324 Introduction to PDEs
Instructor: Petronela Radu
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Review Problems for Exam 2

1. Are there any values a such that the functions $f(x) = x$ and $g(x) = \sin x$ are orthogonal in $L^2[0, a]$? Explain.

2. Use Duhamel's principle to find a solution for the problem

$$u_t - 2u_x + 3u = e^{3x-1}, \quad x \in \mathbb{R}, t > 0, \quad u(x, 0) = 0, \, x \in \mathbb{R}.$$

3. Use an energy argument to show that the solutions to the BVP

$$\begin{cases} u_t - k\Delta u = f(x,t) \text{ in } \Omega \times (0, \infty) \\ \dfrac{\partial u}{\partial n} + \alpha u = g \text{ on } \partial\Omega \\ u(x, 0) = h(x) \text{ in } \Omega \end{cases}$$

are unique for every constants $k, \alpha > 0$ and for every fnctions f, g, h.

4. Use Parseval's identity to compute

$$\sum_{k=2}^{\infty} c_k^2$$

where c_k are the Fourier coefficients on the interval $[-1, 1]$ for the function

$$f(x) = \begin{cases} -2x + 1, -1 \leq x \leq 0 \\ e^x, \quad 0 < x \leq 1. \end{cases}$$

Take $c_n^2 = a_n^2 + b_n^2, \quad n \geq 1$ and $c_0^2 = a_0^2/2$ where a_n and b_n are the classical Fourier coefficients. See pages 108-109 in the textbook for more details ($c_n = \gamma_n$ in the textbook notation).

Assume we extend f by periodicity to the entire axis, what type of convergence do we have for the Fourier series of f? To what value does the Fourier series of f converge for $x = 0$? How about for $x = 3$?

5. Use separation of variables to solve:

$$\begin{cases} u_t - ku_{xx} + mu = 0, \quad 0 < x < L, t > 0 \\ u(0,t) = 0, \quad u(L,t) = 0, t > 0 \\ u(x, 0) = f(x), \quad 0 < x < L. \end{cases}$$

6. Use separation of variables to solve the IBVP:

$$\begin{cases} u_t - ku_{xx} = 0, \quad 0 < x < 2, t > 0 \\ u_x(0,t) = 0, \quad u(2,t) = 0, \quad t > 0 \\ u(x,0) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 \leq x \leq 2. \end{cases} \end{cases}$$

7. Given that $\mathcal{F}(e^{-|x|}) = \frac{2}{1+\xi^2}$, use Fourier transforms to find a solution of

$$u''(x) - u(x) = f(x), \quad -\infty < x < \infty$$

Recall that $\mathcal{F}(f^{(k)}) = (-i\xi)^k \mathcal{F}(f)$.

8. Use Poisson's integral formula to solve:

$$\begin{cases} \Delta u = 0, \quad r < R, 0 \le \theta < 2\pi \\ u(R,\theta) = 2 - \sin 3\theta, 0 \le \theta < 2\pi. \end{cases}$$

Give a physical interpretation to the above BVP.

Recall Poisson's formula:

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(\phi)}{R^2 + r^2 - 2rR\cos(\theta - \phi)} d\phi.$$