u_1(\pi), \quad u_2'(\pi) = u_1'(\pi)$$

Now u_2 is uniquely determined, so we can solve the IVP for u_3:

$$u_3'' + u_3 = 0, \quad u_3(2\pi) = u_2(2\pi), \quad u_3'(2\pi) = u_2'(2\pi).$$

The solution will be:

$$u = \begin{cases} F_0(t - \sin t), & 0 \le t \le \pi, \\ F_0[(2\pi - t) - 3\sin t], & \pi < t \le 2\pi, \\ -4F_0 \sin t, & 2\pi < t. \end{cases}$$

6. Find the integrating factor and then solve the following IVPs:

   (a) $(2x + 3)y' + (2y - 2) = 0, \quad y(1) = 3.$
   Solution: The DE can be written as $y' + \dfrac{2}{2x+3}y = \dfrac{2}{2x+3}.$, so the integrating factor is $e^{\int \frac{2}{2x+3}dx} = 2x + 3$. Multiply the equation by $2x + 3$ and write $(y(2x + 3))' = 2$. Integrate to find $y = \frac{2x+C}{2x+3}$ From the initial condition, we find $C = 13$.

   (b) $y' = e^{2x} + y - 1, \quad y(0) = 2.$
   Solution: The integrating factor is e^{-x}. Multiply the equation by e^{-x} to obtain $(ye^{-x})' = e^x - e^{-x}$. Integrate and find $y = e^{2x} + 1 + Ce^x$. From the initial condition we get $C = 0$.

7. Use the method of variation of parameters to determine the solution of the given IVP:

$$y'' + y = \sec t; \quad y(0) = 2, \quad y'(0) = 1.$$

Solution The solution of the homogeneous equation is a linear combination of $y_1 = \sin t$ and $y_2 = \cos t$. The solution of the nonhomogeneous equation is $y = Ay_1 + By_2 + u_1y_1 + u_2y_2$, where u_1, u_2 satisfy:

$$u_1' = \frac{-y_2 \sec t}{W[y_1, y_2]}, \quad u_2' = \frac{y_1 \sec t}{W[y_1, y_2]}$$

The Wronskian is equal to 1, so $u_1 = -t, u_2 = -\ln|\cos t|$. By plugging in the initial conditions we obtain:

$$y = -t \sin t - \cos t \ln|\cos t| + 2(\sin t + \cos t).$$

8. Use the method of undetermined coefficients to solve the following DEs:

   (a) $2y'' + 3y' + y = t^2 + 3\sin t$
   Solution The complementary solution is $y_c = Ae^{-t} + Be^{-t/2}$. A trial solution for (DE1)$2y'' + 3y' + y = t^2$ is of the form $Y_1 = Ct^2 + Dt + E$, and for (DE2)$2y'' + 3y' + y = 3\sin t$ we take the trial solution $Y_2 = F\sin t + G\cos t$. For (DE1) and (DE2) we find the particular solutions y_{p_1}, y_{p_2}, so the solution to our problem will be

$$y = y_c + y_{p_1} + y_{p_2}.$$

   (b) $y'' + 2y' + 5y = 4e^{-t}\cos 2t$
   Solution The complementary solution is $y_c = e^{-t}(A\cos 2t + B\sin 2t)$. Take the trial solution $Y = te^{-t}(C\cos 2t + D\sin 2t)$.

9. If an undamped spring-mass system with a mass that weighs 6lb and a spring constant 1lb/in is suddenly set in motion at $t = 0$ by an external force of $4\cos t$ lb, determine the position of the mass at any time. Find the amplitude of the motion and the maximum velocity of the system.
Solution: The equation is $6y'' + y = 4\cos t$. with initial conditions $y(0) = 0, y'(0) = 0$. A particular solution is $y_p = -4/5\cos t$, so the solution is $y = A\sin(t/\sqrt{6}) + B\cos(t/\sqrt{6}) + y_p$. From the initial conditions we get: $A = 0, B = 4/5$. The amplitude is $\max|y(t)|$, the maximum velocity is $\max|y'(t)| = \max|4/5\sin t - 4/(5\sqrt{6})\sin t/\sqrt{6}|$. None of these quantities can be easily computed explicitly, so leave your answer in this form.

10. In the absence of damping the motion of a spring-mass system satisfies the initial value problem

$$mu'' + ku = 0, \quad u(0) = a, \quad u'(0) = b.$$

(a) Show that the kinetic energy initially imparted to the mass is $mb^2/2$ and that the potential energy initially stored in the spring is $ka^2/2$, so that initially the total energy in the system is $(ka^2 + mb^2)/2$.

(b) Solve the given initial value problem.

(c) Using the solution in part b), determine the total energy in the system at any time. Your result should confirm the principle of conservation of energy for this system.

Solution:
a)The kinetic energy for a body is $mv^2/2$, where v is the velocity. The potential energy for a spring is $kx^2/2$, where k is the elasticity constant and x is the displacement. These facts imply that the initial kinetic energy is $mu'(0)^2/2 = mb^2/2$, and the initial potential energy is $ku(0)^2/2 = ka^2/2$.
b) $u = a\cos(\sqrt{k/mt}) + b\sqrt{m/k}\sin(\sqrt{k/mt})$.
c) The kinetic energy is $mu'(t)^2/2$, the potential energy is $ku(t)^2/2$.