221 Differential Equations
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Sample Problems for Test 3

1. Detrmine if the systems below could be interpreted as models for predatory-prey, competing, or cooperating species. Motivate your answer.

   (a) $x' = x(1 - x + y), \quad y' = y(4 - 3y - x).$
   (b) $x' = x(1 - x - y), \quad y' = y(4 - 3y - x).$
   (c) $x' = x(1 - x + y), \quad y' = y(4 - 3y + x).$

2. Consider a population of bacteria whose growth is modeled by the differential equation $p' = p^3 - 4p^2 + 3p$. Using a phase line analysis decide what is the long term prognosis of this population depending on the initial value of the population $p_0 \geq 0$.

3. For the differential system **x**'(t)=A**x**(t):

   (a) Draw the nullclines and do a nullcline analysis;
   (b) Find the general solution;
   (c) Discuss the stability of the origin based on parts (a) and (b);
   (d) Draw some trajectories to illustrate what type of a critical point the origin is.

   where

   (a) $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$
   (b) $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$
   (c) $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$

4. Decide if the following statements are TRUE or FALSE. Motivate your answers.

   (a) A center is a stable point.
   (b) A center is an asymptotically stable point.
   (c) A saddle point is an asymptotically unstable point, but some trajectories move towards it.
   For the next statements assume that the origin is the only critical point of $x' = Ax$.
   (d) If A has real negative eigenvalues, then the origin is a sink.
   (e) If the origin is a source then all trajectories that start outside the origin are unbounded.
   (f) If one of the eigenvalues has the real part equal to 0, then the origin is a center.

5. Solve the following initial value problems:

   (a) $x' = 5x - y, \quad y' = 3x + y$ with $x(0) = 2, \quad y(0) = -1.$
   (b) $x' = x - 5y, \quad y' = x - 3y$ with $x(0) = 1, \quad y(0) = 1.$
   (c) $x' = 3x + 9y, \quad y' = -x - 3y$ with $x(0) = 2, \quad y(0) = 4.$

6. The following model can be interpreted as describing the interaction of two species with population densities x and y:

   $$x' = x - 0.5y \quad y' = 0.25x + y.$$

   Use nullcline analysis to determine the long-time behavior of a solution whose trajectory passes through the point (1,1).

7. Nondimensionalize the system

$$\frac{dw}{dt} = kp, \quad \frac{dp}{dt} = rp(1 - \frac{p}{M}) - bpw$$

with the change of variables

$$P = \frac{p}{M}, \quad W = \frac{bw}{r}, \quad \tau = rt$$

to obtain

$$W' = KP, \quad P' = P(1 - P - W),$$

where K is a dimensionless parameter. Change the dimensionless equation for P such that P is not limited by logistic growth, but could have exponential growth.

8. In a city the number of jobs j is proportional with the population p, and both depend on time. The population is restricted by the environment to logistic growth and is also proportional with the number of jobs. Assuming that the jobs are cut every year by the local administration by a constant number b, write a differential system for this model.