221 Differential Equations
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Solutions to Sample Problems for Test 3

1. For each of the linear systems below find an interval in which the general solution is defined.

   (a) $x' = x + \dfrac{2}{\cos t}y, \quad y' = (\ln t)x - \sqrt{3-t}\,y;$

   (b) $(t+1)u' = \dfrac{t}{3-t}u - \sqrt{3}v, \quad v' = (\sin t)u - (\cos t)v.$

   Solution We need **continuity** of the coefficients in the **interior** of an interval which contains the initial conditions. Since here we do not have initial conditions, we specify where the data can be given so we are guaranteed existence and uniqueness.

   (a) (discussed in class) The solution exists on $(0, \pi/2)$ or on $(\pi/2, 3)$, depending on where the initial conditions are given.

   (b) The solution exists on $(-\infty, -1)$, $(-1, 3)$, or $(3, \infty)$ depending on where the initial time is taken.

2. For the differential system **x**'(t)=A**x**(t):

   (a) Perform a phase plane analysis;

   (b) Find the general solution;

   (c) Discuss the stability of the origin based on parts (a) and (b);

   (d) Draw some trajectories to illustrate what type of a critical point the origin is.

   where

   (a) $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 2 & 5 & 7 \\ 0 & 2 & 9 \\ 0 & 0 & 3 \end{bmatrix}.$

   Solution:

   (a) c) The characteristic equation is $(3-\lambda)(8-\lambda) - 6 = 0$ with roots 2 and 9. The eigenvector $\bar{v}$ corresponding to $\lambda = 2$ satisfies $v_1 + 2v_2 = 0$, so we take $\bar{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. The eigenvector $\bar{w}$ corresponding to $\lambda = 9$ satisfies $3w_1 - w_2 = 0$, so one can take $\bar{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The general solution is given by:

   $$\bar{x}(t) = c_1 e^{2t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{9t} \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

   so the origin is a source (hence, unstable critical point) and an improper node.

   (b) c) The characteristic equation is $(1-\lambda)(2-\lambda) - 30 = 0$ with roots -4 and 7. The eigenvector $\bar{v}$ corresponding to $\lambda = -4$ satisfies $v_1 + 2v_2 = 0$, so we take $\bar{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$. The eigenvector $\bar{w}$

corresponding to $\lambda = 7$ satisfies $-6w_1 + 6w_2 = 0$, so one can take $\bar{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution is given by:

$$\bar{x}(t) = c_1 e^{-4t} \begin{bmatrix} 6 \\ -5 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so the origin is a saddle (hence, unstable critical point).

(c) c) The characteristic equation is $(1-\lambda)(3-\lambda) + 10 = 0$ with complex conjugate roots $2+3i$ and $2-3i$. The eigenvector $\bar{v}$ corresponding to $\lambda = 2+3i$ satisfies $(-1-3i)v_1 + 5v_2 = 0$, so we take $\bar{v} = \begin{bmatrix} 5 \\ 1+3i \end{bmatrix}$. We compute the real and the imaginary parts of $e^{(2+3i)t}\bar{v}$ and obtain

$$e^{2t}(\cos t + i \sin 3t)\bar{v} = e^{2t}\left(\begin{bmatrix} 5\cos 3t \\ \cos 3t - 3\sin 3t \end{bmatrix} + \begin{bmatrix} 5\sin 3t \\ \sin 3t + 3\cos 3t \end{bmatrix}\right).$$

The general solution is given by:

$$\bar{x}(t) = c_1 e^{2t} \begin{bmatrix} 5\cos 3t \\ \cos 3t - 3\sin 3t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 5\sin 3t \\ \sin 3t + 3\cos 3t \end{bmatrix},$$

so the origin is a spiral source (hence, unstable critical point).

(d) c) The characteristic equation is $(3-\lambda)(2-\lambda)^2 = 0$ with roots 3 and 2 with multiplicity 2 . The eigenvector $\bar{v}$ corresponding to $\lambda = 3$ satisfies $-v_1 + 5v_2 + 7v_3 = 0and - v_2 + 9v_3 = 0$. By solving the system we get $v_1 = 52v_3$, $v_2 = 9v_3$, so one can take $\bar{v} = \begin{bmatrix} 52 \\ 9 \\ 1 \end{bmatrix}$. The eigenvector $\bar{w}$ corresponding to $\lambda = 2$ satisfies $5v_2 + 7v_3 = 0$, $v_3 = 0$, so take $\bar{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Since we have only two eigenvectors, we need to take the generalized eigenvector $\bar{u}$ corresponding to $\lambda = 2$ satisfies $5u_2 = 1$, so one can take $\bar{w} = \begin{bmatrix} 0 \\ 1/5 \\ 0 \end{bmatrix}$. The general solution is given by:

$$\bar{x}(t) = c_1 e^{3t} \begin{bmatrix} 52 \\ 9 \\ 1 \end{bmatrix} + c_2 e^{2t}(t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/5 \\ 0 \end{bmatrix}).$$

The origin is a source (hence, unstable critical point).

3. Decide if the following statements are TRUE or FALSE. Motivate your answers.

   (a) A center is a stable point.
   (b) A center is an asymptotically stable point.
   (c) A node can be a sink, a source, or a saddle.
   (d) A saddle point is an asymptotically unstable point, but some trajectories move towards it.
   For the next statements assume that the origin is the only critical point of $x' = Ax$.
   (e) If A has real negative eigenvalues, then the origin is a sink.
   (f) If the origin is a source then all trajectories that start outside the origin are unbounded.
   (g) If one of the eigenvalues has the real part equal to 0, then the origin is a center.

   Solution: - discussed in class.

4. Solve the following initial value problems:

(a) $x' = 5x - y, \quad y' = 3x + y$ with $x(0) = 2, \quad y(0) = -1$.
The eigenvalues and corresponding eigenvectors are:

$$\lambda_1 = 2, \bar{v_1} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\lambda_2 = 4, \bar{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. From initial conditions we get $c_1 = 3/4$, $c_2 = 5/4$.

(b) $x' = x - 5y, \quad y' = x - 3y$ with $x(0) = 1, \quad y(0) = 1$.
The eigenvalues are $-1 \pm 2i$. For $-1 + 2i$ we get the eigenvector

$$\begin{bmatrix} 5 \\ 2 - i \end{bmatrix}.$$

The general solution is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{-t}(c_1 \begin{bmatrix} 5\cos t \\ 2\cos t - \sin t \end{bmatrix} + c_2 \begin{bmatrix} 5\sin t \\ 2\sin t - \cos t \end{bmatrix}).$$

From the initial conditions $c_1 = 1/5, c_2 = -3/5$.

(c) $x' = 3x + 9y, \quad y' = -x - 3y$ with $x(0) = 2, \quad y(0) = 4$.
The eigenvalue is 0 has multiplicity 2 and the corresponding eigenvector

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

A generalized eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The generalized solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 (t \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix})$$

From the initial conditions we get $c_1 = -1/2$, $c_2 = 7/2$.

5. The following model can be interpreted as describing the interaction of two species with population densities x and y:

$$x' = x - 0.5y \quad y' = 0.25x + y.$$

Use a phase plane analysis to determine the long-time behavior of a solution whose trajectory passes through the point (1,1).
Solution: discussed in class.

6. Detrmine if the systems below could be interpreted as models for predatory-prey, competing, or cooperating species. Motivate your answer.

(a) $x' = x(1 - x + y), \quad y' = y(4 - 3y - x)$.
(b) $x' = x(1 - x - y), \quad y' = y(4 - 3y - x)$.
(c) $x' = x(1 - x + y), \quad y' = y(4 - 3y + x)$.

Solution: We look for the sign of "the other" term in an equation.

(a) Predator-prey, x =predator, y= prey. Both x, y have logistic growth.
(b) Competitive.
(c) Cooperating.

7. Consider two interconnected tanks such that there is a transfer of mixture between the tanks in both directions through two pipes. Tank 1 initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of of 1.5gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 4 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

   (a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process. Observe that the system in nonhomogeneous.

   (b) Find the values of Q_1 and Q_2 for which the system is in equilibrium, and denote them by Q_1^E and Q_2^E. Can you predict which tank will approach its equilibrium state more rapidly?

   (c) Let $x_1(t) = Q_1(t) - Q_1^E$ and $x_2(t) = Q_2(t) - Q_2^E$. Determine an initial value problem for x_1 and x_2. Observe that the system is homogeneous.

   Solution:

   (a)
$$Q_1' = 1.5 - 3\frac{Q_1}{30} + 1.5\frac{Q_2}{20},\ Q_1 = 25$$
$$Q_2' = 3 + 3\frac{Q_1}{30} - 4\frac{Q_2}{20},\ Q_2 = 15.$$

   (b) Critical points are given by:
$$1.5 - 3\frac{Q_1}{30} + 1.5\frac{Q_2}{20} = 0$$
$$3 + 3\frac{Q_1}{30} - 4\frac{Q_2}{20} = 0.$$
   Therefore the equilibrium solutions are $Q_1^E = 42$, $Q_2^E = 36$. Compute Q_1' and Q_2' at the initial values, so $Q_1'(0) = 1.625, Q_2'(0) = 2.5$. The second tank will approach equilibrium more rapidly.

   (c) With the change of variables the system becomes:
$$x_1'(t) = -\frac{x_1}{10} + \frac{3x_2}{40},\ x_1(0) = -17$$
$$x_2' = \frac{x_1}{10} - \frac{x_2}{5},\ x_2(0) = -21.$$

8. Compute the Laplace transform of the function
$$f(t) = \begin{cases} \sin t, & t < \pi \\ 3 - t, & t \geq \pi. \end{cases}$$
   in two ways: by using the definition, and by using the Heaviside (unit step) function.

   Solution By the definition we have that
$$L[f](s) = \int_0^\infty e^{-st} f(t)dt = \int_0^\pi e^{-st} \sin t dt + \int_\pi^\infty e^{-st}(3-t)dt$$
   For both integrals we use integration by parts and obtain:
$$L[f](s) = \frac{1}{s^2+1}(e^{-s\pi} + 1) + e^{-\pi s}(\frac{3-\pi}{s} - \frac{1}{s^2}).$$
   In order to compute the Laplace transform using tables we write:
$$f(t) = \sin t(1 - H(t-\pi)) + (3-t)H(t-\pi).$$

So

$$L[f](s) = \frac{1}{s^2+1} - L[\sin tH(t-\pi)](s) + 3L[H(t-\pi)](s) - L[tH(t-\pi)](s) =$$

$$\frac{1}{s^2+1} - e^{-\pi s}L[\sin(t+\pi)](s) + 3\frac{e^{-\pi s}}{s} - e^{-\pi s}L[t+\pi](s)$$

Since $\sin(t+\pi) = -\sin t$ we get

$$L[f](s) = \frac{1}{s^2+1} + e^{-\pi s}\frac{s}{s^2+1} + 3\frac{e^{-\pi s}}{s} - e^{-\pi s}(1/s^2 + \pi/s).$$

9. Use the tables to find the inverse Laplace transform of

$$\frac{s+2}{4s^2+9}, \ \frac{e^{-3s}}{4s}, \ \frac{s}{s^2+2s+5}.$$

Solution

$$L^{-1}[\frac{s+2}{4s^2+9}] = \frac{1}{4}L^{-1}[\frac{s}{s^2+9/4}] + \frac{2}{4}L^{-1}[\frac{1}{s^2+9/4}] = \frac{1}{4}\frac{2}{3}\cos\frac{3}{2}t + \frac{1}{2}\frac{2}{3}\sin\frac{3}{2}t.$$

From the tables we have that

$$L^{-1}[\frac{e^{-3s}}{4s}] = 1/4H(s-3),$$

where H is the Heaviside (unit-step) function.