Trisections of Surface Bundles over Surfaces

January 17, 2019

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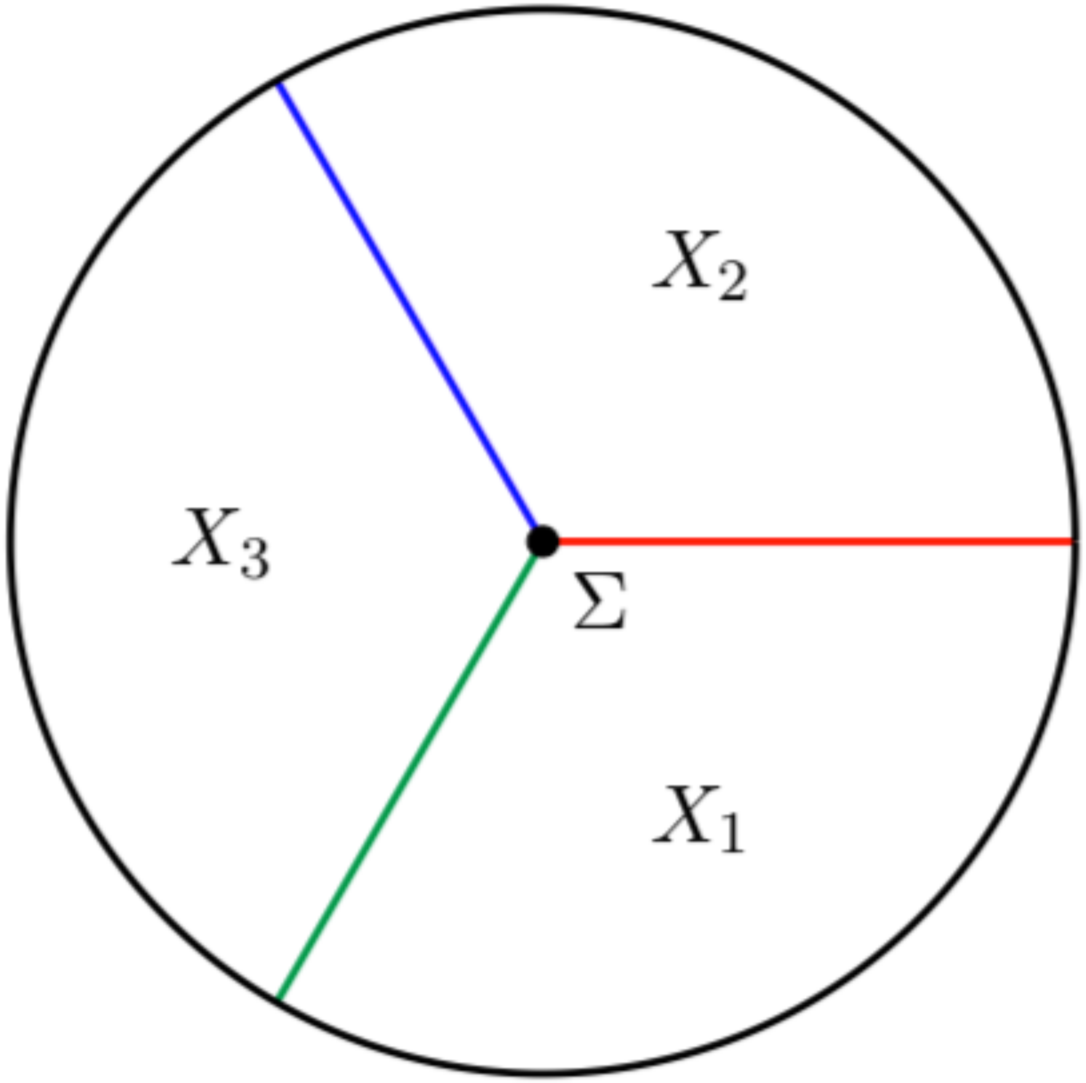
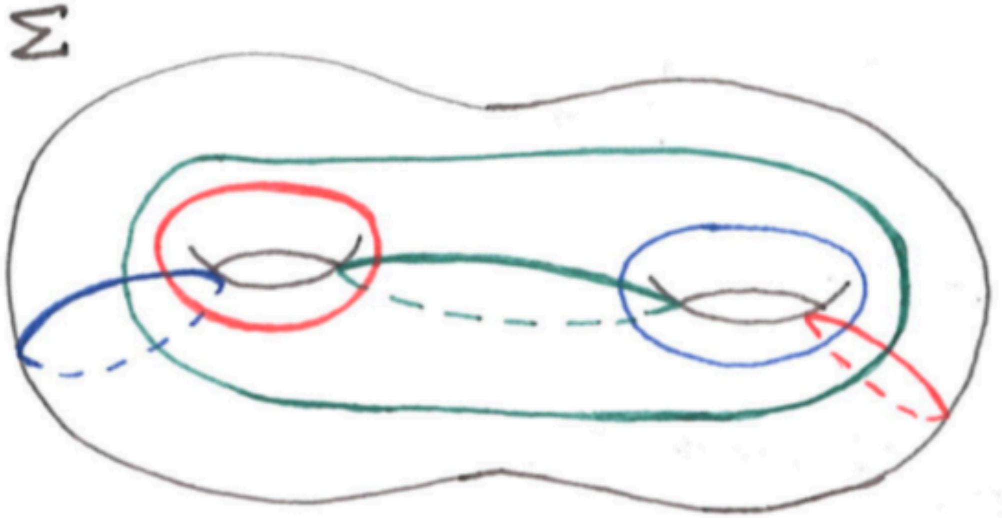
Trisections

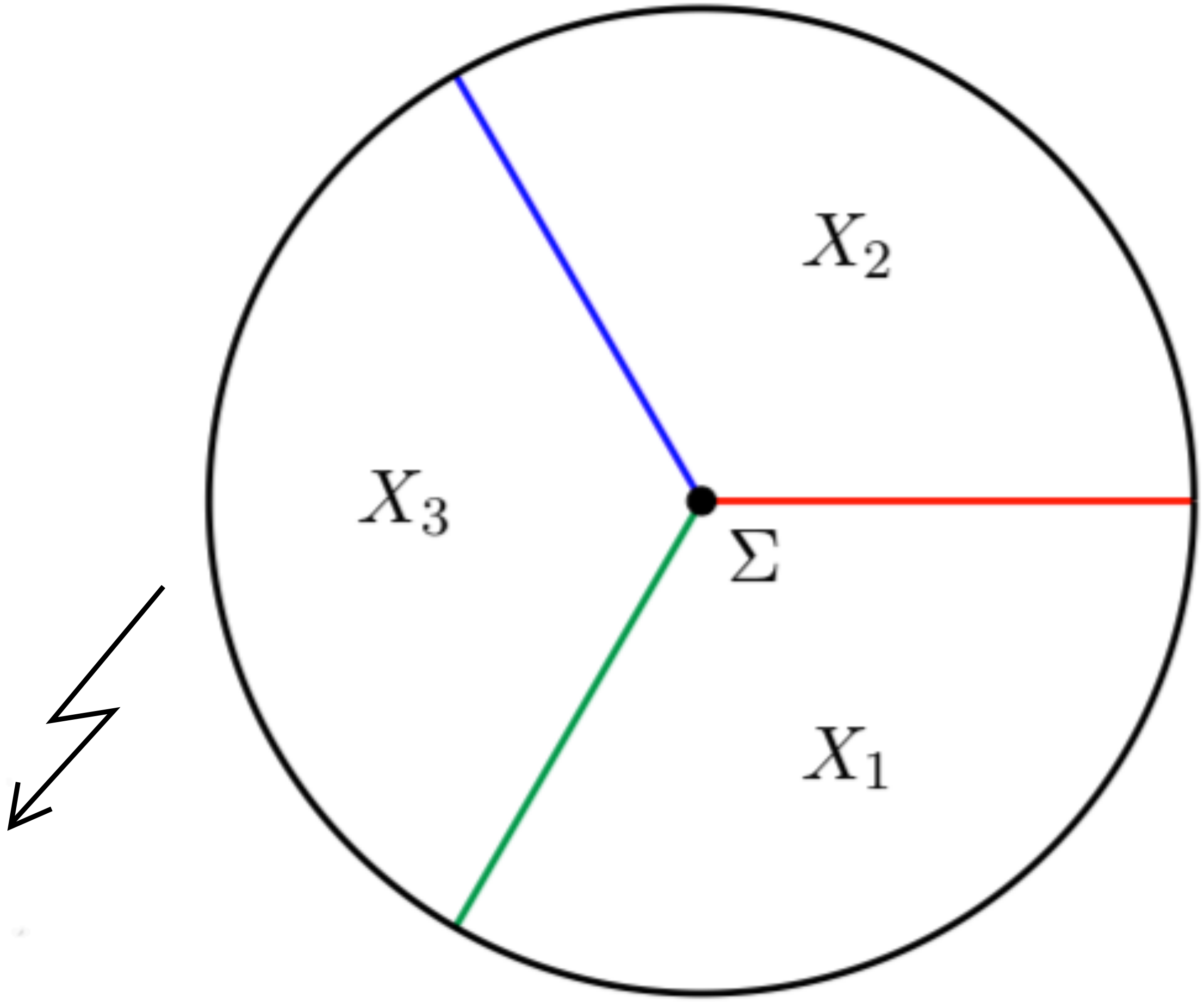
- Trisection: decomposes a 4-manifold into three simple pieces (handlebodies)
- Existence: Gay/Kirby 2012
    - smooth
    - closed
    - orientable
    - connected



Trisections

- Trisection: decomposes a 4-manifold into three simple pieces (handlebodies)
- Existence: Gay/Kirby 2012
- Diagrams:





Past Work

- Existence: smooth, orientable, closed (Gay/Kirby 2012); smooth, orientable, compact (Castro/Gay/Pinzón 2016, 2018)
- Genus 2 trisections standard (Meier/Zupan 2014)
- Lefschetz fibrations (Gay 2015, Castro/Ozbagci 2017, Baykur/Saeki 2017)
- Diagrams: 3-manifold bundles over S^1 (Koenig 2017)
- Diagrams: spun 4-manifolds (Meier 2017)
- Bridge trisections of knotted surfaces: in S^4 (Meier/Zupan 2015); in 4-manifolds (Meier/Zupan 2017); in $\mathbb{CP}^2$ (Lambert-Cole 2018)

$\underbrace{\underline{\text{Surface}}}_{\textit{fiber}}$ Bundle over a $\underbrace{\underline{\text{Surface}}}_{\textit{base}}$

- Bundle: locally a product of the *base* manifold with the *fiber* manifold; not necessarily a global product.

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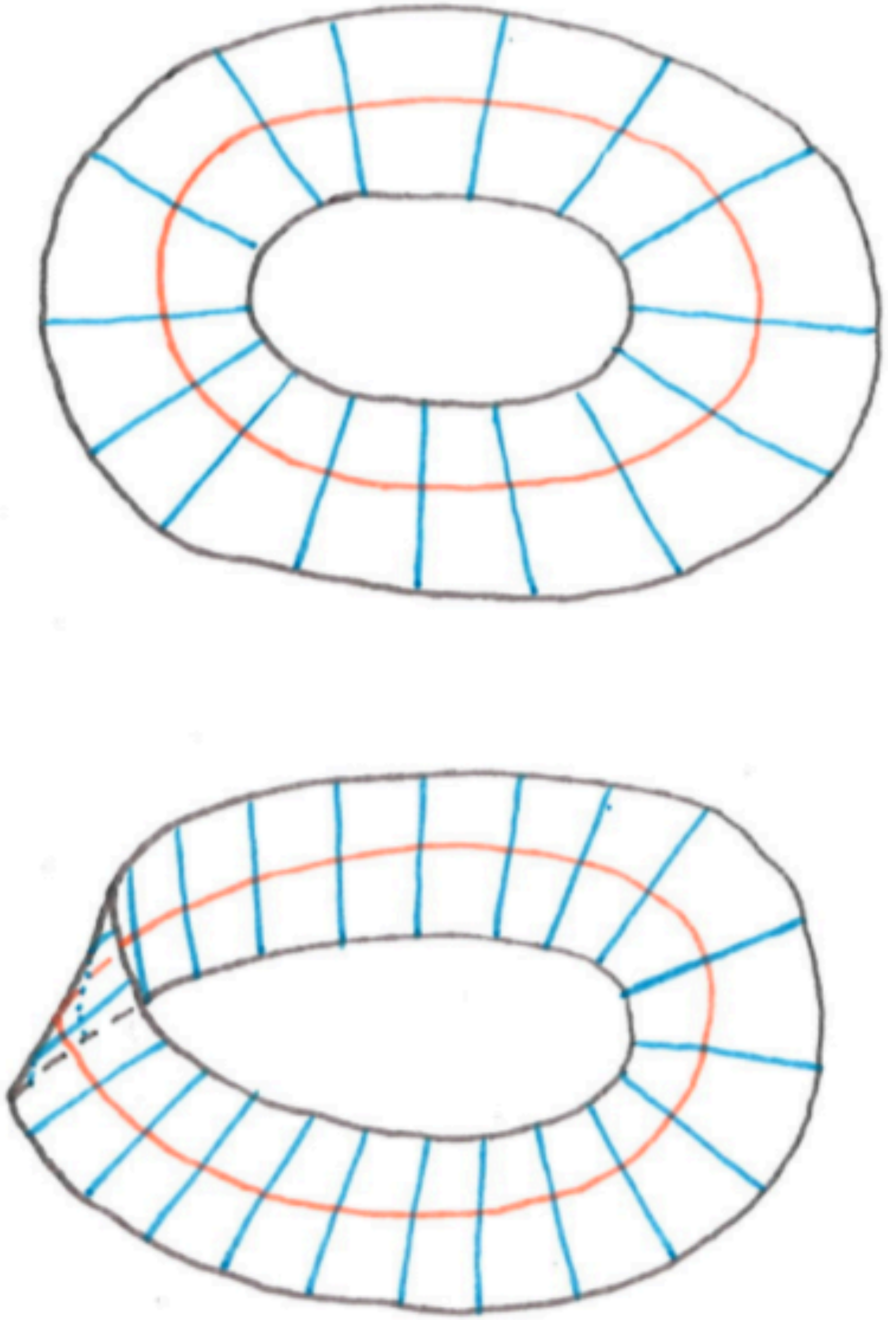
- Examples:

Base: S^1

Fiber: I



Current Work

- Gives an algorithm for drawing trisection diagrams of surface bundles over surfaces, given the monodromy representation of the bundle

Theorem (W. 2018) These diagrams are minimal when

$$\mathrm{rk}\,\pi_1 = 4 - \chi_b - \chi_f$$

where χ_b and χ_f are the euler characteristics of base and fiber

Diagram Construction

$$n := 3 - \chi_b$$

- Decompose base into three $2n$-gons
  - pairwise intersect at n edges
  - triply intersect at $2n$ vertices

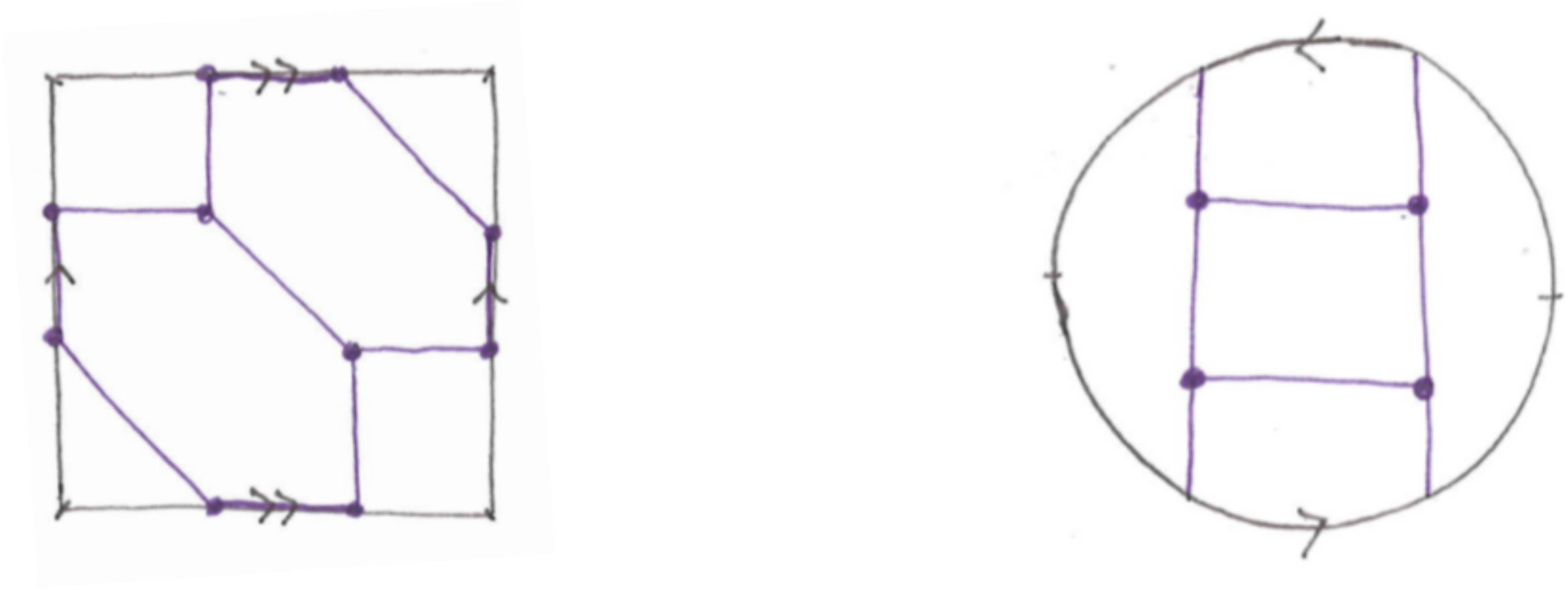


Diagram Construction

Example: $S^2 \times T^2$

$$n := 3 - \chi_b$$

- Decompose base into three $2n$-gons
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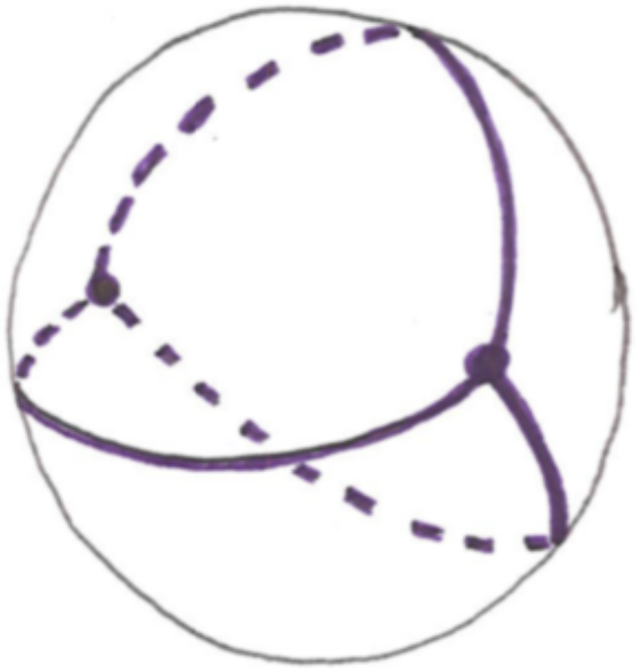
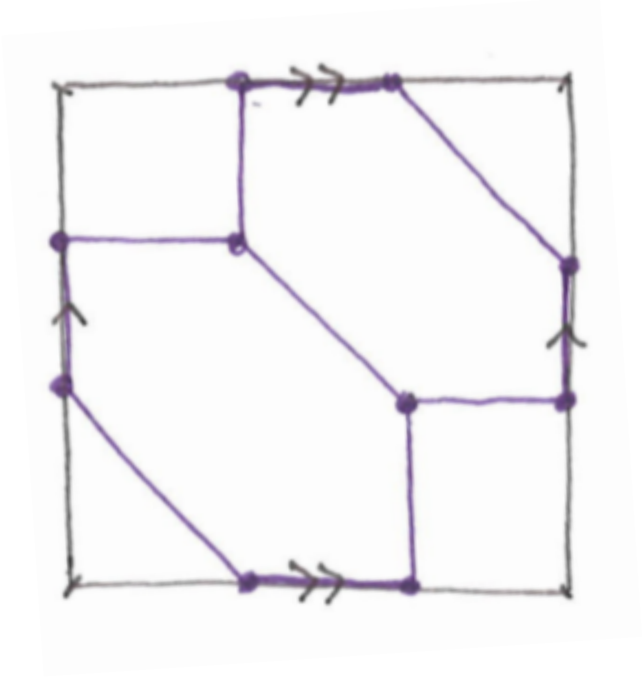




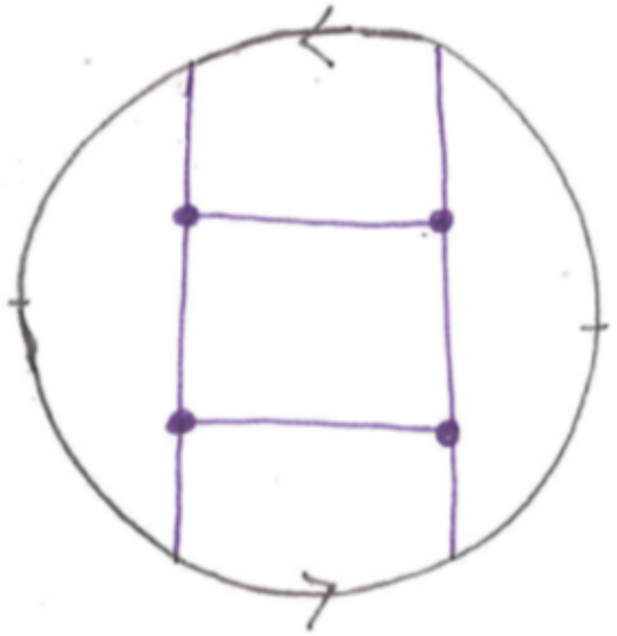
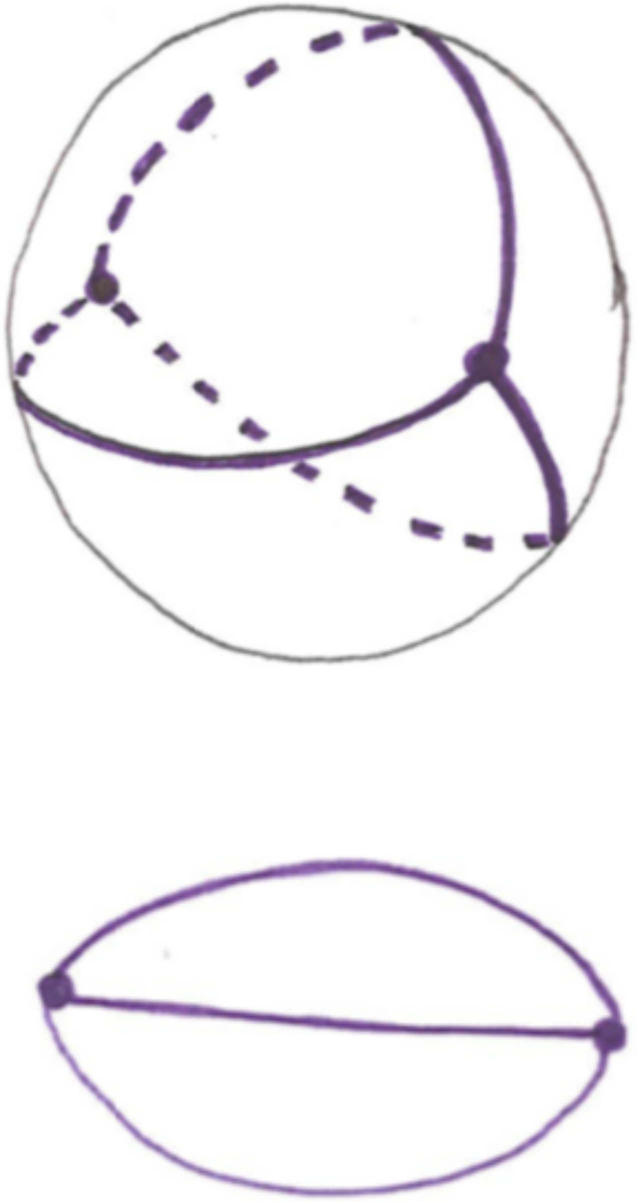


Diagram Construction

Example: $S^2 \times T^2$

$$n := 3 - \chi_b$$

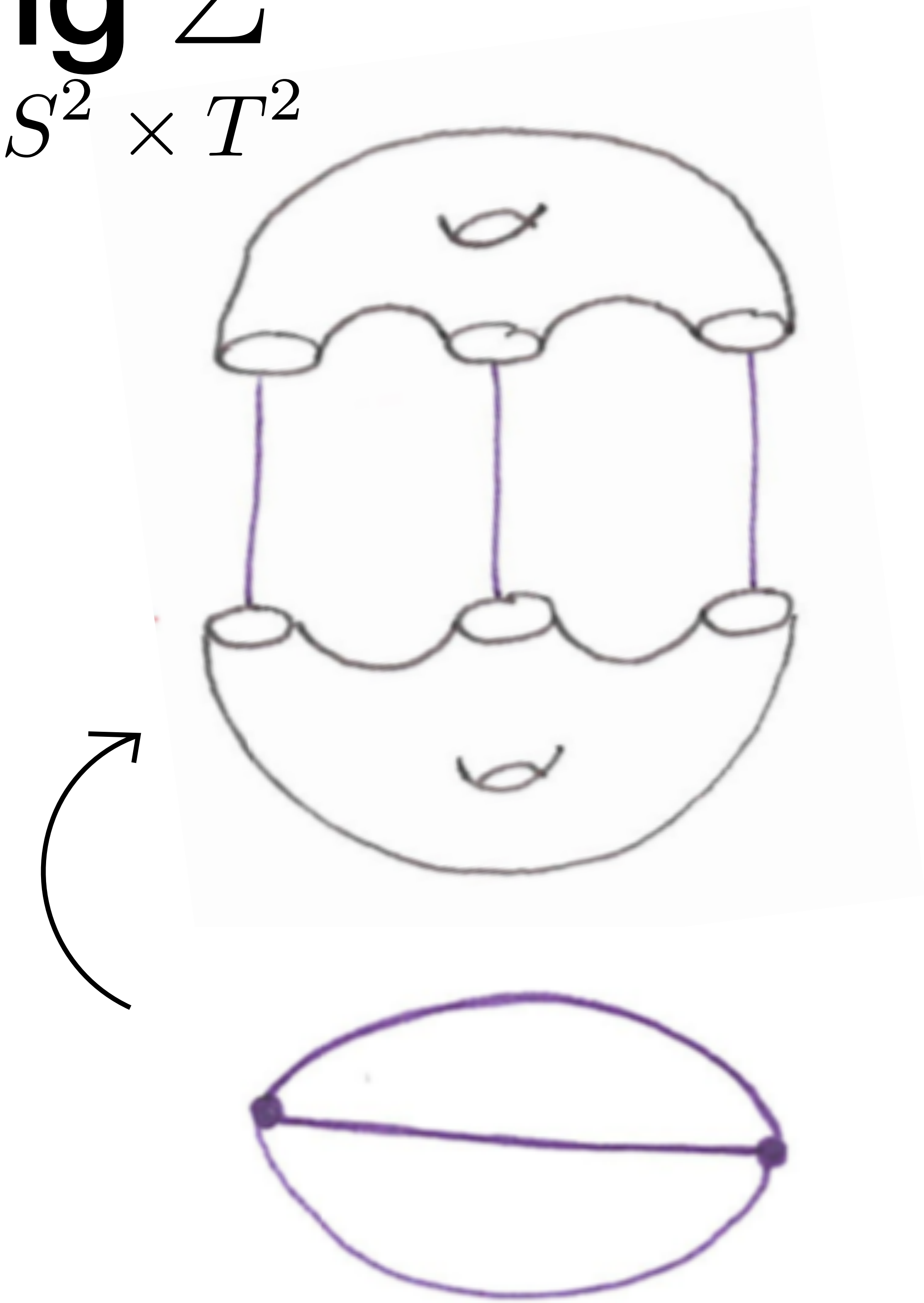
- Decompose base into three $2n$-gons
  - pairwise intersect at n edges
  - triply intersect at $2n$ vertices
- Build surface from 1-skeleton



Building Σ

Example: $S^2 \times T^2$

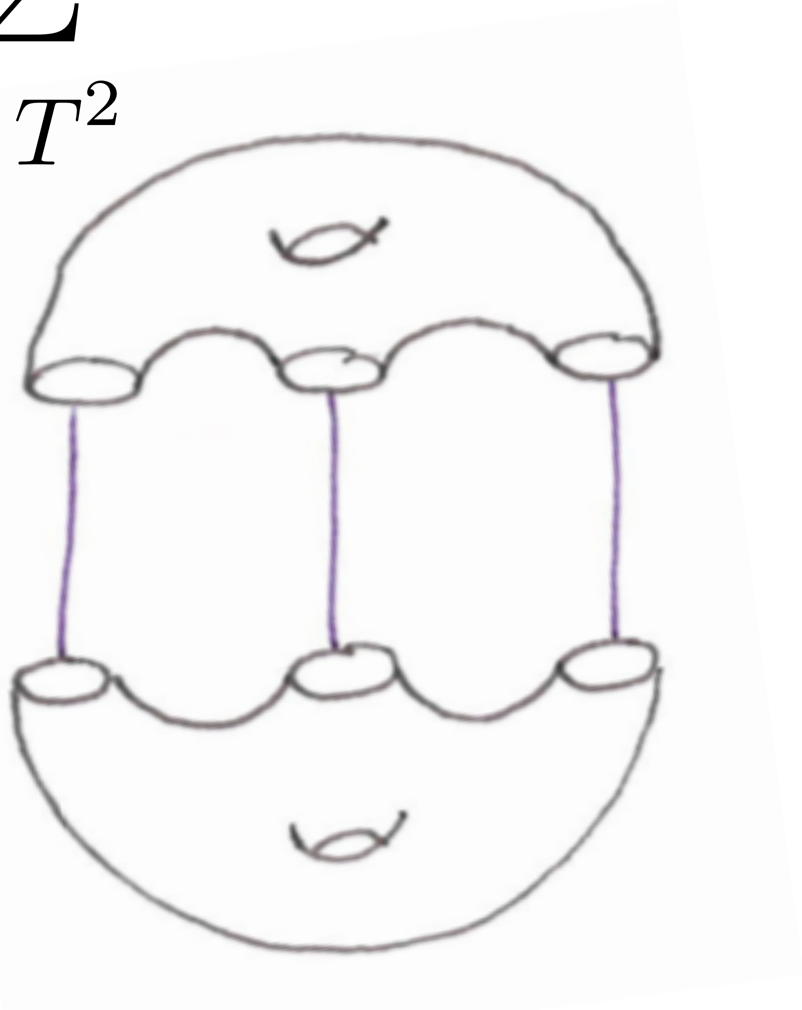
- Replace each vertex with 3-punctured fiber

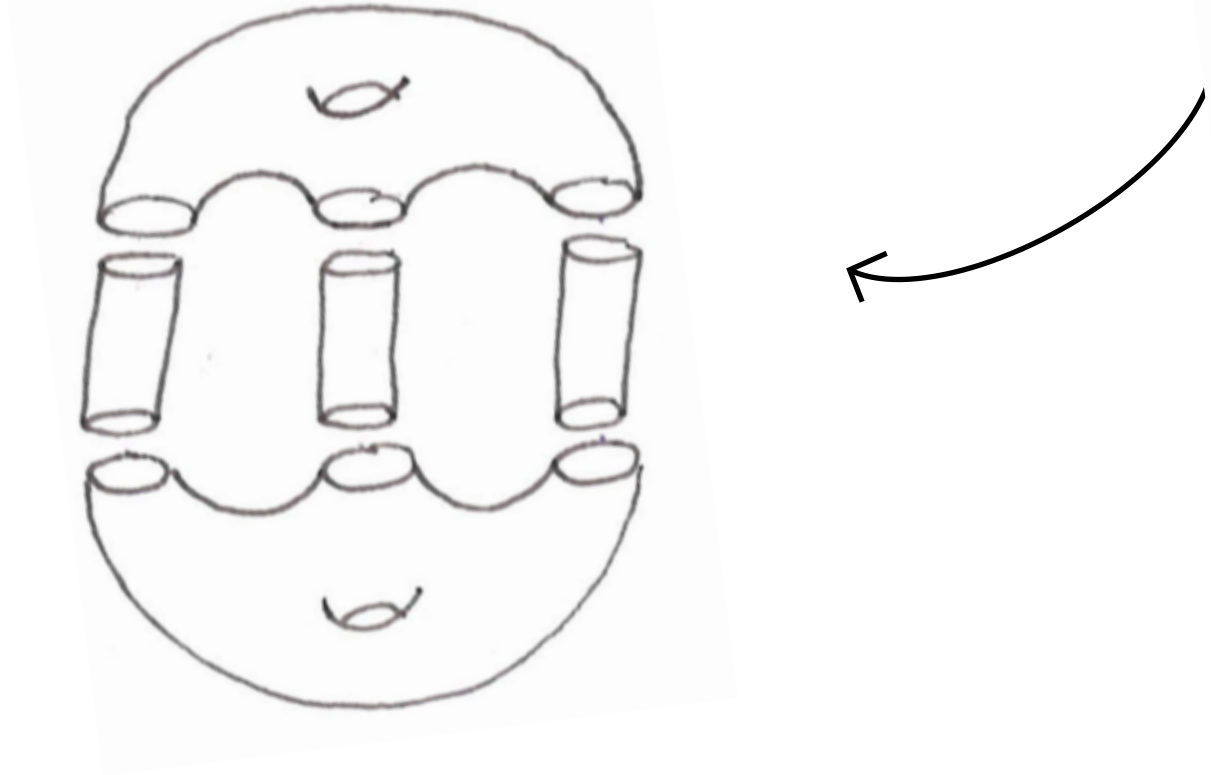


Building Σ

Example: $S^2 \times T^2$

- Replace each vertex with 3-punctured fiber
- Replace each edge with $I \times S^1$

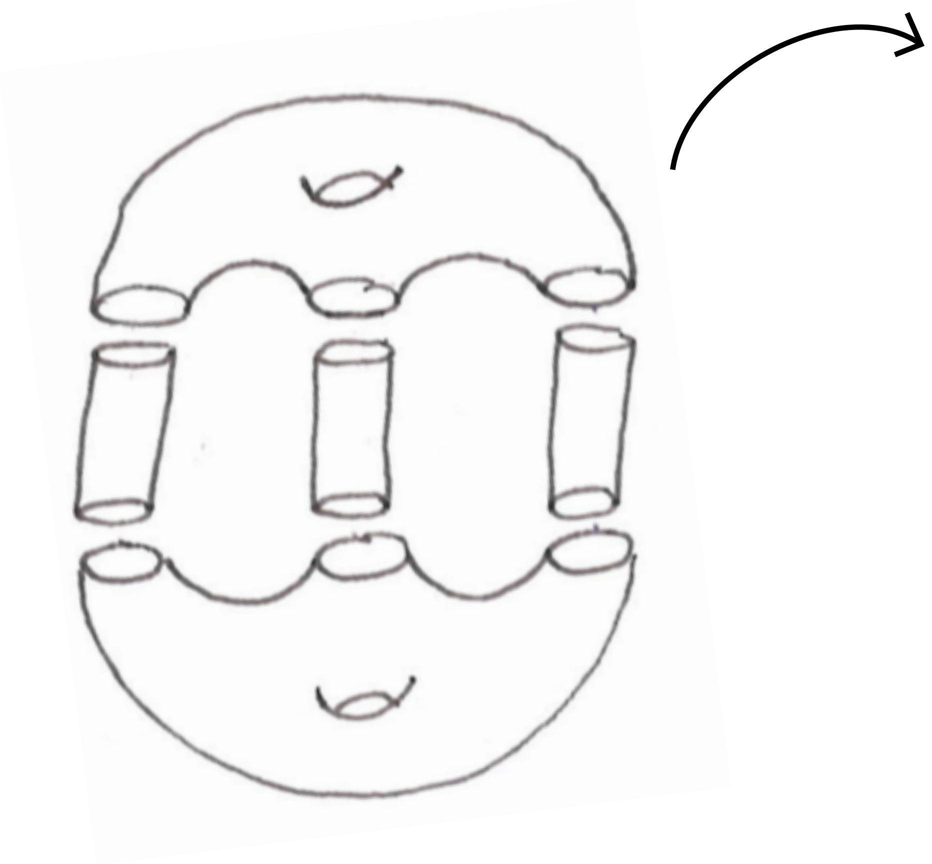




Building Σ

Example: $S^2 \times T^2$

- Replace each vertex with 3-punctured fiber
- Replace each edge with $I \times S^1$
- Glue



Building Σ

Example: $S^2 \times T^2$

- Replace each vertex with 3-punctured fiber
- Replace each edge with $I \times S^1$
- Glue
- This yields a genus

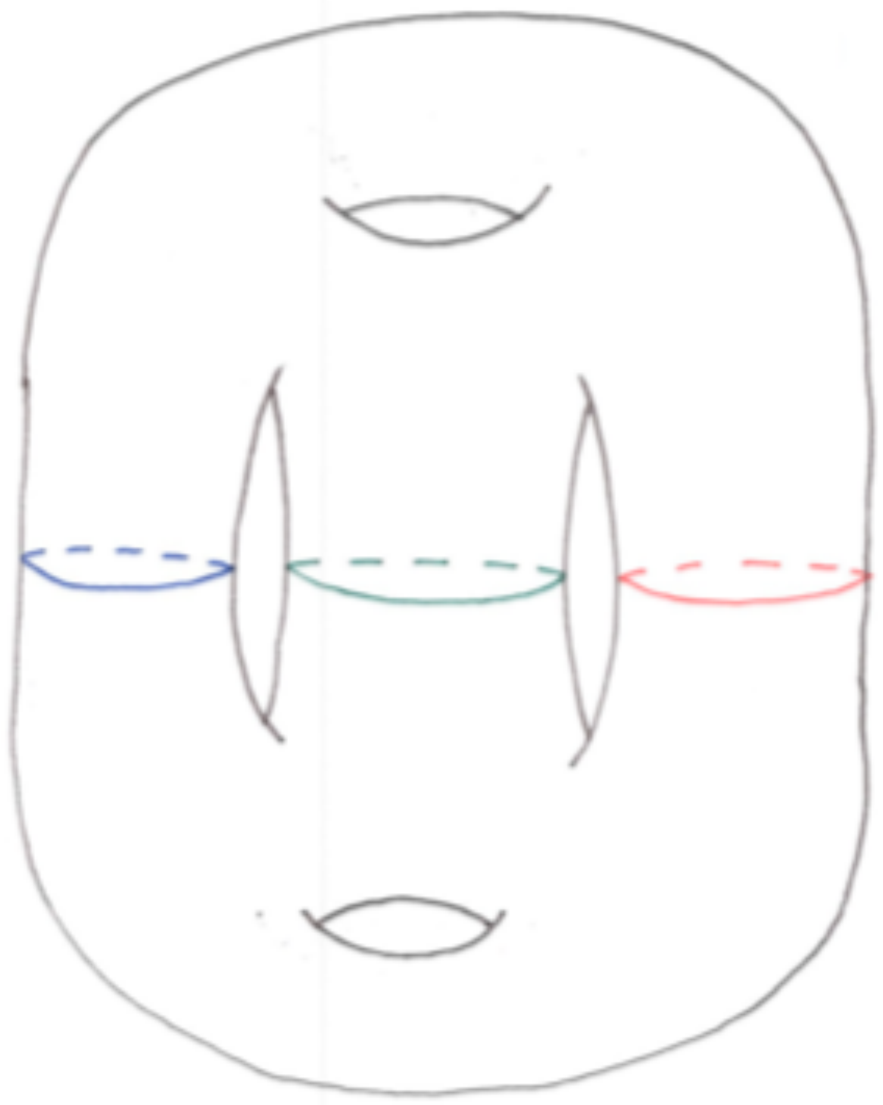
  $$(3 - \chi_b)\,(3 - \chi_f) + 1$$

  surface, where χb and χf are the euler characteristics of base and fiber

Curve Placement

Example: $S^2 \times T^2$

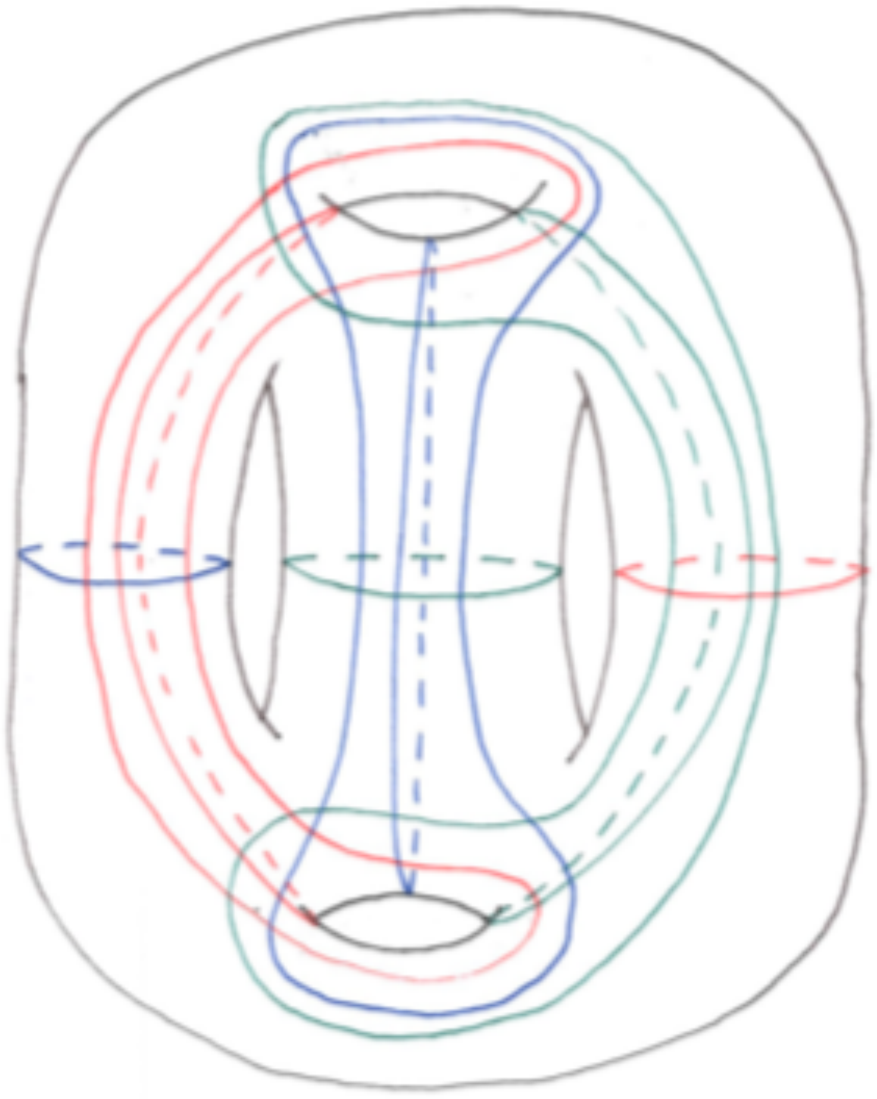
- Meridional curves on tubes



Curve Placement

Example: $S^2 \times T^2$

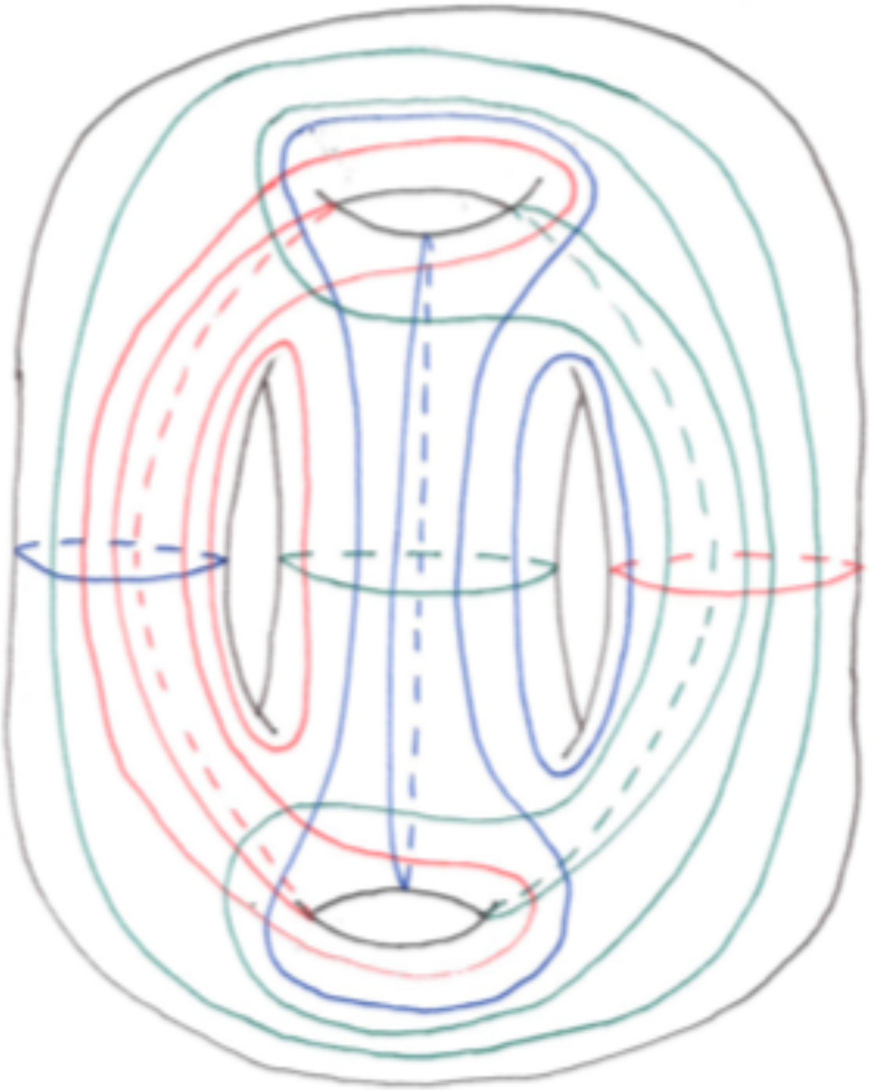
- Meridional curves on tubes
- Paired arcs across tubes, for fiber genus



Curve Placement

Example: $S^2 \times T^2$

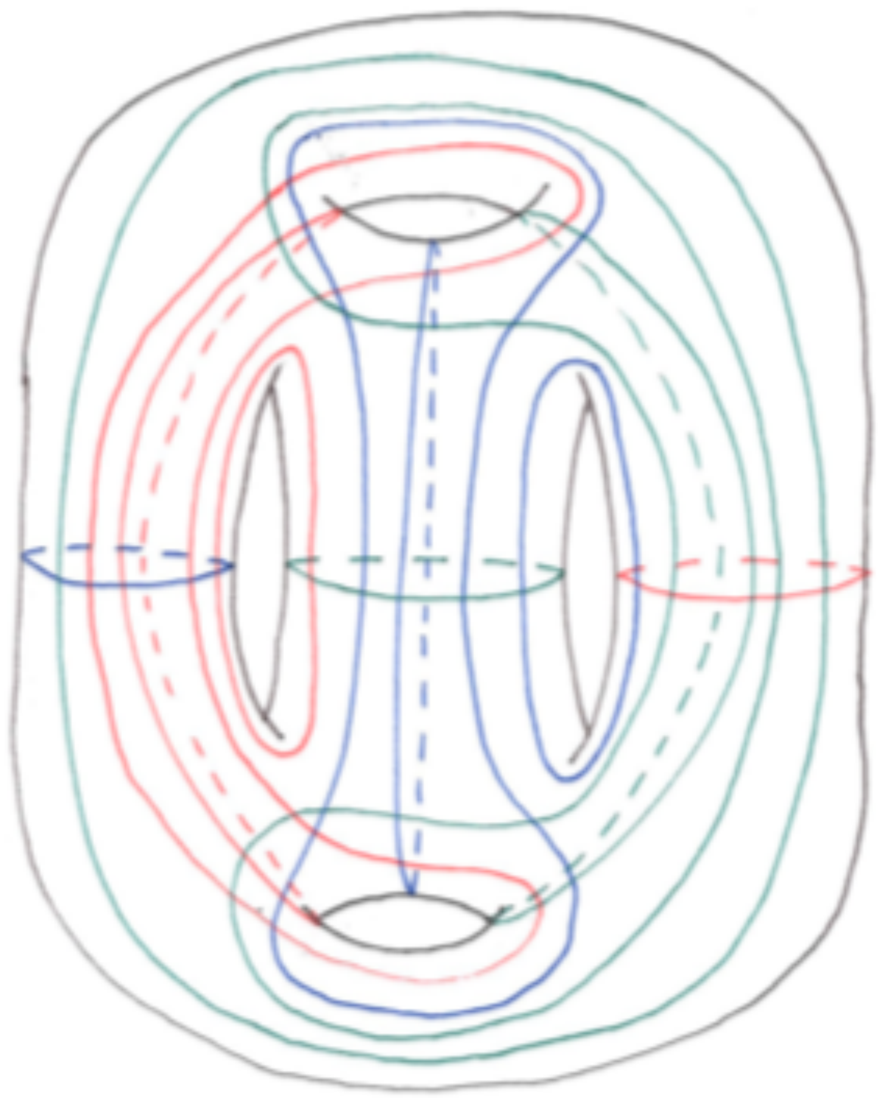
- Meridional curves on tubes
- Paired arcs across tubes, for fiber genus
- Long curves



Curve Placement

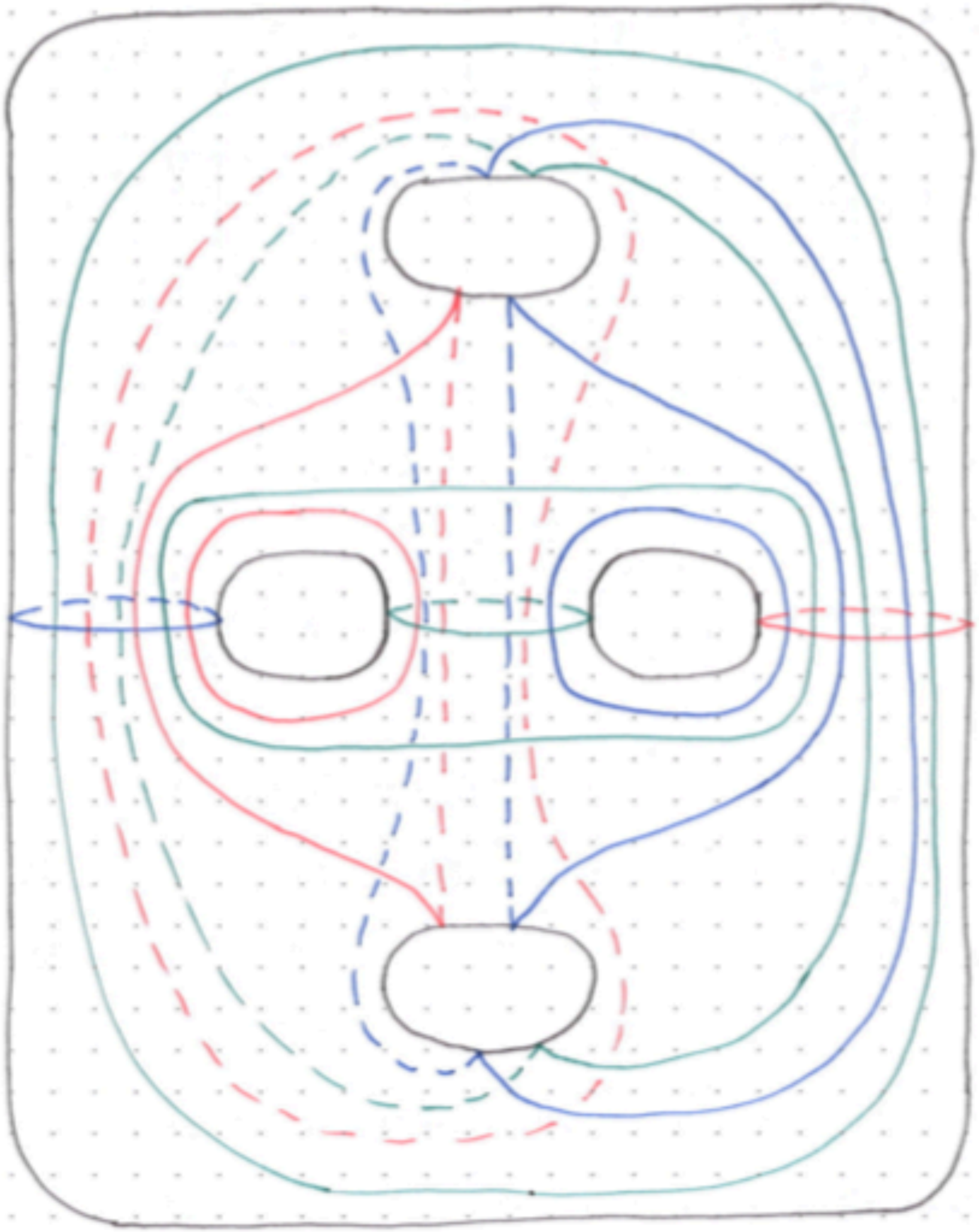
Example: $S^2 \times T^2$

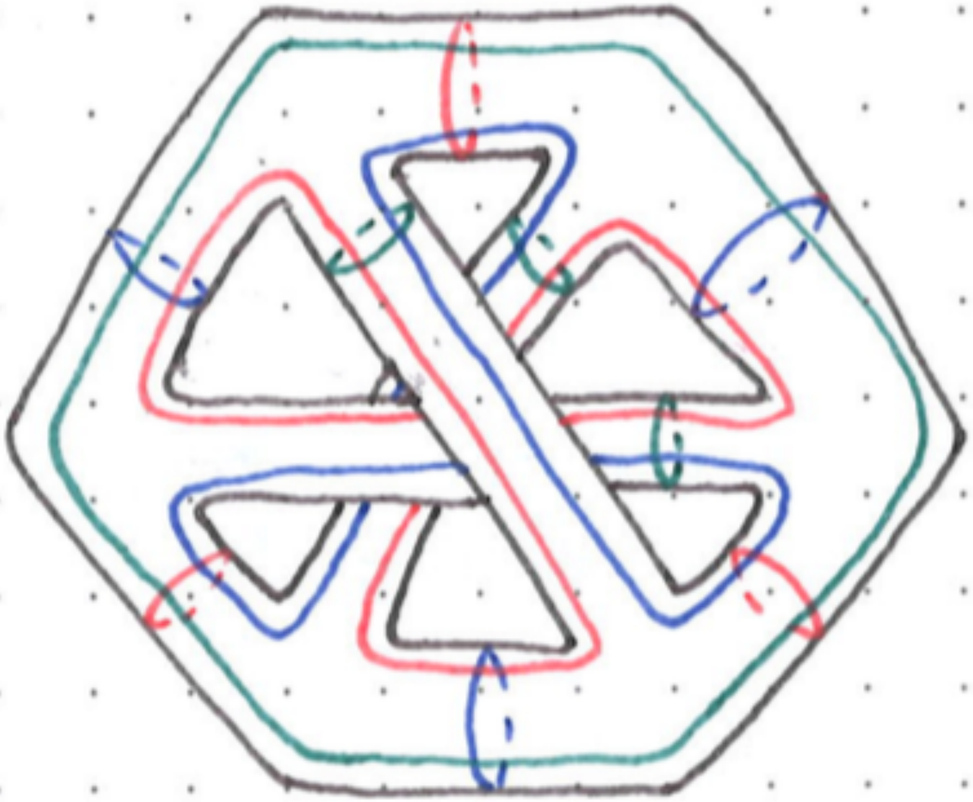
- Meridional curves on tubes
- Paired arcs across tubes, for fiber genus
- Long curves
- Adjust for nontrivial monodromy

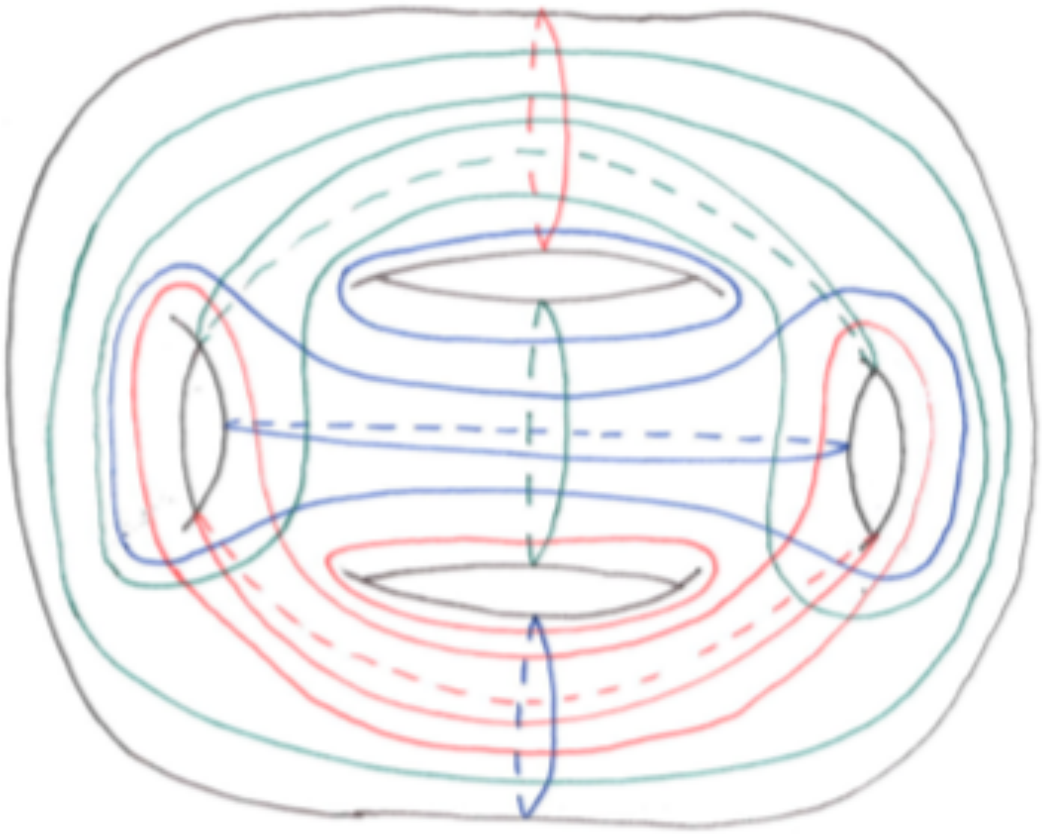


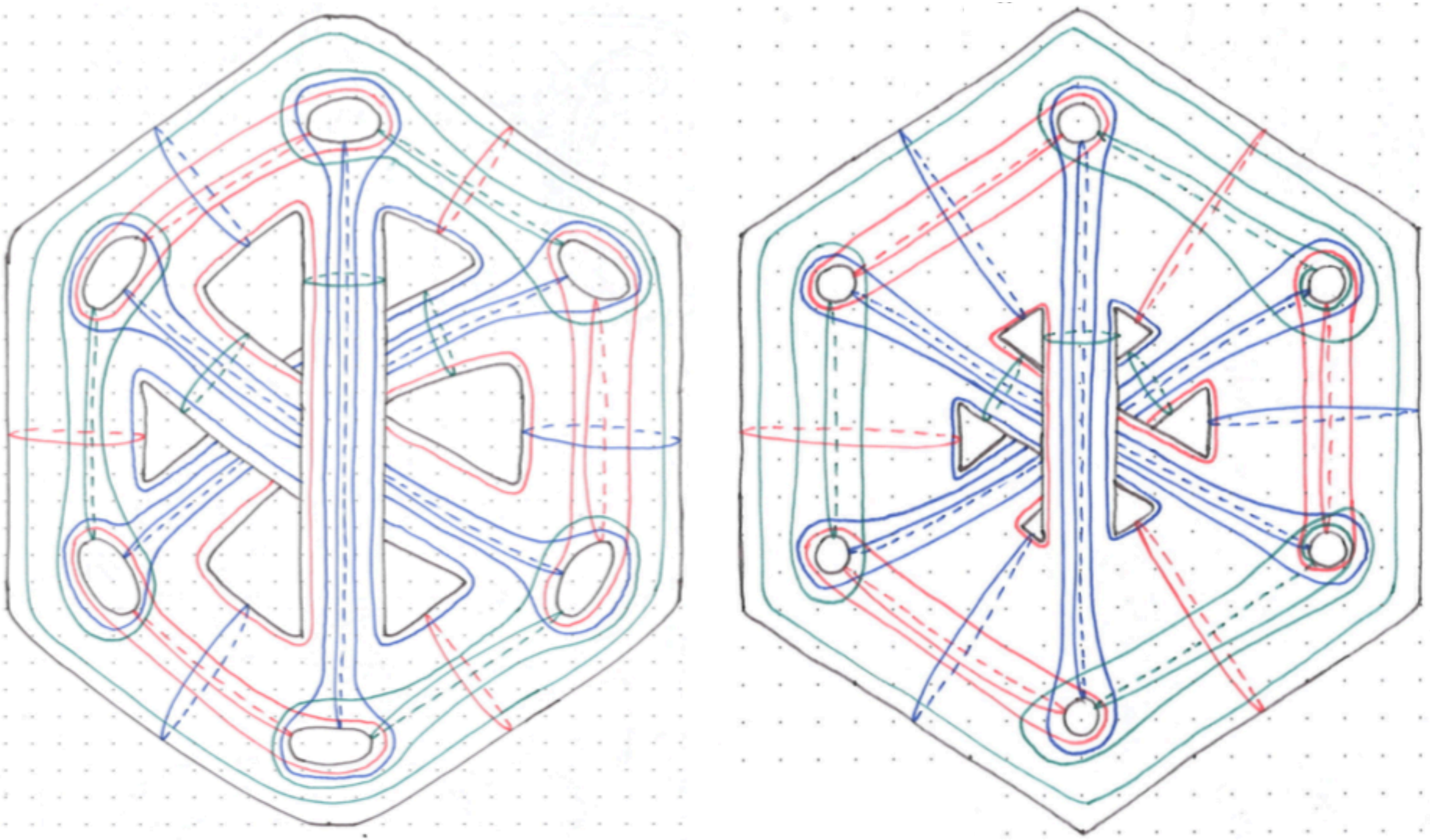
Future Work

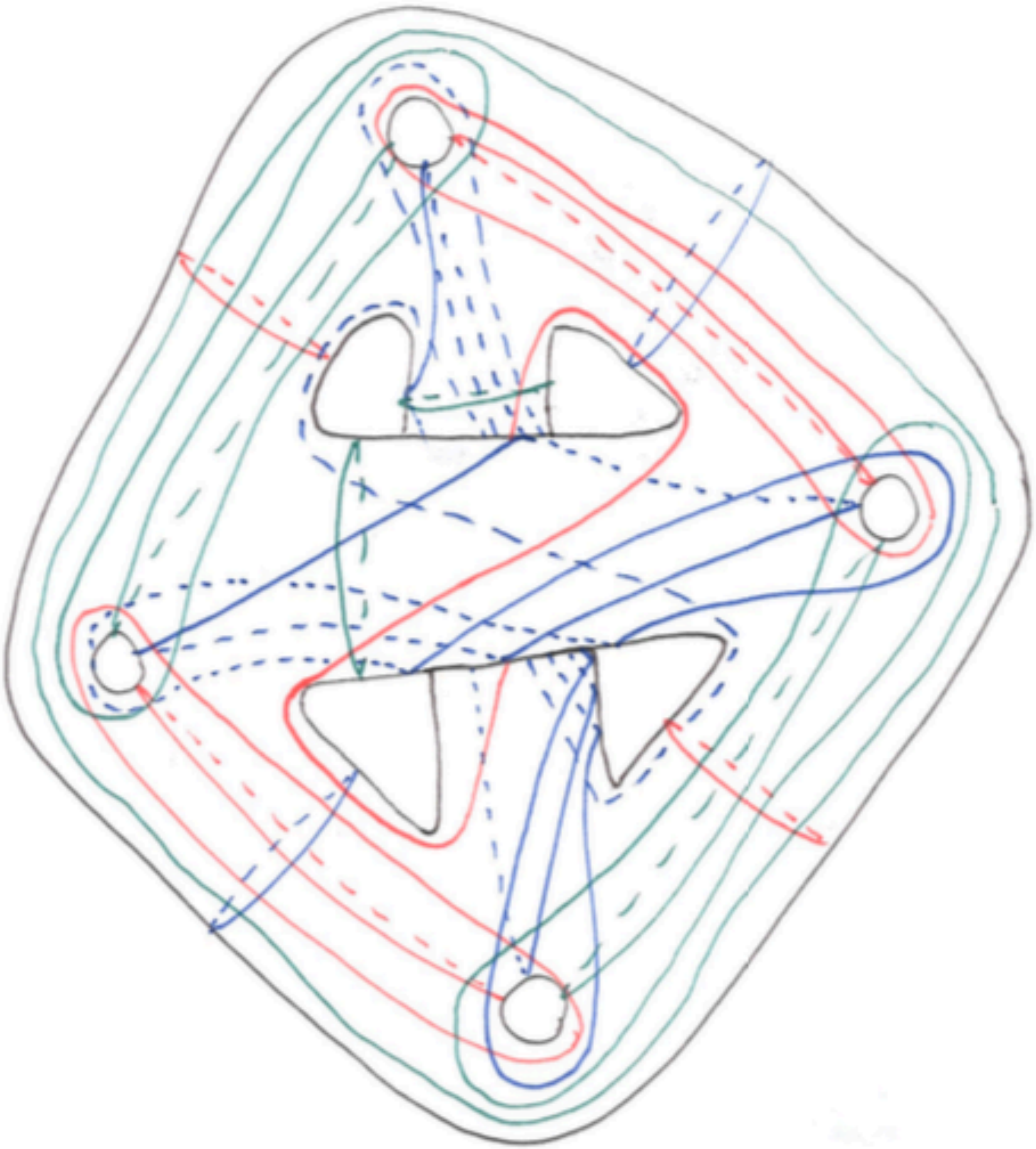
- Adapting to the relative case (manifolds with boundary)
- Bridge trisections of fiber or base within these trisections
- Submanifolds seen in the diagram (e.g. T^3 inside T^4)
- Monodromy conditions that give full rank π_1
- Find small genus surface bundles from small genus fibered 3-manifolds

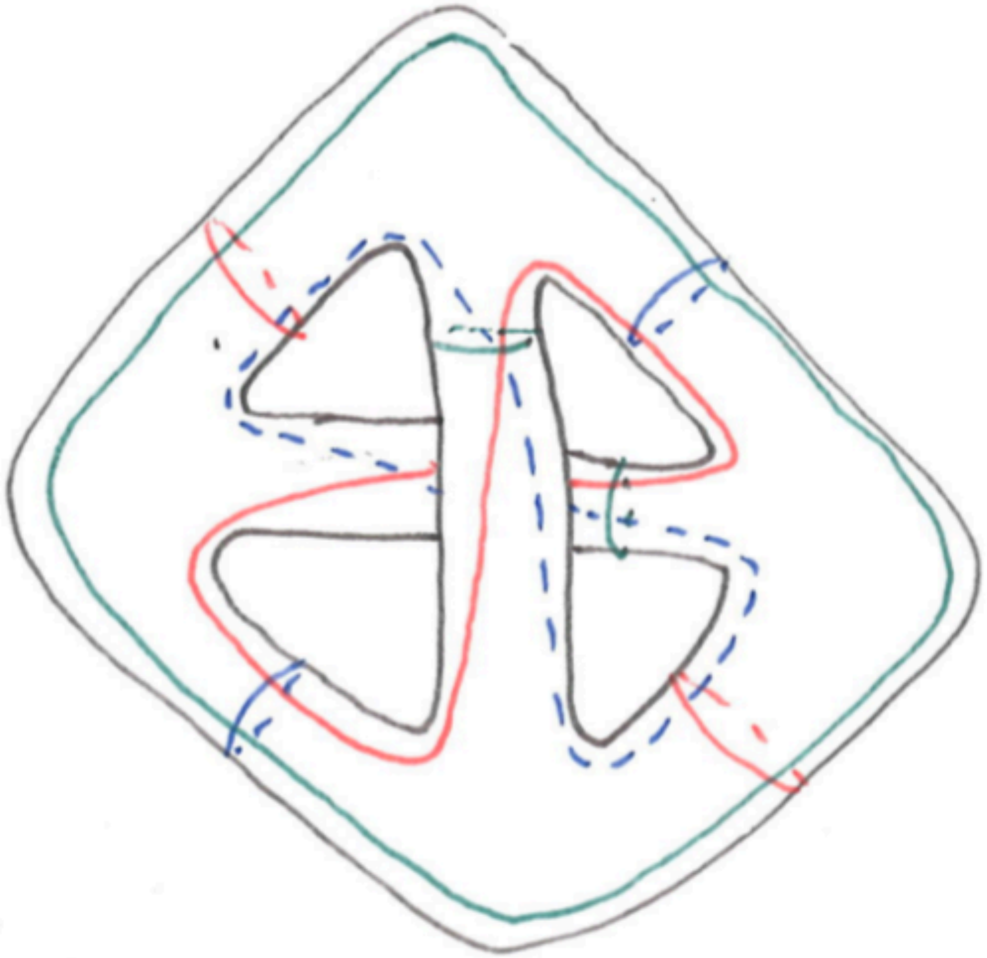


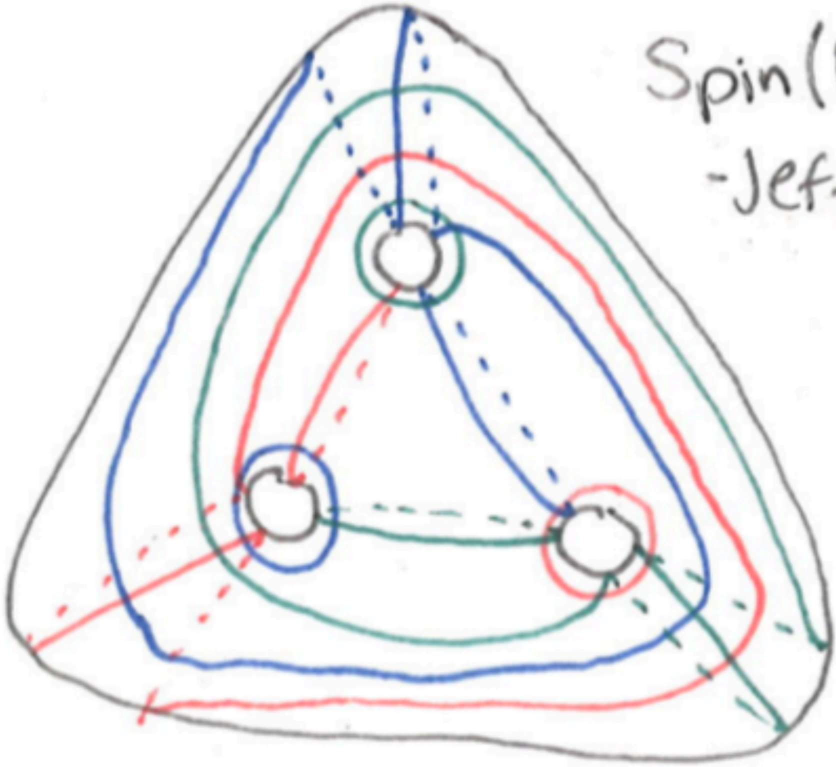












Spin($\mathbb{RP}^3$)

-Jeff Meier