221 Differential Equations
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November 15, 2005

Sample Problems for Test 3

1. For each of the linear systems below find an interval in which the general solution is defined.
   (a) $x' = x + \dfrac{2}{\cos t} y, \quad y' = (\ln t)x - \sqrt{3-t}\, y;$
   (b) $(t+1)u' = \dfrac{t}{3-t} u - \sqrt{3} v, \quad v' = (\sin t)u - (\cos t)v.$
2. For the differential system **x**'(t)=A**x**(t):
   (a) Perform a phase plane analysis;
   (b) Find the general solution;
   (c) Discuss the stability of the origin based on parts (a) and (b);
   (d) Draw some trajectories to illustrate what type of a critical point the origin is.

   where

   (a) $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$
   (b) $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$
   (c) $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$
   (d) $A = \begin{bmatrix} 2 & 5 & 7 \\ 0 & 2 & 9 \\ 0 & 0 & 3 \end{bmatrix}.$
3. Decide if the following statements are TRUE or FALSE. Motivate your answers.
   (a) A center is a stable point.
   (b) A center is an asymptotically stable point.
   (c) A node can be a sink, a source, or a saddle.
   (d) A saddle point is an asymptotically unstable point, but some trajectories move towards it.
   For the next statements assume that the origin is the only critical point of $x' = Ax$.
   (e) If A has real negative eigenvalues, then the origin is a sink.
   (f) If the origin is a source then all trajectories that start outside the origin are unbounded.
   (g) If one of the eigenvalues has the real part equal to 0, then the origin is a center.
4. Solve the following initial value problems:
   (a) $x' = 5x - y, \quad y' = 3x + y$ with $x(0) = 2, \quad y(0) = -1.$
   (b) $x' = x - 5y, \quad y' = x - 3y$ with $x(0) = 1, \quad y(0) = 1.$
   (c) $x' = 3x + 9y, \quad y' = -x - 3y$ with $x(0) = 2, \quad y(0) = 4.$
5. The following model can be interpreted as describing the interaction of two species with population densities x and y:
   $$x' = x - 0.5y \quad y' = 0.25x + y.$$
   Use a phase plane analysis to determine the long-time behavior of a solution whose trajectory passes through the point (1,1).

6. Determine if the systems below could be interpreted as models for predatory-prey, competing, or cooperating species. Motivate your answer.

   (a) $x' = x(1 - x + y), \quad y' = y(4 - 3y - x)$.

   (b) $x' = x(1 - x - y), \quad y' = y(4 - 3y - x)$.

   (c) $x' = x(1 - x + y), \quad y' = y(4 - 3y + x)$.

7. Consider two interconnected tanks such that there is a transfer of mixture between the tanks in both directions through two pipes. Tank initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of of 1.5gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 4 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

   (a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process. Observe that the system in nonhomogeneous.

   (b) Find the values of Q_1 and Q_2 for which the system is in equilibrium, and denote them by Q_1^E and Q_2^E. Can you predict which tank will approach its equilibrium state more rapidly?

   (c) Let $x_1(t) = Q_1(t) - Q_1^E$ and $x_2(t) = Q_2(t) - Q_2^E$. Determine an initial value problem for x_1 and x_2. Observe that the system is homogeneous.

8. Compute the Laplace transform of the function

$$f(t) = \begin{cases} \sin t, & t < \pi \\ 3 - t, & t \geq \pi. \end{cases}$$

   in two ways: by using the definition, and by using the Heaviside (unit step) function.

9. Use the tables to find the inverse Laplace transform of

$$\frac{s+2}{4s^2+9}, \quad \frac{e^{-3s}}{4s}, \quad \frac{s}{s^2+2s+5}.$$