On Trisections of 4-manifolds

Constructing Diagrams for Products of Surfaces

Marla Williams

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Disclaimer

For the purposes of this talk, "4-manifold" means "smooth, closed, orientable, and connected 4-manifold".

Trisections of 4-manifolds
Marla Williams
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Definition

Given a 4-manifold X and integers $0 \leqslant k \leqslant g$, a (g,k)-*trisection* of X is a decomposition $X = X_1 \cup X_2 \cup X_3$, satisfying:

1. Each X_i is diffeomorphic to $\natural^k(S^1 \times B^3)$
2. Each pairwise intersection $X_i \cap X_j = \partial X_i \cap \partial X_j$ is diffeomorphic to $\natural^g(S^1 \times B^2)$
3. The triple intersection is a genus g surface: $X_1 \cap X_2 \cap X_3 = \partial X_1 \cap \partial X_2 \cap \partial X_3 = \Sigma_g$

Remarks

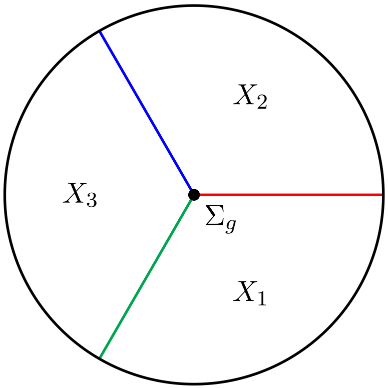
Theorem (Gay-Kirby 2013)

Every (smooth closed orientable connected) 4-manifold has a trisection.

- Trisections can be related to handle decompositions
- $\min(k_i) \geqslant \mathrm{rk}\pi_1(X)$

Schematic

Schematically, we may think of a trisection of X as the following:



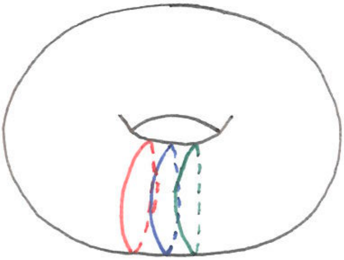
Trisection Diagrams

Definition

A (g,k)-*trisection diagram* is a 4-tuple $(\Sigma_g, \alpha, \beta, \gamma)$ where each triple $(\Sigma_g, *, *)$ is a genus g Heegaard diagram for

$$\partial X_i = \#^{k_i}(S^1 \times S^2).$$

Example: A (1,1)-trisection diagram for $S^1 \times S^3$: $(T^2, \alpha, \beta, \gamma)$



Surface Products & Bundles over S^2

Previous work: Gay and Kirby gave a construction for a $(8g+5, 4g+1)$-trisection of a surface bundle over S^2 with fiber Σ_g.

This talk: The following construction is adapted from their work and yields a $((2g+1)(2h+1)+1, 2g+2h)$-trisection of $\Sigma_h \times \Sigma_g$.

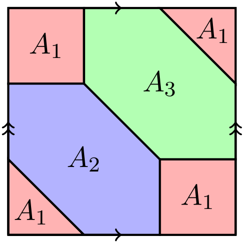
In the case where $h = 0$, we get a $(2g+2, 2g)$-trisection of $S^2 \times \Sigma_g$.

Next step: It's likely the construction carries over to nontrivial surface bundles.

Trisecting $\Sigma_h \times \Sigma_g$

1. Trisect Σ_g as $\Sigma_g = A_1 \cup A_2 \cup A_3$, where
   1. each A_i is a disk
   2. $A_i \cap A_{i+1} = \partial A_i \cap \partial A_{i+1}$ is a collection of $2g+1$ arcs
   3. $A_1 \cap A_2 \cap A_3$ is a set of $4g+2$ distinct points.

   For $\Sigma_1 \cong T^2$, this looks like the diagram on the right; in general, taking Σ_g as the connect sum of g copies of this diagram will suffice.

2. Choose three disjoint disks $D_1^2, D_2^2, D_3^2 \subset \Sigma_h$.

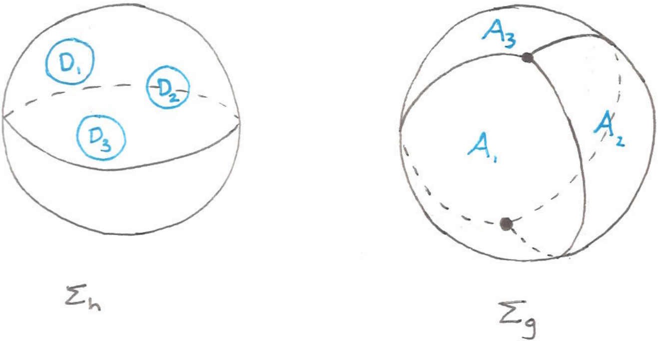
Trisecting $\Sigma_h \times \Sigma_g$

1. $\Sigma_g = A_1 \cup A_2 \cup A_3$
2. $D_1, D_2, D_3 \subset \Sigma_h$ are disjoint
3. Define $X_i := \left[\overline{\Sigma_h \backslash D_i} \times A_i\right] \cup [D_{i+1} \times A_{i+1}]$
   1. $X_i = \natural^{2h+2g}(S^1 \times B^3)$
   2. $X_i \cap X_{i+1} = \natural^{(2g+1)(2h+1)+1}(S^1 \times B^2)$

We now have a $((2g+1)(2h+1)+1, 2h+2g)$-trisection of $\Sigma_h \times \Sigma_g = X_1 \cup X_2 \cup X_3$.

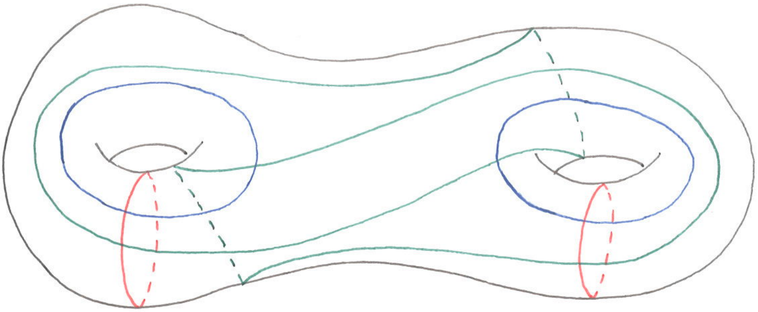
Remark: This is a minimal genus trisection, since $\text{rk}\pi_1(\Sigma_h \times \Sigma_g) = 2g + 2h$.

$S^2 \times S^2$

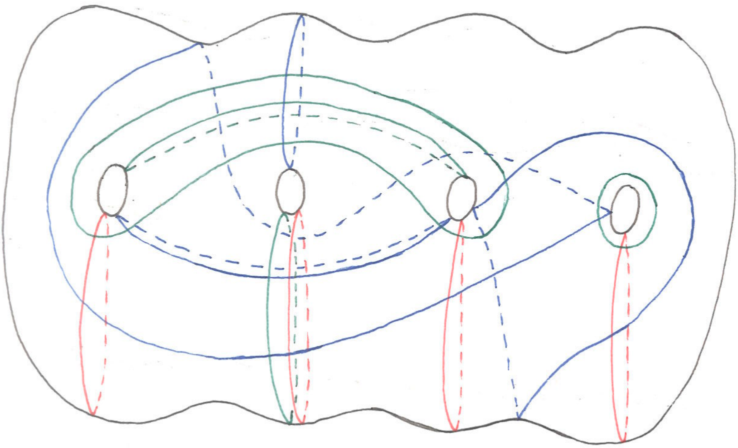


Since $g = h = 0$, we will get a (2,0)-trisection of $S^2 \times S^2$:

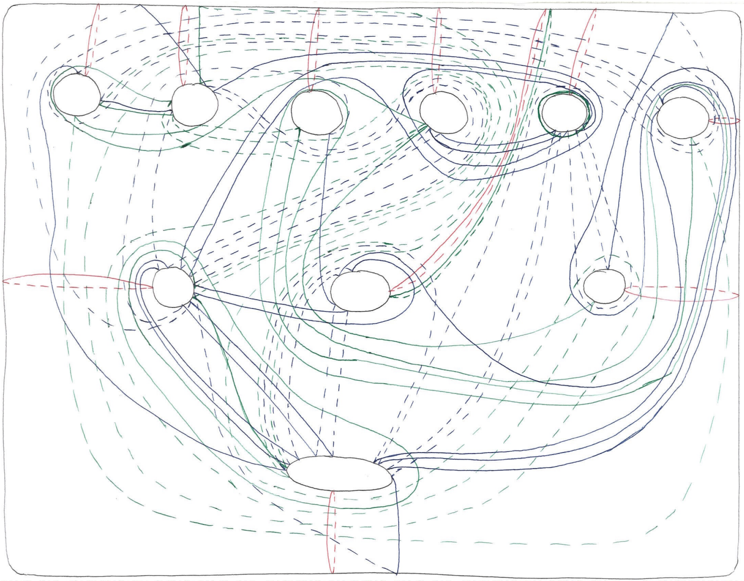
A (2,0)-trisection diagram for $S^2 \times S^2$



A (4,2)-trisection diagram for $S^2 \times T^2$



A (10,4)-trisection diagram for T^4



Thank you!