314 - Matrix Algebra
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Sample Problems for Exam 2

1. Use Cramer's Rule to find the inverse of the matrix A associated with the system:

$$\begin{cases} 2x & +y & -3z = 1 \\ & -y & +z = 0 \\ x & -y & -2z = 2. \end{cases}$$

   Use A^{-1} obtained above to find the solution of the system.

2. Given the matrix A

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -2 & 0 \\ 6 & 0 & 0 \end{bmatrix}$$

   find a basis for Col A, Row A and Nul A. What does the rank theorem says about rank A?

3. Determine if the following statements are TRUE or FALSE. Explain.

   (a) The volume of the parallelepiped determined by the columns of A is det A.

   (b) If two rows of a 3×3 matrix are the same, then det A=0.

   (c) If $A^5 = 0$, then det A =0.

   (d) If A is 3×3 and det A=2, then det A^3=6.

   (e) Any system of n linear equations in n variables can be solved by Cramer's rule.

   (f) The set of all solutions of a homogeneous linear differential equation is the kernel of a linear transformation.

   (g) The rows of an $n \times n$ invertible matrix form a basis for $\mathbb{R}^n$.

   (h) Every subspace of $\mathbb{R}^3$ of dimension 1 is a line that passes through the origin.

   (i) The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ have the same dimension.

   (j) The dimensions of the row space and the column space of A are the same, even if A is not square.

   (k) The eigenvalues of an upper triangular matrix A are exactly the nonzero entries on the diagonal of A.

4. Consider the matrix

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

   (a) Find the eigenvalues of the matrix A.

   (b) Find a bsis for each of the eigenspaces of A.

   (c) Write the characteristic equation for A and explain why A is diagonalizable.

(d) Diagonalize A.

5. Show that if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

6. Show that the set of vectors (x, y, z) in $\mathbb{R}^3$ whose cordinates satisfy $2x - y + 3z = 0$ is a vector subspace of $\mathbb{R}^3$. Find a basis for this vector subspace.

7. Show that the set of vectors $B = \{3 - t, -1 + t, 1 - t^2\}$ is a basis for $\mathbb{P}_2$. Find the change-of-coordinates matrix from the standard basis to B. Use this matrix to find the B-coordinates of $\mathbf{p} = 3 + 2 + t - 4t^2$.

8. Find the dimension of the subspace of all vectors (x, y, z, w) in $\mathbb{R}^4$ whose coordinates satisfy $x = 2z$ and $y = 3w$.

9. Suppose that A is a 4×3 matrix and that Nul A $= \{\ \mathbf{0}\}$. Find rank A and explain your answer.

10. Consider the two sets below contained in $\mathbb{R}^4$. Decide whether or not they are subspaces of $\mathbb{R}^4$. If so, find a basis. If not, prove it.

(a) $V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \text{ where } a = 3b + 1,\ c = -d, \text{ and } a, b, c, d \in \mathbb{R} \right\}.$

(b) $W = \left\{ \begin{bmatrix} a - b \\ b + c \\ c - d \\ d + a \end{bmatrix}, \text{ where } a, b, c, d \in \mathbb{R} \right\}.$

11. Let $B = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$. Find $[v]_B$, where $v = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. Use the change of coordinates matrix, to find $[v]_C$.