CHERN CLASSES FOR TWISTED K-THEORY

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ABSTRACT. We define a total Chern class map for the K-theory of a variety X twisted by a central simple algebra A. This includes defining a suitable notion of the motivic cohomology of X twisted by A to serve as the target for such a map. Our twisted motivic groups turn out to be different than those defined and studied by Kahn and Levine.

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This paper is dedicated to Eric Friedlander, collaborator, mentor, and friend, on the occasion of his 60th birthday.

1. INTRODUCTION

Let k be a field, A a (finite dimensional) central simple k-algebra, and X a smooth k-variety. In the paper we address the question: What is the proper notion of Chern classes for coherent $\mathcal{O}_X \otimes A$-modules that are locally free as $\mathcal{O}_X$-modules? If $A = \mathrm{Mat}_n(k)$, the algebra of $n \times n$ matrices over k, then, by Morita equivalence, the category of such modules is equivalent to the category of vector bundles on X, and the theory of Chern classes is well understood. In general, it turns out that

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while we can define a total Chern class map, which in the case $A = k$ sends a vector bundle E to $c(E) = 1 + c_1(E) + c_2(E) + \cdots$, the individual Chern classes are not defined. That is, the target of the total Chern class map, written $CH(X, A)^\times$, is not a graded group.

A recent take on the usual theory of Chern classes is that there is a map of infinite loop spaces from the K-theory space of X, $\mathcal{K}(X)$, taking values is an infinite loop space, $\mathcal{H}_{mult}(X)$, whose homotopy groups give the group of units of the motivic cohomology of X. This map is a spectrum-level version of the total Chern class map in the sense that the induced map on homotopy groups is the usual total Chern class map from K-theory to motivic cohomology. In the topological context, the construction of such a map of spectra was achieved by Boyer et al [3], settling in the affirmative a conjecture of Graeme Segal. In the algebraic and semi-topological contexts, Eric Friedlander and the author constructed such a map of spectra in [11]. In this paper, we also construct such a spectrum-level version of the twisted total Chern class map.

Let $\mathcal{K}(X, A)$ be the K-theory spectrum of the exact category of coherent $\mathcal{O}_X \otimes A$-modules that are locally free as $\mathcal{O}_X$-modules; we seek a total Chern class map defined on $\mathcal{K}(X, A)$. A more basic question is then: What is the target for such a map — i.e., what is the twisted form of motivic cohomology associated to A? One possible answer to this latter question is provided by Kahn and Levine, who construct a spectral sequence converging to $K_*(X, A)$ whose E_2-terms are "twisted" forms of Bloch's higher Chow groups. Perhaps surprisingly, the E_2-terms of the Kahn-Levine spectral sequence turn out not to be the target for the total Chern class map. We describe the correct target in this paper, and explain how it is related to the Kahn-Levine twisted Chow groups.

One source of inspiration for this paper is the work of Pedro dos Santos and Paulo Lima-Filho [6], who study Chern classes of quaternionic topological bundles — i.e., they address the topological version of the question posed here in the case $k = \mathbb{R}$ and $A = \mathbb{H}$. An important distinction is that dos Santos and Lima-Filho work at the level of equivariant spectra by thinking of a quaternionic bundle on a Real space as being a complex bundle equipped with an anti-linear involution τ satisfying $\tau^2 = -1$. Dos Santos and Lima-Filho have, more recently, been pursuing such an equivariant notion of Chern classes for twisted forms of algebraic K-theory, $\mathcal{K}(X, A)$, in the context of the Morel-Voevodsky $\mathbb{A}^1$-homotopy category. We make no attempt to work at this level of generality, but is seems likely that the map of spectra we construct, which induces a twisted form of the total Chern class map, should coincide with the result of taking fixed-points of a suitable map of equivariant spectra.

For the purposes of motivation, it is useful to consider a possible generalization of what we do in this paper. Namely, one might seek to generalize the situation by fixing a smooth k-variety Y, an Azumaya algebra $\mathcal{A}$ over Y, and defining for a Y-variety $f : X \to Y$ the spectrum $\mathcal{K}(X, \mathcal{A})$ to be the K-theory of coherent $f^*(\mathcal{A})$-modules locally free on X. The goal would then be to define a total Chern class map from $\mathcal{K}(X, \mathcal{A})$ taking values in a suitable target. Although we have restricted attention to the case $Y = \operatorname{Spec} k$ in this paper, it seems likely that such a generalization could be defined and that similar results as those presented in this paper should be attainable. Note that $\mathcal{K}(X, \mathcal{A})$ depends only on the class of $\mathcal{A}$ in the Brauer group $Br(Y)$ of Y and recall that we have $Br(Y) \subset H^2_{et}(Y, \mathcal{O}^*)_{tor}$. Fixing now $k = \mathbb{C}$ and using that $H^2_{et}(Y, \mathcal{O}^*) \cong H^2_{an}(Y, \mathcal{O}^*)$, the boundary map induced by the exponential sequence defines a homomorphism $Br(Y) \to H^3_{sing}(Y, \mathbb{Z})$. Given an arbitrary class $\delta \in H^3_{sing}(Y, \mathbb{Z})$, there is a way of defining "twisted" topological K-theory $K^*_\delta(Y)$. (See [5] for the original construction and [1] for a more recent discussion.) Presumably if δ comes from an element $\mathcal{A}$ of the Brauer group of Y under this boundary homomorphism, then there is a natural map $K_*(Y, \mathcal{A}) \to K^*_\gamma(Y)$. Since twisted topological K-theory has become a topic of interest in mathematical physics, one might hope the results of this paper will find applications in that field.

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2. Twisted K-theory, motivic cohomology, and Chern classes

Throughout this paper, k will be a field, A a finite dimensional central simple k-algebra ("csa", for short) of degree n (defined below), and *variety* will refer to a quasi-projective k-variety. For many of the results, we will need to assume that X is a smooth k-variety and that k admits resolutions of singularities (which holds, for example, if the characteristic of k is zero).

By Wedderburn's Theorem, we have $A \cong \mathrm{Mat}_l(D)$ for some division algebra D with center k. Recall that the *degree* of A is defined to be $\deg(A) = \sqrt{\dim_k(A)}$, the *index* of A is defined as $\mathrm{ind}(A) = \deg(D) = \sqrt{\dim_k(D)}$, and the *exponent* of A, $\exp(A)$, is the order of the class of A (equivalently, the class of D) in the Brauer group $Br(k)$ of k. Since A has degree n, it is a twisted form of $\mathrm{Mat}_n(k)$, in the sense that $A \otimes_k k' \cong \mathrm{Mat}_n(k')$ for some finite Galois field extension $k \subset k'$.

2.1. Twisted K-theory.

An A-*bundle* on a variety X is defined to be a coherent left $\mathcal{O}_X \otimes_k A$-module that is locally free as an $\mathcal{O}_X$-module. If $X = \operatorname{Spec} R$, an A-bundle is the same things as a left $R \otimes_k A$-module that is finitely generated and projective as an R-module (equivalently, finitely generated and projective as a left $R \otimes_k A$-module).

Definition 2.1. Let $\mathcal{P}(X, A)$ denote the exact category of A-bundles on X, let $\mathcal{K}(X, A)$ denote the associated K-theory spectrum (as defined, say, by Quillen [22]) of the exact category $\mathcal{P}(X, A)$, and let $K_n(X, A)$ denote the n-th homotopy group of $\mathcal{K}(X, A)$.

By a theorem of Quillen [22], we can relate $\mathcal{K}(X, A)$ to the usual K-theory of the appropriate Brauer-Severi variety. First, we introduce some notation, which will be used throughout the paper. We let $\mathbb{P}(A)$ denote the Brauer-Severi variety associated to A. Recall that $\mathbb{P}(A)$ represents the functor sending an affine k-variety $\operatorname{Spec} R$ to the collection of quotient objects $R \otimes_k A \twoheadrightarrow P$ of $R \otimes_k A$-modules such that P is projective of rank n as an R-module. The variety $\mathbb{P}(A)$ is a twisted form of $\mathbb{P}^{n-1}$ in the sense that it is locally for the étale topology on $\operatorname{Spec} k$ isomorphic to $\mathbb{P}^{n-1}$ — i.e., $\mathbb{P}(A) \times_k k' \cong \mathbb{P}^{n-1}_{k'}$ for some finite Galois extension $k \subset k'$. Dropping the condition on the rank of P in the definition of $\mathbb{P}(A)$ gives a functor represented by the projective variety we write as $\mathrm{Grass}(A)$, a twisted form of the variety $\mathrm{Grass}(k^n)$ parameterizing all quotient (or, equivalently, sub) k vector spaces of k^n. More generally, if M is a finitely generated left A-module, $\mathrm{Grass}_A(M)$ (resp., $\mathbb{P}_A(M)$) denotes the k-variety representing the functor whose value at $\operatorname{Spec} R$ is the collection of quotient objects $R \otimes_k M \twoheadrightarrow P$ of $R \otimes_k A$-modules such that P is projective (resp., projective of rank n) as an R-module. For example, if $A = \mathrm{Mat}_n(k)$ and $M = (k^n)^j$ with k^n being the unique simple A-module, then $\mathrm{Grass}_A(M) = \mathrm{Grass}_k(k^j)$ and $\mathbb{P}_A(M) = \mathbb{P}^{j-1}$, by Morita equivalence. For an arbitrary A and taking $M = A^t$, we write $\mathrm{Grass}(A^t) = \mathrm{Grass}_A(A^t)$ and $\mathbb{P}_A(A^t) = \mathbb{P}(A^t)$ (i.e., we usually drop the subscripts from the notation in this case). The varieties $\mathrm{Grass}(A^t)$ and $\mathbb{P}(A^t)$ are twisted forms of $\mathrm{Grass}(k^{nt})$ and $\mathbb{P}^{nt-1}$, respectively. By Morita equivalence, we have canonical isomorphisms

$$\mathbb{P}(A^t) = \mathbb{P}_A(A^t) \cong \mathbb{P}_{\mathrm{Mat}_t(A)}(\mathrm{Mat}_t(A)) = \mathbb{P}(\mathrm{Mat}_t(A))$$

and

$$\mathrm{Grass}(A^t) = \mathrm{Grass}_A(A^t) \cong \mathrm{Grass}_{\mathrm{Mat}_t(A)}(\mathrm{Mat}_t(A)) = \mathrm{Grass}(\mathrm{Mat}_t(A)).$$

There is a canonical vector bundle J on $\mathbb{P}(A)$ of rank n coming from the universal property of $\mathbb{P}(A)$. The bundle J is a twisted form of the bundle $\mathcal{O}(-1)^n$ on $\mathbb{P}^{n-1}$ and the pushforward of the algebra $\underline{\mathrm{End}}_{\mathbb{P}(A)}(J)^{op}$ to $\mathrm{Spec}\, k$ is A and the pullback of A to $\mathbb{P}(A)$ is $\underline{\mathrm{End}}_{\mathbb{P}(A)}(J)^{op}$ (see [22, §8]). Consequently, given $P \in \mathcal{P}(X, A)$, we may form the object $J \otimes_A P = J \otimes_{\pi^* A} \pi^* P$ in $\mathcal{P}(X \times \mathbb{P}(A))$. More generally, given P in $\mathcal{P}(X, A^{\otimes i})$, we may form the object $J^{\otimes i} \otimes_{A^{\otimes i}} P$ in $\mathcal{P}(X \times \mathbb{P}(A))$.

Theorem 2.2 (Quillen)**.** [22, 4.1] *For a variety X, we have a natural decomposition up to weak homotopy equivalence*

$$\mathcal{K}(X \times \mathbb{P}(A)) \sim \mathcal{K}(X) \oplus \mathcal{K}(X, A) \oplus \mathcal{K}(X, A^{\otimes 2}) \oplus \cdots \oplus \mathcal{K}(X, A^{\otimes n-1}),$$

where $A^{\otimes i}$ denotes the csa given by tensoring A over k with itself i times. The equivalence is induced by the collection of exact functors

$$\mathcal{P}(X, A^{\otimes i}) \to \mathcal{P}(X \times \mathbb{P}(A))$$

sending P to $J^{\otimes i} \otimes_{A^{\otimes i}} P$, for $0 \leq i \leq n-1$.

In fact, Quillen states his theorem in the more general context of an Azumaya algebra $\mathcal{A}$ defined on X, but we will only use the special case presented here. Notice that if $A = \mathrm{Mat}_n(k)$, this theorem gives the usual formula for the K-theory of $X \times \mathbb{P}^{n-1}$.

2.2. **Twisted Chow Groups and Twisted Motivic Groups.** Our motivation for the definition of Chow groups (more generally, motivic groups) twisted by A comes from K-theory, in that such groups should be the target of the total Chern class map. Another possible motivation along these lines would be to seek an Atiyah-Hirzebruch like spectral sequence converging to $K_*(X, A)$, the E_2-terms of which would be versions of twisted motivic groups. This latter approach is carried out by Bruno Kahn and Marc Levine. It turns out these two approaches lead to different (but related) definitions of twisted motivic groups, and this might be something of a surprise. More will be said in §3 about the Kahn-Levine groups and their relation to those studied in this paper.

Returning to the point of view adopted in this paper, given an A-bundle P on a smooth variety X, we seek to define its Chern classes. In the un-twisted context, when $A = k$, one way of defining the Chern classes of P (assuming P is generated by its global sections) is to choose a surjection $\mathcal{O}_X^{M+1} \twoheadrightarrow P$ and form associated projective bundles to obtain a closed immersion

$$\mathbb{P}(P) \subset X \times \mathbb{P}^M.$$

In fact, it simpler to stabilize by composing with the canonical surjection $\mathcal{O}_X^\infty \twoheadrightarrow \mathcal{O}_X^{M+1}$ to get a sub-variety $\mathbb{P}(P)$ of the ind-variety $X \times \mathbb{P}^\infty$. We then view this integral subvariety as defining an effective cycle in $X \times \mathbb{P}^\infty$ equidimensional of relative dimension $\operatorname{rank}(P) - 1$ over X. In general, for Y smooth, let $Z_r(Y, W)$ denote the collection of cycles on $Y \times W$ consisting of sums of integral, closed subvarieties of $Y \times W$ each of which is equidimensional of relative dimension r over a component of Y. The cycle $\mathbb{P}(P)$ defines an element of the abelian group

$$\pi_0 Z_{r-1}(X \times \Delta^\bullet, \mathbb{P}^\infty),$$

where $Z_{r-1}(X \times \Delta^\bullet, \mathbb{P}^\infty)$ denotes the simplicial abelian group $d \mapsto Z_{r-1}(X \times \Delta^d, \mathbb{P}^\infty)$. The following theorems allow one to relate this group to the Chow groups. (The first two of these are actually proven by Friedlander and Lawson [8] in the context of morphic cohomology, but the proofs are algebraic in nature and give with only superficial modifications the corresponding algebraic results.)

Theorem 2.3. [8, 3.3] *For a smooth variety X, the map given by linear join with $X \times \mathbb{P}^0$ defines a natural homotopy equivalence*

$$Z_s(X \times \Delta^\bullet, \mathbb{P}^\infty) \xrightarrow{\sim} Z_{s+1}(X \times \Delta^\bullet, \mathbb{P}^\infty)$$

for any $s \geq 0$.

Theorem 2.4. [8, 2.10] *For a smooth variety X, there is a natural decomposition up to homotopy equivalence of the form*

$$Z_s(X \times \Delta^\bullet, \mathbb{P}^\infty) \sim \bigoplus_{q \geq s} Z_s(X \times \Delta^\bullet, \mathbb{P}^q)/Z_s(X \times \Delta^\bullet, \mathbb{P}^{q-1}),$$

for any $s \geq 0$.

Theorem 2.5. [10, 8.1] *For a smooth k-variety X with k a field admitting resolutions of singularities, we have*

$$H^p_{\mathcal{M}}(X, \mathbb{Z}(q)) \cong \pi_{2q-p}\left(Z_0(X \times \Delta^\bullet, \mathbb{P}^q)/Z_0(X \times \Delta^\bullet, \mathbb{P}^{q-1})\right)$$

and, in particular,

$$CH^q(X) \cong \pi_0\left(Z_0(X \times \Delta^\bullet, \mathbb{P}^q)/Z_0(X \times \Delta^\bullet, \mathbb{P}^{q-1})\right).$$

Using these facts, we obtain the formula

$$\pi_0 Z_{r-1}(X \times \Delta^\bullet, \mathbb{P}^\infty) \cong \pi_0 Z_0(X \times \Delta^\bullet, \mathbb{P}^\infty) \cong \bigoplus_{q \geq 0} CH^q(X)$$

and hence the cycle $\mathbb{P}(P) \subset X \times \mathbb{P}^\infty$ determined by the quotient $\mathcal{O}_X^\infty \twoheadrightarrow P$ produces the element $s(P) = 1 + s_1(P) + s_2(P) + \cdots$ of $\bigoplus_{q \geq 0} CH^q(X)$, the total Segre class of P. The total Segre class of P is related to its total Chern class by the formula

$$s(P) = c(P)^{-1}, \tag{2.6}$$

where $c(P) = 1 + c_1(P) + c_2(P) + \cdots$.

Remark 2.7. Actually, the correct formula here depends on how one chooses the isomorphism

$$\pi_0\left(Z_0(X \times \Delta^\bullet, \mathbb{P}^q)/Z_0(X \times \Delta^\bullet, \mathbb{P}^{q-1})\right) \cong CH^q(X)$$

above, which affects the precise definition of the individual Segre classes. With the choice used in [12], the correct formula would instead be $s(P) = c(P^*)^{-1}$, where $P^* = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(P, \mathcal{O}_X)$. The difference, however, amounts to merely a choice of sign convention, in that we have $c_n(P^*) = (-1)^n c_n(P)$. We will use the sign convention that gives the cleaner formula $s(P) = c(P)^{-1}$.

To clarify this point, consider the case of a line bundle L on X that is generated by its global sections, so that there is a quotient $\mathcal{O}_X^\infty \twoheadrightarrow \mathcal{O}_X^{N+1} \twoheadrightarrow L$. Such a quotient is the same thing as a map $X \xrightarrow{f} \mathbb{P}^N \subset \mathbb{P}^\infty$ which gives a map

$$f^* : \mathbb{Z}[[t]] = CH^*(\mathbb{P}^\infty) \to CH^*(X),$$

where $t \in CH^1(\mathbb{P}^\infty)$ is the class of a hyperplane (i.e., restricts to the class of a hyperplane along $\mathbb{P}^M \rightarrowtail \mathbb{P}^\infty$ for all M). We choose the above isomorphism to be so that $s(L) = f^*(1 - t + t^2 - \cdots) \in CH^*(X)$ and hence that $c(L) = s(L)^{-1} = f^*(1+t) = 1 + c_1(L) \in CH^*(X)$, where $c_1(L)$ is the divisor of zeroes associated to any non-zero global section of L.

Higher Chern classes associated to elements in the higher K-groups are defined in a similar manner by taking higher homotopy groups of an appropriately defined map from the K-theory space of X to $Z_0(X \times \Delta^\bullet, \mathbb{P}^\infty)$. The details of this are in [11], but will also be recalled below.

To define the Chern classes of an A-bundle, we mimic the above procedure as closely as possible. Assume P is an A-bundle on a smooth variety X and assume in addition that it is generated by its global sections. (This condition turns out to be insignificant.) Choose a surjection $(\mathcal{O}_X \otimes A)^M \twoheadrightarrow P$ of A-bundles. To proceed, we need to define the analogue of the construction of projectivized $\mathcal{O}_X$-bundles for A-bundles.

Let E be an A-bundle on X. Then E is locally free of rank nj on X, for some j. (To see this, observe that one may pass to a finite field extension $k \subset k'$ so that $A \otimes_k k' \cong \mathrm{Mat}_n(k')$, in which case the assertion follows from Morita equivalence.) There is a projective X-variety $\mathbb{P}_{(X,A)}(E) \to X$ representing the functor on X-schemes that sends $g : Y \to X$ to the collection of quotients of A-bundles on Y of the form $g^*E \twoheadrightarrow Q$ with Q locally free on Y of rank n (cf. [19]). Then $\mathbb{P}_{(X,A)}(E)$ is a twisted form of $X \times \mathbb{P}^{j-1}$, and, in particular, the map $\mathbb{P}_{(X,A)}(E) \to X$ is equidimensional of relative dimension $j - 1$.

For example, if $E = \mathcal{O}_X \otimes M$ for an A-module M, then $\mathbb{P}_{(X,A)}(E) = X \times \mathbb{P}_A(M)$, where $\mathbb{P}_A(M)$ was defined above.

This construction determines a functor $\mathbb{P}_{(X,A)}(-)$ from $\mathcal{P}(X, A)$ to the category of projective X-varieties that are twisted forms of $X \times \mathbb{P}^N$ for some N. In particular, given a quotient $(\mathcal{O}_X \otimes A)^M \twoheadrightarrow P$ of objects in $\mathbb{P}(X, A)$, we apply $\mathbb{P}_{(X,A)}$ to obtain a closed immersion

$$\mathbb{P}_{(X,A)}(P) \subset X \times \mathbb{P}(A^M),$$

and letting M go to infinity as before, we have the closed immersion $\mathbb{P}_{(X,A)}(P) \subset X \times \mathbb{P}(A^\infty)$. This gives an element of $Z_{j-1}(X, \mathbb{P}(A^\infty))$, where $\mathrm{rank}_{\mathcal{O}_X}(P) = nj$, and hence an element of

$$\pi_0 Z_{j-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)).$$

By way of analogy with the usual setting, we wish to view this element as the total Segre class of P. To proceed, however, we need to relate the groups $\pi_0 Z_{j-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))$ for different values of j. The following analogue of Theorem 2.3 serves this purpose.

Theorem 2.8. *Assume X is a smooth k-variety and k is a field admitting resolutions of singularities. Let E an A-bundle of rank nj on X that is generated* *by its global sections. For any choice of surjection $(\mathcal{O}_X \otimes A)^m \twoheadrightarrow E$, which we extend to a surjection $(\mathcal{O}_X \otimes A)^\infty \twoheadrightarrow E$, linear join with the associated cycle $\mathbb{P}_{(X,A)}(E) \subset X \times \mathbb{P}(A^\infty)$ in $Z_{j-1}(X, \mathbb{P}(A^\infty))$ determines a homotopy equivalence*

$$Z_r(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \xrightarrow{\sim} Z_{r+j}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))$$

for all $r \geq 0$.

Before proving the Theorem, we introduce some notation for cycle groups and prove a lemma. For a morphism $V \to W$ of quasi-projective varieties with W smooth and an integer r, write $Z_r(V/W)$ for the group of cycles on V that are sums of closed, integral subvarieties of V each of which is equidimensional of relative dimension r over some connected component of W. Note that we have $Z_r(X, Y) = Z_r(X \times Y/X)$. We view $Z_r(V/W)$ as a presheaf on smooth k-varieties by sending such a variety U to the group $Z_r(V \times_k U/W \times_k U)$ and by sending a morphism $U' \to U$ to the map on cycle groups given by pullback. In order to employ the *cdh* topology in the proof of Lemma 2.9 below, we need to extend $Z_r(V/W)$ to a presheaf defined on all k-varieties. This is accomplished by using the results of [25, §3]. In the notation of that paper, the presheaf $Z_r(V/W)$ is given by restricting the domain of the presheaf $z_{equi}(V/W, r)$ (which is defined on all Noetherian schemes over W) along the functor sending a k-variety U to the W-scheme $U \times_k W$ (see [25, 3.4.5]).

If W is smooth of pure dimension d, there is an injective morphism of presheaves

$$Z_r(V/W) \rightarrowtail Z_{r+d}(V/\operatorname{Spec} k) = Z_{r+d}(-, V).$$

At a smooth variety U, the map is given by regarding a cycle on $V \times U$ that is equidimensional of relative dimension r over $W \times U$ is as a cycle equidimensional of relative dimension $r + d$ over U. More generally, the definition of this map is a special case of [25, 3.7.5].

Let $Z_r(V/W)(\Delta^\bullet)$ denote the result of applying $Z_r(V/W)$ to the standard simplicial object — that is, $Z_r(V/W)(\Delta^\bullet)$ is the simplicial abelian group $d \mapsto Z_r(V \times \Delta^d/W \times \Delta^d)$.

Lemma 2.9. *Assume W is a smooth, quasi-projective k-variety of pure dimension d, where k is a field admitting resolutions of singularities. Let E be a vector bundle on W. Then for any integer r the map*

$$Z_r(\mathbb{P}(E)/W)(\Delta^\bullet) \to Z_{r+d}(\mathbb{P}(E)/\operatorname{Spec} k)(\Delta^\bullet) = Z_{r+d}(\Delta^\bullet, \mathbb{P}(E))$$

is a homotopy equivalence.

Proof. The proof amounts to a slight modification of the proof of the Friedlander-Voevodsky duality theorem [10, 7.4]. Let $i : W \subset X$ be a projective closure of W and choose a coherent sheaf M on X such that $i^*(M) \cong E$. By platification par eclatement [23], there is a blow-up $X' \to X$ with center in $X - W$ such that the proper transform M' of M is flat (and hence a vector bundle, since M is coherent). By resolutions of singularities, we may find a blow-up $X'' \to X'$ with center missing W and such that X'' is smooth, and we take M'' to be the pullback of M' to X''. Since W is disjoint from the center of $X'' \to X$, the restriction of M'' to W is isomorphic to E. Thus, we may assume X is smooth and M is a vector bundle.

Let $\alpha : Z_{r+d}(\mathbb{P}(M)/\operatorname{Spec} k) \to Z_{r+d}(\mathbb{P}(E)/\operatorname{Spec} k)$ be the morphism of presheaves defined on all k-varieties given by pullback along the open immersion $\mathbb{P}(E) \subset \mathbb{P}(M)$ and let G be the subpresheaf of $Z_{r+d}(\mathbb{P}(M)/\operatorname{Spec} k)$ given as the inverse image of

$Z_r(\mathbb{P}(E)/W) \subset Z_{r+d}(\mathbb{P}(E)/\operatorname{Spec} k)$ under α. That is, $G(U)$ (at least for a smooth k-variety U) is the group of cycles on $U \times \mathbb{P}(M)$ generated by closed, integral subschemes $Z \subset U \times \mathbb{P}(M)$ whose fibers over points of $U \times W$ have dimension r. Define K, K', C, and C' so that we have a commutative diagram with exact rows of preshaves defined on all k-varieties of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Z_r(\mathbb{P}(E)/W) & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K' & \longrightarrow & Z_{r+d}(\mathbb{P}(M)/\operatorname{Spec} k) & \xrightarrow{\alpha} & Z_{r+d}(\mathbb{P}(E)/\operatorname{Spec} k) & \longrightarrow & C' & \longrightarrow & 0. \end{array}$$

Note that $K \cong K'$ and the remaining vertical maps are injections. It suffices to prove that

$$G(\Delta^\bullet) \rightarrowtail Z_{r+d}(\mathbb{P}(M)/\operatorname{Spec} k)(\Delta^\bullet)$$

is a homotopy equivalence and that $C(\Delta^\bullet)$ and $C'(\Delta^\bullet)$ are contractible. The latter property follows from [10, 5.5(2)], since C'_{cdh} (the *cdh* sheafification of C') and hence C_{cdh} vanish, using the argument in the proof of [10, 5.11].

The homotopy equivalence is established using the Friedlander-Lawson moving lemma [7] (see also [10, 6.3]), just as in the proof of [10, 7.3]. That is, the moving lemma provides a family of homotopies that move cycles in $Z_{r+d}(\mathbb{P}(M)/\operatorname{Spec} k)(\Delta^n)$ of some bounded degree to cycles that meet all the fibers of $\mathbb{P}(M) \times \Delta^n \to X \times \Delta^n$ over points of $W \times \Delta^n$ properly. □

Proof of Theorem 2.8. We claim that for any M, the map

$$Z_r(X, \mathbb{P}(A^M))(\Delta^\bullet) = Z_r(\mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M)/X)(\Delta^\bullet) \to Z_{r+j}(\mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus E)/X)(\Delta^\bullet)$$

given by linear join with $\mathbb{P}_{(X,A)}(E)$ is a homotopy equivalence. By the Lemma, it suffices to prove

$$Z_s(\Delta^\bullet, \mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M)) \to Z_{s+j}(\Delta^\bullet, \mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus E))$$

is a homotopy equivalence, where $s = r + \dim(X)$.

If one composes this map by pullback along the open immersion

$$Z_{s+j}(\Delta^\bullet, \mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus E)) \to Z_{s+j}(\Delta^\bullet, \mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus E) - \mathbb{P}_{(X,A)}(E)),$$

the result is a homotopy equivalence, since it coincides with the map given by pullback along

$$\mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus E) - \mathbb{P}_{(X,A)}(E) \to \mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M),$$

which is a vector bundle. The sequence

$$Z_{s+j}(\Delta^\bullet, \mathbb{P}_{(X,A)}(E)/X) \to Z_{s+j}(\Delta^\bullet, \mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus E)) \to Z_{s+j}(\Delta^\bullet, \mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus E) - \mathbb{P}_{(X,A)}(E))$$

is a fibration sequence by localization [10, 5.11], and we have $Z_{s+j}(\Delta^\bullet, \mathbb{P}_{(X,A)}(E)) = 0$, since $s + j = \dim(X) + r + j > d + j - 1 = \dim(\mathbb{P}_{(X,A)}(E))$.

It remains to show the limit as $M \to \infty$ of maps

$$Z_{r+j}(\mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus E)/X)(\Delta^\bullet) \to Z_{r+j}(\mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus \mathcal{O}_X \otimes A^m)/X)(\Delta^\bullet)$$

induced by the surjection $\mathcal{O}_X \otimes A^m \twoheadrightarrow E$ is a weak equivalence. Using Lemma 2.9 and localization [10, 5.11] again, it suffices to prove that for a fixed s and q, we have

$$\pi_q\Big(Z_s\big(\Delta^\bullet, \mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus \mathcal{O}_X \otimes A^m) - \mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus E)\big)\Big) = 0$$

for $M \gg 0$. Let F be the kernel of $\mathcal{O}_X \otimes A^m \twoheadrightarrow E$. Then there is an affine vector bundle

$$\mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus \mathcal{O}_X \otimes A^m) - \mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus E) \to \mathbb{P}_{(X,A)}(F)$$

of relative dimension $nM + j$. Using homotopy invariance and duality [10, 5.9 and 7.4], we obtain

$$\pi_q\Big(Z_s\big(\Delta^\bullet, \mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus \mathcal{O}_X \otimes A^m) - \mathbb{P}_{(X,A)}(\mathcal{O}_X \otimes A^M \oplus E)\big)\Big)$$
$$\cong \pi_q\Big(Z_0(\Delta^\bullet \times \mathbb{P}_{(X,A)}(F), \mathbb{A}^t)\Big) = H^{2t-q}_{\mathcal{M}}(\mathbb{P}_{(X,A)}(F), \mathbb{Z}(t))$$

where $t = n(M+m) - s - 1$. For $M \gg 0$, we have $H^{2t-q}_{\mathcal{M}}(\mathbb{P}_{(X,A)}(F), \mathbb{Z}(t)) = 0$. □

Definition 2.10. For a smooth variety X, define

$$j(X, A) = \gcd\{\mathrm{rank}_{\mathcal{O}_X}(P)/n \ \mid \ P \in \mathcal{P}(X, A)\}.$$

Theorem 2.11. [20, Theorem 1] *For a smooth, irreducible variety X with field of rational functions $k(X)$, we have*

$$j(X, A) = \mathrm{ind}(A \otimes_k k(X)).$$

In other words, $j(X, A)$ is the index of the image of the map

$$K_0(X, A) \to K_0(F, A) \rightarrowtail K_0(F)$$

where $K_0(F, A) \rightarrowtail K_0(F)$ is the reduced norm homomorphism (see §3.1 below). In particular, $j(X, A)$ is a birational invariant for smooth varieties. It is also an invariant of the Brauer group, since $j(X, A) = j(X, \mathrm{Mat}_r(A))$ for all r.

Theorem 2.8 gives us homotopy equivalences

$$Z_{j(X,A)-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \sim Z_{j-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \sim Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)),$$

for any j such that there is a $P \in \mathcal{P}(X, A)$ of rank jn, and thus the cycle $\mathbb{P}_{(X,A)}(P) \subset X \times \mathbb{P}(A^\infty)$ can be interpreted as living in $Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))$. A bit of care must be taken, however, to make the choices involved canonical. (In the case $A = k$, this problem does not arise since one can always use the cycle $X \times \mathbb{P}^0 \subset X \times \mathbb{P}^\infty$ to relate these cycle spaces.)

Definition 2.12. For a smooth variety X, the *motivic cohomology space of X twisted by A* is defined to be

$$\mathcal{H}_{\mathcal{M}}(X, A) = Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)),$$

and the *motivic cohomology groups of X twisted by A* are the homotopy groups of $\mathcal{H}_{\mathcal{M}}(X, A)$. In particular, the *Chow group of X twisted by A* is defined to be

$$CH(X, A) = \pi_0 \mathcal{H}_{\mathcal{M}}(X, A).$$

The *twisted total Segre class* of an A-bundle P on X that is generated by its global sections and has rank jn as an $\mathcal{O}_X$-module is the element

$$s_A(P) \in CH(X, A)$$

defined as follows. Let $\gamma \in Z_{j-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))$ denote the cycle $\mathbb{P}_{(X,A)}(P) \subset X \times \mathbb{P}(A^\infty)$ and let $c \in Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))$ denote the cycle $X \times \mathbb{P}(A) \subset X \times \mathbb{P}(A^\infty)$. Then $s_A(P)$ corresponds to the class of γ in $\pi_0 Z_{j-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))$ under the homotopy equivalences

$$Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \xrightarrow{\# c^{j-1}} Z_{jn-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \xleftarrow{\# \gamma^{n-1}} Z_{j-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)).$$

In the classical situation, when $A = k$, we have $\mathcal{H}_{\mathcal{M}}(X) = \mathcal{H}_{\mathcal{M}}(X, k) = Z_0(X \times \Delta^\bullet, \mathbb{P}^\infty)$, which, thanks to Theorem 2.4, admits a decomposition by weight

$$\mathcal{H}_{\mathcal{M}}(X) \sim \bigoplus_{q \geq 0} \mathcal{H}_{\mathcal{M}}(X, \mathbb{Z}(q))$$

where

$$\mathcal{H}_{\mathcal{M}}(X, \mathbb{Z}(q)) = Z_0(X \times \Delta^\bullet, \mathbb{P}^q)/Z_0(X \times \Delta^\bullet, \mathbb{P}^{q-1}).$$

Thus

$$\pi_q \mathcal{H}_{\mathcal{M}}(X) \cong \bigoplus_t H^{2t-q}_{\mathcal{M}}(X, \mathbb{Z}(t))$$

where $H^*_{\mathcal{M}}(X, \mathbb{Z}(*))$ is the bigraded ring-valued motivic cohomology functor. In particular, the classical Chow group is graded by weight (i.e., codimension of support) and the total Segre class defines the individual Segre classes thanks to this decomposition. In the twisted setting, we do not have such a nice decomposition, but rather something a bit weaker:

Proposition 2.13. *Assume X is a smooth k-variety and k is a field admitting resolutions of singularities. For any $r \geq 0$, there is a natural (in X) decomposition up to homotopy*

$$Z_r(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \sim \bigoplus_{t \geq 0} Z_r(X \times \Delta^\bullet, \mathbb{P}(A^{t+1}))/Z_r(X \times \Delta^\bullet, \mathbb{P}(A^t)).$$

Proof. We claim there is an element $p_t \in \pi_0 Z_0(\mathbb{P}(A^{t+1}), \mathbb{P}(A^t))$ whose image under the pullback map

$$\pi_0 Z_0(\mathbb{P}(A^{t+1}), \mathbb{P}(A^t)) \to \pi_0 Z_0(\mathbb{P}(A^t), \mathbb{P}(A^t))$$

is the class Δ of the diagonal. This will suffice to establish the result, for then $(p_t)_* \circ i_*$ is homotopic to $\Delta_* = id$.

As will be explained below in Section 2.4, for all j, the variety $\mathbb{P}(A^j)$ admits an A-bundle L_j of rank n and hence

$$\mathbb{P}(A^j) \times \mathbb{P}(A^i) \cong \mathbb{P}((L_j^*)^i)$$

for any i, where $\mathbb{P}((L_j^*)^i)$ denotes the usual projectivized vector bundle of the dual of L_j^i. Moreover, for $l < j$, the pullback of L_j to $\mathbb{P}(A^l)$ is isomorphic to L_l. In particular,

$$\mathbb{P}(A^{t+1}) \times \mathbb{P}(A^t) \cong \mathbb{P}((L_{t+1}^*)^t)$$

and the pullback of $\mathbb{P}((L_{t+1}^*)^i)$ to $\mathbb{P}(A^t)$ is isomorphic to $\mathbb{P}((L_t^*)^t)$. We thus have

$$\begin{aligned}
\pi_0 Z_0(\mathbb{P}(A^{t+1}) \times \Delta^\bullet, \mathbb{P}(A^t)) &\cong \pi_0 Z_0(\mathbb{P}((L_{t+1}^*)^t)/\mathbb{P}(A^{t+1}))(\Delta^\bullet) \\
&\cong \pi_0 Z_{nt+n-1}(\mathbb{P}((L_{t+1}^*)^t))(\Delta^\bullet) \\
&\cong \bigoplus_{i=0}^{nt-1} CH^i(\mathbb{P}((L_{t+1}^*)^t))
\end{aligned}$$

with the second isomorphism coming from Lemma 2.9. Likewise,

$$\pi_0 Z_0(\mathbb{P}(A^t) \times \Delta^\bullet, \mathbb{P}(A^t)) \cong \bigoplus_{i=0}^{nt-1} CH^i(\mathbb{P}((L_t^*)^t)).$$

It thus suffices to show the pullback map

$$CH^i(\mathbb{P}((L_{t+1}^*)^t)) \to CH^i(\mathbb{P}((L_t^*)^t))$$

is onto, for all $i \leq nt-1$. Note that

$$CH^i(\mathbb{P}((L^*_{t+1})^t)) \cong CH^i(\mathbb{P}(A^{t+1})) \oplus CH^{i-1}(\mathbb{P}(A^{t+1})) \oplus \cdots$$

and

$$CH^i(\mathbb{P}((L^*_t)^t)) \cong CH^i(\mathbb{P}(A^t)) \oplus CH^{i-1}(\mathbb{P}(A^t)) \oplus \cdots ,$$

and thus it suffices to prove the pullback map

$$i^* : CH^i(\mathbb{P}(A^{t+1})) \to CH^i(\mathbb{P}(A^t))$$

is onto. Recall that the map $p : \mathbb{P}(A^{t+1}) - \mathbb{P}(A) \to \mathbb{P}(A^t)$ is a vector bundle and let $j : \mathbb{P}(A^t) \to \mathbb{P}(A^{t+1}) - \mathbb{P}(A)$ be the zero section. Then $i^* = j^* \circ \alpha^*$, where α is the open immersion $\mathbb{P}(A^{t+1}) - \mathbb{P}(A) \rightarrowtail \mathbb{P}(A^{t+1})$. Since j^* is an isomorphism and α^* is a surjection, i^* is also a surjection. □

The decomposition of Proposition 2.13 is not as complete as in the un-twisted setting; for example, if $A = \mathrm{Mat}_n(k)$, then $\mathbb{P}(A^t) \cong \mathbb{P}^{nt-1}$ and so the Proposition asserts merely that

$$Z_r(X \times \Delta^\bullet, \mathbb{P}^\infty) \sim \bigoplus_{t \geq 0} Z_r(X \times \Delta^\bullet, \mathbb{P}^{t(n+1)-1})/Z_r(X \times \Delta^\bullet, \mathbb{P}^{tn-1}).$$

Thus, the Proposition does not provide a sense of the individual Segre classes nor does not endow $CH(X, A)$ with the structure of a graded ring. (The un-graded ring structure on $CH(X, A)$ is described below in §2.3.) But the proposition does allow one to related the twisted Chow groups of X with the ordinary Chow groups of $X \times \mathbb{P}(A)$:

Proposition 2.14. *For a k-variety X, there is a natural homotopy equivalence*

$$Z_{n-1}(X, \mathbb{P}(A^{t+1}))(\Delta^\bullet)/Z_{n-1}(X, \mathbb{P}(A^t))(\Delta^\bullet) \sim Z_0(X \times \mathbb{P}(A), \mathbb{A}^{nt}).$$

In particular, is X is smooth and k admits resolutions of singularities, we have

$$\mathcal{H}_{\mathcal{M}}(X, A) \sim \bigoplus_{q \geq 0} \mathcal{H}_{\mathcal{M}}(X \times \mathbb{P}(A), \mathbb{Z}(nq))$$

and

$$CH(X, A) \cong \bigoplus_{q \geq 0} CH^{qn}(X \times \mathbb{P}(A)).$$

Proof. By localization and homotopy invariance [10, 5.9 and 5.11], we have natural homotopy equivalences

$$Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^{t+1}))/Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^t)) \xrightarrow{\sim} Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^{t+1}) - \mathbb{P}(A^t))$$

and

$$Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^{t+1}) - \mathbb{P}(A^t)) \xrightarrow{\sim} Z_{n+nt-1}(X \times \Delta^\bullet, (\mathbb{P}(A^{t+1}) - \mathbb{P}(A^t)) \times \mathbb{A}^{nt}).$$

Since the map $\mathbb{P}(A^{t+1}) - \mathbb{P}(A^t)) \to \mathbb{P}(A)$ is a vector bundle of rank nt, we have the homotopy equivalence

$$Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A) \times \mathbb{A}^{nt}) \xrightarrow{\sim} Z_{n+nt-1}(X \times \Delta^\bullet, (\mathbb{P}(A^{t+1}) - \mathbb{P}(A^t)) \times \mathbb{A}^{nt}).$$

Finally, duality [10, 7.4] gives the natural homotopy equivalence

$$Z_0(X \times \Delta^\bullet \times \mathbb{P}(A), \mathbb{A}^{nt}) \xrightarrow{\sim} Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A) \times \mathbb{A}^{nt}).$$

□

It is important to remark that the isomorphism $CH(X,A) \cong \bigoplus_q CH^{qn}(X \times \mathbb{P}(A))$ does not preserve ring structures. The multiplication rule for $CH(X,A)$ will be introduced shortly, but it is already clear in the case $A = \mathrm{Mat}_n(k)$ that one should not expect the isomorphism of the Proposition to preserve multiplication. For in this case it becomes the isomorphism

$$CH^*(X) \cong \bigoplus_q CH^{qn}(X \times \mathbb{P}^{n-1}),$$

arising from the projective space formula for Chow groups, given by associating $(\delta_{n-1}, \dots, \delta_0) \in CH^{qn-n+1}(X) \oplus \cdots \oplus CH^{qn}(X)$ to $\sum \pi^*\delta_i \cap (X \times \mathbb{P}^{n-1-i})$, and this isomorphism is clearly not multiplicative.

2.3. **The Total Segre Class Map and Multiplicative Structure.** In this section we define the multiplication rule making the twisted motivic groups into a ring, and we show that the function given by taking total Segre classes of A-bundles generated by their global sections extends to a group homomorphism $s_A : K_0(X,A) \to CH(X,A)^\times$, taking values in the group of units of $CH(X,A)$. We then define the twisted total Chern class map by the formula $c_A(x) = s_A(x)^{-1}$.

Just as for usual motivic cohomology, cup product for its twisted counterpart may be given by linear join of cycles. That is, given integers r, s, we have the pairings

$$Z_{r-1}(X \times \Delta^\bullet, \mathbb{P}(A^M)) \times Z_{s-1}(X \times \Delta^\bullet, \mathbb{P}(A^N)) \to Z_{r+s-1}(X \times \Delta^\bullet, \mathbb{P}(A^{M+N})),$$

for each $M, N \geq 0$, given by linear join. To stabilize, one chooses (arbitrarily) an k-linear surjection $k^\infty \twoheadrightarrow k^\infty \oplus k^\infty$, which in turn determines an A-linear surjection $A^\infty \twoheadrightarrow A^\infty \oplus A^\infty$ and hence an embedding $\mathbb{P}(A^\infty) \amalg \mathbb{P}(A^\infty) \subset \mathbb{P}(A^\infty \oplus A^\infty) \subset \mathbb{P}(A^\infty)$, so that the above pairings stabilize to give the pairing

$$Z_{r-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \times Z_{s-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \to Z_{r+s-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)).$$

In particular, we have the pairing

$$Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \times Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \to Z_{2n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)).$$

By choosing a homotopy inverse for the suspension equivalence of Theorem 2.8

$$Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \xrightarrow{\sim} Z_{2n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))$$

given by linear join with the cycle $X \times \mathbb{P}(A)$, we obtain the operation

$$\mathcal{H}_{\mathcal{M}}(X,A) \times \mathcal{H}_{\mathcal{M}}(X,A) \to \mathcal{H}_{\mathcal{M}}(X,A) \tag{2.15}$$

It can be shown that the choices made do not affect the homotopy class of this map.

Proposition 2.16. *Assume X is a smooth k-variety and k is a field admitting resolutions of singularities. The operation (2.15) makes $\mathcal{H}_{\mathcal{M}}(X,A)$ into a commutative H-ring — i.e., it satisfies the axioms of a commutative ring up to homotopy. Consequently, the twisted motivic groups $\pi_*\mathcal{H}_{\mathcal{M}}(X,A)$ form a graded commutative ring and $CH(X,A) = \pi_0\mathcal{H}_{\mathcal{M}}(X,A)$ is a commutative ring.*

Proof. The pairing we defined is part of a family of operations of the form

$$\mathcal{I}(j) \times Z_{t_1 n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \times \cdots \times Z_{t_j n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \to Z_{tn-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))$$

where $t = \sum_i t_i$ and $j \mapsto \mathcal{I}(j)$ is the simplicial E_∞-operad defined in 4.1 below. The necessary homotopies required for $\mathcal{H}_{\mathcal{M}}(X, A)$ to be an H-ring follow from the contractibility of $\mathcal{I}(j)$. $\square$

The following result can be thought of as a Whitney sum formula for twisted Segre classes, except that we cannot make sense of the individual Segre classes.

Theorem 2.17. *Assume X is a smooth k-variety and k is a field admitting resolutions of singularities. The twisted total Segre class map induces a group homomorphism*

$$s_A : K_0(X, A) \to CH(X, A)^\times$$

whose value on the class of a A-bundle generated by its global sections is as given in Definition 2.12.

Proof. Let $\mathcal{P}^{gl,\oplus}(X, A)$ denote the full subcategory of $\mathcal{P}(X, A)$ consisting of A-bundles that are generated by their global sections, endowed with the structure of an exact category by declaring just the split exact sequences to be short exact sequences, and let $K_0^{gl,\oplus}(X, A)$ denote the associated Grothendieck group. The proof of [15, Lemma 2.1] shows that

$$K_0^{gl,\oplus}(X \times \mathbb{A}^1, A) \xrightarrow{i_0^* - i_1^*} K_0^{gl,\oplus}(X, A) \to K_0(X, A) \to 0$$

is exact. Note that this uses that $K_0(-, A)$ is homotopy invariant on smooth varieties, a fact which holds since $K_0(X, A)$ is a natural summand of $K_0(X \times \mathbb{P}(A))$ and $\mathbb{P}(A)$ is smooth.

Likewise, $CH(-, A)$ is homotopy invariant on smooth varieties, and hence we just need to show the function from isomorphism classes of objects in $\mathcal{P}^{gl,\oplus}(X, A)$ to $CH(X, A)^\times$ is additive on split short exact sequences. This is a consequence of the fact that, given surjections $(\mathcal{O}_X \otimes A)^\infty \twoheadrightarrow P$ and $(\mathcal{O}_X \otimes A)^\infty \twoheadrightarrow Q$, the linear join

$$\mathbb{P}_{(X,A)}(P) \# \mathbb{P}_{(X,A)}(Q) \subset X \times \mathbb{P}(A^\infty) \amalg \mathbb{P}(A^\infty) \subset X \times \mathbb{P}(A^\infty)$$

of $\mathbb{P}_{(X,A)}(P) \subset X \times \mathbb{P}(A^\infty)$ and $\mathbb{P}_{(X,A)}(Q) \subset X \times \mathbb{P}(A^\infty)$ coincides with the cycle

$$\mathbb{P}_{(X,A)}(P \oplus Q) \subset X \times \mathbb{P}(A^\infty)$$

induced by the surjection

$$(\mathcal{O}_X \otimes A)^\infty \twoheadrightarrow (\mathcal{O}_X \otimes A)^\infty \oplus (\mathcal{O}_X \otimes A)^\infty \twoheadrightarrow P \oplus Q.$$

These cycles live in $Z_{j+k-1}(X, \mathbb{P}(A^\infty))$, where $\operatorname{rank}(P) = jn$ and $\operatorname{rank}(Q) = jk$, but the convention we have adopted in Definition 2.12 ensures that $s_A(P) \cup s_A(Q) = s_A(P \oplus Q)$ as classes in $\pi_0 Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))$. $\square$

Definition 2.18. For a smooth k-variety X with k admitting resolutions of singularities, the *twisted total Chern class map*

$$c_A : K_0(X, A) \to CH(X, A)^\times$$

is defined by the formula

$$c_A(\alpha) = s_A(\alpha)^{-1}.$$

2.4. **Splitting Principle.** In this section, we describe twisted motivic cohomology in the case where A splits over a smooth variety Y, and we use this description to characterize the twisted total Chern class map in terms of the classical one.

We say that A *splits* over a variety Y if there exists an A-bundle L on Y that has rank n (the least possible non-zero value) as an $\mathcal{O}_Y$-module. If A splits over Y, then $j(Y,A) = 1$ (but not necessarily conversely). In this case, the canonical map $\mathcal{O}_Y \otimes A \to \underline{\mathrm{End}}_{\mathcal{O}_Y}(L)$ is an isomorphism, since it is so locally for the étale topology on Y. Let $L^* = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(L, \mathcal{O}_Y)$, a coherent right $\mathcal{O}_Y \otimes A$-module that is locally free as an $\mathcal{O}_Y$-module. Morita equivalence gives that the functor $P \mapsto L^* \otimes_{\underline{\mathrm{End}}_{\mathcal{O}_Y}(L)} P$, from A-bundles to ordinary vector bundles, is an equivalence of categories, and hence

$$Y \times \mathbb{P}(A^m) \cong \mathbb{P}((L^*)^m),$$

for all $m \geq 0$. Here, $\mathbb{P}((L^*)^m)$ refers to the usual projectivized bundle over Y associated to locally free $\mathcal{O}_Y$-module $(L^*)^m$.

For example, if there exists a morphism $Y \to \mathbb{P}(A^t)$, for any t, then such a morphism determines a quotient $\mathcal{O}_Y \otimes A^t \twoheadrightarrow L$ of A-bundles with L locally free of rank n as an $\mathcal{O}_Y$-module, so that $Y \times \mathbb{P}(A^m) \cong \mathbb{P}((L^*)^m)$ in this case. In particular, taking $Y = \mathbb{P}(A^t)$ itself, we have that A splits over Y and hence

$$\mathbb{P}(A^t) \times \mathbb{P}(A^m) \cong \mathbb{P}((L_t^*)^m),$$

where L_t is the canonical A-bundle on $\mathbb{P}(A^t)$ of rank n. (The existence of L_t was used already in the proof of Proposition 2.13.)

If A splits over Y, we have

$$\mathcal{H}_{\mathcal{M}}(Y,A) \cong Z_{n-1}(\mathbb{P}((L^*)^\infty)/Y)(\Delta^\bullet)$$

(The notation $Z_r(-/-)$ used here was defined in the proof of Theorem 2.8.) More generally, suppose E is any vector bundle of rank n over a variety Y, and consider the simplicial abelian group

$$Z_{n-1}(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet).$$

The proof of Theorem 2.8 shows that suspension by $\mathbb{P}(E) \subset \mathbb{P}(E^\infty)$ determines a homotopy equivalence

$$Z_{n-1}(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet) \xrightarrow{\sim} Z_{2n-1}(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet).$$

We have a binary operation on $Z_{n-1}(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet)$ defined by first taking the linear join pairing

$$Z_{n-1}(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet) \times Z_{n-1}(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet) \to Z_{2n-1}(\mathbb{P}(E^\infty \oplus E^\infty)/Y)(\Delta^\bullet),$$

then choosing a surjection $k^\infty \twoheadrightarrow k^\infty \oplus k^\infty$ so as to obtain a map closed immersion $\mathbb{P}(E^\infty \oplus E^\infty) \rightarrowtail \mathbb{P}(E^\infty)$ and thus a map

$$Z_{2n-1}(\mathbb{P}(E^\infty \oplus E^\infty)/Y)(\Delta^\bullet) \to Z_{2n-1}(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet),$$

and finally picking a homotopy inverse to the suspension equivalence

$$Z_{n-1}(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet) \xrightarrow{\sim} Z_{2n-1}(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet)$$

given by Theorem 2.8. This operation endows $Z_{n-1}(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet)$ with the structure of a homotopy commutative H-ring. When A splits over Y and $E = L^*$, with L as above, the isomorphism

$$\mathcal{H}_{\mathcal{M}}(Y,A) \cong Z_{n-1}(\mathbb{P}((L^*)^\infty)/Y)(\Delta^\bullet)$$

above is an isomorphism of H-rings.

Theorem 2.19. *Assume Y is a smooth k-variety and k is a field admitting resolutions of singularities. Let E be a vector bundle on Y of rank n. For any integer r, we have a natural homotopy equivalence*

$$Z_r(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet) \sim Z_r(Y \times \Delta^\bullet, \mathbb{P}^\infty).$$

Moreover, we have a natural (up to homotopy) homotopy equivalence

$$Z_{n-1}(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet) \sim \mathcal{H}_{\mathcal{M}}(Y)$$

of H-rings.

Proof. We may assume E is generated by its global section since $\mathbb{P}(F) \cong \mathbb{P}(F(j))$ for all integers j and bundles F. Choose a surjection $q : \mathcal{O}_Y^M \twoheadrightarrow E$. For each $t \geq 1$, the surjection $q^{\oplus t} : \mathcal{O}_Y^{tM} \twoheadrightarrow E^t$ determines a closed immersion $i_t : \mathbb{P}(E^t) \rightarrowtail \mathbb{P}^{tM-1}$ such that evident square involving i_t and i_{t+1} commutes. Let $j_t : \mathbb{P}(E^t) \rightarrowtail Y \times \mathbb{P}^{tM-1}$ be the map (π, i_t). Then, for any integer r, the closed immersion j_t defines a map

$$Z_r(\mathbb{P}(E^t)/Y)(\Delta^\bullet) \to Z_r(Y \times \mathbb{P}^{tM-1}/Y)(\Delta^\bullet) = Z_r(Y \times \Delta^\bullet, \mathbb{P}^{tM-1})$$

such that the evident square involving j_t and j_{t+1} commutes, for all t. Taking limits gives a map

$$Z_r(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet) \to Z_r(Y \times \Delta^\bullet, \mathbb{P}^\infty) = \mathcal{H}_{\mathcal{M}}(Y),$$

which we claim is a homotopy equivalence.

Using Lemma 2.9, it suffice to show

$$Z_s(\Delta^\bullet, \mathbb{P}(E^\infty) \to Z_s(\Delta^\bullet, Y \times \mathbb{P}^\infty)$$

is a homotopy equivalence, where $s = \dim(Y) + r$. Let F_t be the kernel of $\mathcal{O}_Y^{tM} \twoheadrightarrow E^t$ and set $U_t = Y \times \mathbb{P}^{tM-1} - \mathbb{P}(E^t)$. Then localization [10, 5.11] gives the fibration sequence

$$Z_s(\Delta^\bullet, \mathbb{P}(E^t)) \to Z_s(\Delta^\bullet, Y \times \mathbb{P}^{tM-1}) \to Z_s(\Delta^\bullet, U_t),$$

and thus it suffices to show that for a fixed q,

$$\pi_q Z_s(\Delta^\bullet, U_t) = 0,$$

for $t \gg 0$. Note that U_t is a vector bundle over $\mathbb{P}(F_t)$ of rank t, so that, using homotopy invariance and duality [10, 5.9 and 7.4], we have a homotopy equivalence

$$Z_s(\Delta^\bullet, U_t) \sim Z_0(\mathbb{P}(F_t) \times \Delta^\bullet, \mathbb{A}^{d_t+t-s}),$$

where $d_t = \dim(\mathbb{P}(F_t))$, and thus we have an isomorphism

$$\pi_q Z_s(\Delta^\bullet, U_t) \cong H_{\mathcal{M}}^{2d_t+2t-2s-q}(\mathbb{P}(F_t), \mathbb{Z}(d_t + t - s)).$$

Using the bundle formula [26, 3.5.1], we obtain

$$\pi_q Z_s(\Delta^\bullet, U_t) \cong \bigoplus_{i=0}^{\dim(\mathbb{P}(F_t))} H_{\mathcal{M}}^{2t-2s+2i-q}(Y, \mathbb{Z}(t - s + i)).$$

Since $H_{\mathcal{M}}^{2t-2s+2i-q}(Y, \mathbb{Z}(t-s+i)) = 0$ for $t > \dim(Y)+q+s-i$, we have$\pi_q Z_s(\Delta^\bullet, U_t) = 0$ for $t > \dim(Y) + q + s$.

Now fix $r = n - 1$. Under the homotopy equivalence

$$Z_{n-1}(\mathbb{P}(E^\infty)/Y)(\Delta^\bullet) \xrightarrow{\sim} Z_{n-1}(Y \times \Delta^\bullet, \mathbb{P}^\infty)$$

we have constructed, the binary operation defined on the source corresponds up to homotopy with the operation on the target induced by linear join together with a certain choice of linear surjection $k^\infty \twoheadrightarrow k^\infty \oplus k^\infty$ and a choice of inverse of the suspension homotopy equivalence

$$Z_{n-1}(Y \times \Delta^\bullet, \mathbb{P}^\infty) \xrightarrow{\sim} Z_{2n-1}(Y \times \Delta^\bullet, \mathbb{P}^\infty)$$

given by taking linear join with the cycle $Y \times \mathbb{P}^{n-1}$. Since we have a homotopy equivalence

$$\mathcal{H}_M(Y) = Z_0(Y \times \Delta^\bullet, \mathbb{P}^\infty) \sim Z_{n-1}(Y \times \Delta^\bullet, \mathbb{P}^\infty)$$

and the binary operation on $\mathcal{H}_M(Y)$ is independent (up to homotopy) of all choices made, the result follows. □

Corollary 2.20. *Assume Y is a smooth k-variety and k is a field admitting resolutions of singularities. If A splits over Y, we have a natural isomorphism*

$$\pi_* \mathcal{H}_\mathcal{M}(Y, A) \cong \pi_* \mathcal{H}_\mathcal{M}(Y)$$

of graded rings and hence a natural isomorphism

$$CH(Y, A) \cong CH^*(Y)$$

of rings.

For emphasis, we remark again that, when A splits over Y, the composition

$$CH^{n*}(Y \times \mathbb{P}(A)) \cong CH(Y, A) \cong CH^*(Y)$$

does *not* respect the gradings.

For an arbitrary variety X, let $Y = X \times \mathbb{P}(A)$. The canonical map from Y to $\mathbb{P}(A)$ defines a quotient $\mathcal{O}_Y \otimes A \twoheadrightarrow L$ of A-bundles with L of rank n as an $\mathcal{O}_Y$-module. In particular, A splits over Y. Moreover, the pullback map

$$CH(X, A) \to CH(Y, A),$$

being isomorphic to the pullback map

$$CH^{n*}(X \times \mathbb{P}(A)) \to CH^{n*}(Y \times \mathbb{P}(A)),$$

is a split injection, with splitting given by pullback along the diagonal map $X \times \mathbb{P}(A) \rightarrowtail X \times \mathbb{P}(A) \times \mathbb{P}(A) = Y \times \mathbb{P}(A)$. These observations allow one to uniquely characterize the total Chern class of an A-bundle, via the following "splitting theorem":

Theorem 2.21. *Assume X is a smooth k-variety and k is a field admitting resolutions of singularities. There is a commutative diagram of abelian groups*

$$\begin{array}{ccc} K_0(X, A) & \rightarrowtail & K_0(Y, A) \cong K_0(Y) \\ \downarrow{\scriptstyle s_A} & & \downarrow{\scriptstyle s} \\ CH(X, A)^\times & \rightarrowtail & CH(Y, A)^\times \cong CH^*(Y)^\times, \end{array}$$

in which $Y = X \times \mathbb{P}(A)$, the horizontal arrows are injections given by pullback, and the right-hand vertical map is the classical total Segre class map.

Proof. The only thing left to prove is the commutativity of the diagram, and this amounts to the assertion that

$$\begin{array}{ccc} K_0(Y,A) & \xrightarrow{\cong} & K_0(Y) \\ {\scriptstyle s_A}\downarrow & & {\scriptstyle s}\downarrow \\ CH(Y,A)^\times & \xrightarrow{\cong} & CH(Y)^\times \end{array}$$

commutes, where Y is any variety for which $\mathcal{O}_Y \otimes A$ splits and the isomorphisms are given by Morita equivalence. If L is an A-bundle of rank n on Y, then choosing a surjection $\mathcal{O}_Y^M \twoheadrightarrow L^*$ allows us to identify $CH(Y)$ with $\pi_0 Z_{r-1}(\mathbb{P}((L^*)^\infty/Y)(\Delta^\bullet)$, for any r, as in Theorem 2.19. One may readily verify that each of the maps $K_0(Y,A) \to CH_0(Y)^\times$ sends the class of a quotient object $[\mathcal{O}_Y \otimes A^\infty \twoheadrightarrow P] = [\underline{\mathrm{End}}_{\mathcal{O}_Y}(L)^\infty \twoheadrightarrow P]$ to the class of the cycle in $Z_{r-1}(\mathbb{P}((L^*)^\infty)/Y)$ (where $\mathrm{rank}_{\mathcal{O}_Y}(P) = rn$) given by

$$\mathbb{P}(L^* \otimes_{\underline{\mathrm{End}}_{\mathcal{O}_Y}(L)} P) \subset \mathbb{P}((L^*)^\infty).$$

□

In particular, given a A-bundle P, the image of its total Segre class under the canonical injection $CH(X,A) \rightarrowtail CH^*(Y)$ is the usual total Segre class the vector bundle $J^* \otimes_{\underline{\mathrm{End}}_{\mathcal{O}_Y}(J)} \pi^* P$, where J is the canonical rank n bundle on $Y = X \times \mathbb{P}(A)$. Note, however, that the failure, in general, of $CH(X,A)$ to admit a multiplicative grading (and thus the failure of the total Chern class $c_A(P)$ to decompose into individual Chern classes) corresponds to the fact that the inclusion $CH(X,A) \rightarrowtail CH^*(Y)$ is *not* an inclusion of graded rings.

3. Connections with the work of Kahn and Levine

In this section we discuss the work of Bruno Kahn and Marc Levine concerning a different version of twisted Chow groups and their relation to $K_*(X,A)$. We indicate connections between their work and the topics of this paper and provide a detailed example.

3.1. The Kahn-Levine twisted Chow groups.

We need the following three general results on twisted K-theory, each of which is an easy consequence of Quillen's main theorems on K-theory [22]. (See, for example, [4, §5].)

Theorem 3.1. *Let $\mathcal{M}(X,A)$ denote the abelian category of all coherent left $\mathcal{O}_X \otimes A$-modules, let $\mathcal{G}(X,A)$ denote the associated K-theory spectrum, and let $G_*(X,A)$ denote the associated homotopy groups.*

The inclusion $\mathcal{P}(X,A) \subset \mathcal{G}(X,A)$ induces a weak homotopy equivalence on K-theory spaces

$$\mathcal{K}(X,A) \xrightarrow{\sim} \mathcal{G}(X,A)$$

provided X is smooth.

Theorem 3.2. *Let $\mathcal{M}^q(X,A)$ denote the Serre sub-category of $\mathcal{M}(X,A)$ consisting of those objects whose support, when regarded as coherent sheaves on X, have codimension at least q, and let $\mathcal{G}^{(q)}(X,A)$ denote the associated K-theory spectrum. There is a natural homotopy fibration sequence*

$$\mathcal{G}^{(q+1)}(X,A) \to \mathcal{G}^{(q)}(X,A) \to \bigoplus_{x \in X^{(q)}} \mathcal{K}(k(x),A),$$

where $X^{(q)}$ denotes the collection of points of X whose closures have codimension q.

Theorem 3.3. *If I is a nilpotent ideal in a ring R, then the inclusion $\mathcal{G}(R/I, A) \rightarrowtail \mathcal{G}(R, A)$ induces a homotopy equivalence*

$$\mathcal{G}(R/I, A) \xrightarrow{\sim} \mathcal{G}(R, A).$$

We now describe the Kahn-Levine twisted Chow groups.

Definition 3.4. View $K_0(-, A)$ as a presheaf of abelian groups on Sm/k, the category of smooth, quasi-projective k-varieties, and let $\mathbb{Z}_A$ denote the Zariski sheafification of this presheaf:

$$\mathbb{Z}_A = K_0(-, A)^{\sim}_{Zar}.$$

For a local ring R, there is an isomorphism $K_0(R, A) \cong \mathbb{Z}$, but $\mathbb{Z}_A$ is not the constant sheaf $\mathbb{Z}$, since for an extension $R \subset R'$ of integral domains, the map $K_0(R', A) \to K_0(R', A)$ is injective but not necessarily surjective. Indeed, if R^{sh} denotes the strict henselization of a local domain R, then $R^{sh} \otimes A \cong \mathrm{Mat}_n(R^{sh})$ and so there is a canonical isomorphism $K_0(R^{sh}, A) \cong \mathbb{Z}$. The index of the image of the injective map $K_0(R, A) \rightarrowtail K_0(R^{sh}, A) \cong \mathbb{Z}$ is the index of $F \otimes_k A$, where F is the field of fractions of R. In other words, the étale sheafification of $K_0(-, A)$ is canonically isomorphic to the constant sheaf $\mathbb{Z}$ and the canonical map $\mathbb{Z}_A \to K_0(-, A)^{\sim}_{\text{ét}} \cong \mathbb{Z}$ is an injection, but not an isomorphism, of Zariski sheaves. We regard $\mathbb{Z}_A$ as a subsheaf of $\mathbb{Z}$ in this manner.

For example, if $k = \mathbb{R}$ and $A = \mathbb{H}$, then inclusion $\mathbb{Z}_{\mathbb{H}} \subset \mathbb{Z}$ evaluated on an affine variety $X = \operatorname{Spec} R$ is either $\mathbb{Z} = \mathbb{Z}$ or $2\mathbb{Z} \subset \mathbb{Z}$, depending on whether $R \otimes_k \mathbb{H} \cong \mathrm{Mat}_2(R)$.

The presheaf $\mathbb{Z}_A$ is "motivic" (i.e., it admits transfers in a suitable sense) and in fact determines what is known as a birational motive. That is, for a smooth, connected variety X with field of rational functions $k(X)$, we have $\Gamma(X, \mathbb{Z}_A) = K_0(k(X) \otimes A)$. In particular, we have

$$\Gamma(X, \mathbb{Z}_A) = j(X, A) \cdot \mathbb{Z} \subset \mathbb{Z}$$

by Theorem 2.11, where $j(X, A)$ is defined in Definition 2.10. To put it abstractly, Kahn and Levine define the motivic cohomology of X twisted by A by the formula

$$H^n_{\mathcal{M}}(X, \mathbb{Z}_A(q)) = \mathbb{H}^n_{Zar}(X, \mathbb{Z}_A \otimes_{\mathbb{Z}} \mathbb{Z}(q)),$$

where $\mathbb{Z}(q)$ is is the weight q motivic complex.

More concretely, they define twisted higher Chow groups for smooth varieties (which agree with the twisted motivic groups) using cycle complexes with coefficients in $\mathbb{Z}_A$. In detail, let $C^t(X, q)$ denote the collection of points $z \in X \times \Delta^q$ such that $\overline{z} \subset X \times \Delta^q$ is a codimension t subvariety that meets each face $X \times \Delta^p \subset X \times \Delta^q$ properly. Define

$$Z^t(X, q; \mathbb{Z}_A) = \bigoplus_{z \in C^t(X,q)} K_0(k(z) \otimes A).$$

Then

$$Z^t(X, \bullet; \mathbb{Z}_A) := \big(q \mapsto Z^t(X, q; \mathbb{Z}_A)\big)$$

forms a simplicial abelian group if we define the face and degeneracy maps as follows. Let $Y \subset X \times \Delta^d$ be an integral subvariety of codimension q, let Y' be the (scheme-theoretic) intersection of Y with a face of $X \times \Delta^d$, and let $Y'_1, \cdots, Y'_l$ be the reduced components of Y' that have codimension 1 in Y. To specify the face maps on $Z^t(X, \bullet; \mathbb{Z}_A)$ it suffices to define the map $K_0(k(Y), A) \to \bigoplus_i K_0(k(Y'_i), A)$. Note that $Y' \subset Y$ has finite Tor dimension and hence there is a well-defined map

$$G_0(Y, A) \to G_0(Y', A)$$

given by pullback. From Theorem 3.1, we have a map $G_0(Y', A) \to \bigoplus_i G_0(\mathcal{O}_{Y', Y'_i}, A)$ and a surjection $G_0(Y, A) \twoheadrightarrow K_0(k(Y), A)$, and from Theorems 3.2 and 3.3, we have isomorphisms

$$G_0(\mathcal{O}_{Y', Y'_i}, A) \cong G_0(k(Y'_i), A) \cong K_0(k(Y'_i), A).$$

Define the map $K_0(k(Y), A) \to \bigoplus_i K_0(k(Y'_i), A)$ by lifting along the surjection in the diagram

$$K_0(k(Y), A) \twoheadleftarrow G_0(Y, A) \to G_0(Y', A) \to \oplus_i G_0(\mathcal{O}_{Y', Y'_i}, A) \cong \oplus_i K_0(k(Y'_i), A).$$

The definition of the map is independent of the choice made.

The degeneracy maps are defined more simply by pulling back along the flat maps of the form $X \times \Delta^{d+1} \twoheadrightarrow X \times \Delta^d$.

Definition 3.5. The *Kahn-Levine twisted higher Chow groups* are

$$CH^t(X, q; \mathbb{Z}_A) := \pi_q Z^t(X, \bullet; \mathbb{Z}_A).$$

If $A = k$ so that $\mathbb{Z}_A$ is the constant sheaf $\mathbb{Z}$, we recover the definition of Bloch's higher Chow groups:

$$CH^t(X, q; \mathbb{Z}_k) = CH^t(X, q).$$

More generally, if $A = \mathrm{Mat}_n(k)$ for any n, then $\mathbb{Z}_A \cong \mathbb{Z}$, thanks to Morita equivalence, and thus we have a canonical isomorphism

$$CH^t(X, q; \mathbb{Z}_{\mathrm{Mat}_n(k)}) \cong CH^t(X, q).$$

In general, the inclusion of presheaves $\mathbb{Z}_A \rightarrowtail \mathbb{Z}$ defines a natural map of simplicial abelian groups

$$Z^t(X, \bullet; \mathbb{Z}_A) \rightarrowtail Z^t(X, \bullet; \mathbb{Z})$$

and hence a natural map

$$CH^t(X, q; \mathbb{Z}_A) \to CH^t(X, q)$$

from the Kahn-Levine groups to the usual higher Chow groups.

Kahn and Levine prove the following important result (which is a consequence of much broader theorem of Levine).

Theorem 3.6. *(See* [18]*) For a smooth variety X, there is a natural convergent spectral sequence*

$$CH^t(X, p; \mathbb{Z}_A) \Longrightarrow K_p(X, A).$$

This is the Atiyah-Hirzebruch spectral sequence for twisted K-theory, generalizing, from the case $A = k$, the Atiyah-Hirzebruch spectral sequence for ordinary K-theory established in [2, 9, 17, 24, 14]. In the classical setting, the Chern class map takes values in the E_2-terms of the Atiyah-Hirzebruch spectral sequence. It may be a bit of a surprise that the E_2-terms $CH^t(X, p; \mathbb{Z}_A)$ above do not coincide with the target of the total Chern class map we have defined. That the two notions of twisted Chow groups must differ is actually clear for the following reason. The sheaf $\mathbb{Z}_A$ is not a functor taking values in the category of commutative rings, since the morphism induced by a map of varieties need not be a ring homomorphism. As a consequence, the bi-graded group $CH^*(X,*;\mathbb{Z}_A)$ has no natural ring structure. Since a total Chern class map lands in the group of units of a cohomology theory, we see that the E_2-terms of the Kahn-Levine spectral sequence have little hope of receiving Chern classes in the usual sense.

It is worth noting, however, that if one is primarily interested in the Chern *character* map, the conflict disappears, since the twisted motivic groups in the sense of this paper and those in the sense of Kahn and Levine coincide upon tensoring with $\mathbb{Q}$. In particular, one expects that the Chern character map defines an isomorphism from twisted K-theory to the E_2-terms of the Kahn-Levine spectral sequence. But upon tensoring with $\mathbb{Q}$, all the twisted theories introduced are actually isomorphic with their un-twisted counter-parts, and so this is not a new result.

Just as the classical Chow groups can be described in terms of cycles modulo divisors of rational functions, so too does $CH^q(X,\mathbb{Z}_A)$ admit such a definition. Suppose X is a smooth variety and $Z \subset X$ is an integral subvariety with generic point z. Let $k(z)$ denote the field of rational functions on Z and write $Z^{(1)}$ for the set of codimension 1 points on Z. For $w \in Z^{(1)}$, the ring $R = \mathcal{O}_{Z,w}$ is a dimension one local integral domain. By Theorems 3.2 and 3.1, we have the localization long exact sequence

(3.7)
$$\cdots \to G_1(R,A) \to K_1(k(z),A) \xrightarrow{\partial} K_0(k(w),A) \to G_0(R,A) \to K_0(F,A) \to 0.$$

Now we define

$$\mathrm{div}_A : \bigoplus_{z\in X^{(q-1)}} K_1(k(z)\otimes_k A) \to \bigoplus_{w\in X^{(q)}} K_0(k(w)\otimes A)$$

to be the evident sum of boundary maps.

Recall that for K_1 and K_0, we have reduced norm homomorphism

$$\mathrm{Nrd} : K_0(F\otimes_k A) \rightarrowtail K_0(F) \cong \mathbb{Z},$$

and

$$\mathrm{Nrd} : K_1(F\otimes_k A) \to K_1(F) \cong F^\times$$

which are uniquely determined by the condition that the square

$$\begin{array}{ccc} K_i(F\otimes_k A) & \longrightarrow & K_i(F) \\ \downarrow & & \downarrow \\ K_i(\overline{F}\otimes_k A) & \xrightarrow{\cong} & K_i(\overline{F}) \end{array}$$

commutes, where the isomorphism along the bottom is given by Morita equivalence. (The map Nrd on K_0-groups defines the map of presheaves $\mathbb{Z}_A \rightarrowtail \mathbb{Z}$ above.)

Proposition 3.8 (Kahn-Levine)**.** *For a smooth variety X with field of rational functions F, there is a natural isomorphism*

$$CH^q(X,\mathbb{Z}_A) \cong \mathrm{coker}\left(\mathrm{div}_A : \bigoplus_{w\in X^{(q-1)}} K_1(k(w)\otimes_k A) \to \bigoplus_{z\in X^{(q)}} K_0(k(z)\otimes A)\right).$$

Moreover, the canonical map

$$CH^q(X,\mathbb{Z}_A) \to CH^q(X,\mathbb{Z})$$

is induced by the reduced norm homomorphisms — i.e., we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} \bigoplus_{w\in X^{(q-1)}} K_1(k(w)\otimes_k A) & \xrightarrow{\operatorname{div}_A} & \bigoplus_{z\in X^{(q)}} K_0(k(z)\otimes A) & \longrightarrow & CH^q(X,A) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mathrm{Nrd}} & & \downarrow{\scriptstyle \mathrm{Nrd}} & & \downarrow & & \\ \bigoplus_{w\in X^{(q-1)}} k(w)^\times & \xrightarrow{\operatorname{div}} & \bigoplus_{z\in X^{(q)}} \mathbb{Z} & \longrightarrow & CH^q(X) & \longrightarrow & 0 \end{array}$$

We attribute this result to Kahn and Levine but include a complete proof for lack of a reference:

Proof. The group $CH^q(X,\mathbb{Z}_A)$ may be defined as

$$\operatorname{coker}\left(\bigoplus_{Y\subset X\times\mathbb{P}^1} K_0(k(Y),A) \xrightarrow{i_0^*-i_\infty^*} \bigoplus_{z\in X^{(q)}} K_0(k(z),A)\right),$$

where Y ranges over closed, integral subvarieties of codimension q that map dominantly to $\mathbb{P}^1$ and that map finitely and dominantly onto codimension $q-1$ subvarieties of X. In fact, we can drop the condition that Y map dominantly to $\mathbb{P}^1$, since $i_0^* - i_\infty^*$ is zero on the extra summands.

Let Y be a subvariety of $X\times\mathbb{P}^1$ mapping finitely onto $W\subset X$ with W of codimension $q-1$. The projection map $Y\to\mathbb{P}^1$ determines a rational function $f\in k(Y)^\times = K_1(k(Y))$. Cupping with f gives a map $K_0(k(Y),A)\to K_1(k(Y),A)$ and transfer along the finite field extension $k(W)\subset k(Y)$ defines a map $K_1(k(Y),A)\to K_1(k(W),A)$. We thus have a map

$$\bigoplus_{Y\subset X\times\mathbb{P}^1} K_0(k(Y),A)\to \bigoplus_{w\in X^{(q-1)}} K_1(k(w),A) \tag{3.9}$$

and we claim the composition

$$\bigoplus_{Y\subset X\times\mathbb{P}^1} K_0(k(Y),A)\to \bigoplus_{w\in X^{(q-1)}} K_0(k(x),A) \xrightarrow{\operatorname{div}_A} \bigoplus_{z\in X^{(q)}} K_0(k(z),A)$$

coincides with the map $i_0^*-i_\infty^*$.

Let $k\subset k'$ be a finite field extension splitting A. Since the map $K_0(k(z),A)\to K_0(k'(z),A)\cong\mathbb{Z}$ is injective for all z and the maps under consideration are natural, it suffices to verify the claim upon extending scalars along $k\subset k'$. In other words, we may assume $A=k$ — in this case, the coincidence of these maps amounts to the standard fact that the two common definitions of rational equivalence coincide [13].

To complete the proof of the first assertion, it suffices to show that the composition of the map (3.9) with div_A has the same image as div_A itself. Since $\mathrm{Nrd} : K_0(k(z),A)\rightarrowtail K_0(k(z))$ is injective for all z, it suffices to show that the composition of (3.9) with $\Sigma_w(K_1(k(w),A)\xrightarrow{\mathrm{Nrd}}K_1(k(w)))$ coincides with the image of $\Sigma_w(K_1(k(w),A)\xrightarrow{\mathrm{Nrd}}K_1(k(w)))$ itself.

For this, we may fix a $w\in X^{(q-1)}$ and show the composition of

$$\bigoplus_{Y\subset X\times\mathbb{P}^1} K_0(k(Y),A)\to K_1(k(w),A)\xrightarrow{\mathrm{Nrd}}K_1(k(w))$$

has image equal to $\operatorname{im}(\mathrm{Nrd})$, where now Y ranges over those integral subvarieties mapping dominantly to $W=\overline{w}$. Letting $F=k(w)$, this amounts to showing that the composition

$$\bigoplus_{E,\alpha} K_0(E,A) \to K_1(F,A) \xrightarrow{\mathrm{Nrd}} K_1(F) \tag{3.10}$$

has image equal to im(Nrd), where (E,α) ranges over pairs with E a finite extensions of F and $\alpha \in E$ such that $E = F(\alpha)$ and the map is given as the composition of

$$\bigoplus_{E,\alpha} K_0(E,A) \xrightarrow{\Sigma - \cup\alpha} K_1(E,A) \xrightarrow{N} K_1(F,A).$$

Let $D = F \otimes_k A$ and, without loss, assume D is a division algebra, so that $K_1(F,A) = K_1(D) = D^\times/[D^\times, D^\times]$. Pick $\alpha \in D$ and define $E = F(\alpha) \subset D$. Let $E(\alpha) \subset L \subset D$ be a maximal subfield of D so that $[L : F] = n$ and L splits D. The composition $K_1(E) \to K_1(A) \xrightarrow{\mathrm{Nrd}} K_1(F)$ is the map $(N_{E/F})^{n/[E:F]} = (N_{E/F})^{[L:E]}$, and thus the image of $\alpha \in K_1(D)$ under Nrd is $N_{E/F}(\alpha)^{[L:E]}$. The composition of $K_0(E,D) \to K_1(E,D) \to K_1(D) \to K_1(F)$ coincides with the composition of $K_0(E,A) \rightarrowtail K_0(E) \xrightarrow{-\cup\alpha} K_1(E) \xrightarrow{N_{E/F}} K_1(F)$ and so $N_{E/F}(\alpha)^i$ lies in the image of this map where i is the index of $K_0(E,D) \subset K_0(E)$. That is, by Wedderburn's Theorem, we know $E \otimes_F D \cong \mathrm{Mat}_l(D')$ for some division algebra D' with center E and $i = \sqrt{\mathrm{rank}_E(D')}$. Since L is a maximal subfield of D' too, we have that $[L : E] = i$. This shows that the image of $\alpha \in K_1(D)$ under Nrd lies in the image of the composition $\bigoplus_{E,\alpha} K_0(E,A) \to K_1(F,A) \xrightarrow{\mathrm{Nrd}} K_1(F)$.

To prove $CH^q(X,A) \to CH^q(X)$ is induced by the reduced norm homomorphisms, it suffice to show

$$\begin{array}{ccc} K_1(k(z),A) & \xrightarrow{\partial} & K_0(k(w),A) \\ \downarrow & & \downarrow \\ K_1(k(z)) & \xrightarrow{\partial} & K_0(k(w)) \end{array} \tag{3.11}$$

commutes, using the notation of (3.7). There exists a finite, flat extension $R \subset R'$ of local, dimension one domains such that $R' \otimes_k A \cong \mathrm{Mat}_n(R')$. Letting z' and w' denote the generic and closed points of $\mathrm{Spec}(R')$, we have that

$$\begin{array}{ccc} K_1(k(z),A) & \longrightarrow & K_0(k(w),A) \\ \downarrow & & \downarrow \\ K_1(k(z'),A) & \longrightarrow & K_0(k(w'),A) \\ \cong\downarrow & & \cong\downarrow \\ K_1(k(z')) & \longrightarrow & K_0(k(w')) \end{array}$$

commutes. The commutativity of (3.11) follows from the defining properties of the reduced norm homomorphisms. □

In the special case $k = \mathbb{R}$ and $A = \mathbb{H}$, the quaternions, it is well-known that, for any field extension $\mathbb{R} \subset F$, the reduced norm homomorphism

$$\mathrm{Nrd} : K_1(F \otimes_{\mathbb{R}} \mathbb{H}) \to K_1(F) \cong F^\times$$

is injective with image $\Sigma_4(F)^\times$, the set of non-zero field elements that can be written as a sum of at most four squares in F. In particular, we have the following commutative diagram relating $CH^1(X,\mathbb{H})$ and $CH^1(X)$ for a real variety X.

$$\begin{array}{ccc}
\Sigma_4(F)^\times & \rightarrowtail & F^\times \\
\downarrow{\scriptstyle \mathrm{div}_{\mathbb{R}}} & & \downarrow{\scriptstyle \mathrm{div}} \\
\bigoplus_{x\in X^{(1)}} K_0(k(x)\otimes A) & \rightarrowtail & \bigoplus_{x\in X^{(1)}} \mathbb{Z} \\
\downarrow & & \downarrow \\
CH^1(X,\mathbb{H}) & \longrightarrow & CH^1(X) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Returning to the general case, we mention that there is also a "Gersten resolution" available for the functors $K_q(-,A)$. Let $K_q(-,A)^{\sim}_{Zar}$ denote the Zariski sheafification of the presheaf $U \mapsto K_q(U,A)$.

Theorem 3.12. *If R is the localization of a smooth variety X at a point, the map*

$$\mathcal{G}^{(q+1)}(R,A) \to \mathcal{G}^{(q)}(R,A)$$

is homotopic to the constant map. Consequently, for any $p \geq 0$, there is an exact sequence of Zariski sheaves on X of the form

$$0 \to K_p(-,A)^{\sim}_{Zar} \to K_p(k(X),A) \to \bigoplus_{x\in X^{(1)}} K_{p-1}(k(x),A) \to$$
$$\bigoplus_{x\in X^{(2)}} K_{p-2}(k(x),A) \to \cdots \to \bigoplus_{x\in X^{(p)}} K_0(k(x),A) \to 0.$$

Proof. [4, 5.3] □

As is the classical setting, we deduce immediately from the Gersten resolution the following generalization of Bloch's formula:

Corollary 3.13. *For a smooth variety X, we have a natural isomorphism*

$$CH^p(X,\mathbb{Z}_A) \cong \mathbb{H}^p_{Zar}(X, K_p(-,A)^{\sim}_{Zar}).$$

In particular, the "twisted Picard group" is given by

$$CH^1(A,\mathbb{Z}_A) \cong \mathbb{H}^1_{Zar}(X, K_1(-,A)^{\sim}_{Zar}).$$

3.2. **The case $A = \mathbb{H}$.** Another point of view provided by Kahn and Levine is that $\mathbb{Z}_A$ may also be defined as the image of the morphism of motivic sheaves

$$\pi_* : M(\mathbb{P}(A)) \to M(\operatorname{Spec} k) = \mathbb{Z}$$

induced by the structure map $\pi : \mathbb{P}(A) \to \operatorname{Spec} k$. Using duality [10, 7.4], this induces a natural map

$$H^p_{\mathcal{M}}(X \times \mathbb{P}(A), \mathbb{Z}(q)) \to H^{p-2n+2}_{\mathcal{M}}(X, \mathbb{Z}_A(q-n+1)),$$

which, for Chow groups, is a surjection of the form

$$CH^q(X\times\mathbb{P}(A)) \twoheadrightarrow CH^{q-n+1}(X;\mathbb{Z}_A).$$

When $A = \mathrm{Mat}_n(k)$, this map is the push-forward map

$$H^p_{\mathcal{M}}(X \times \mathbb{P}^{n-1}, \mathbb{Z}(q)) \xrightarrow{\pi_*} H^{p+2n-2}_{\mathcal{M}}(X, \mathbb{Z}(q-n+1)),$$

specializing to the map

$$CH^q(X \times \mathbb{P}^{n-1}) \xrightarrow{\pi_*} CH^{q-n+1}(X)$$

in the case of Chow groups. By Proposition 2.14, we obtain a natural map

$$\pi_m \mathcal{H}_{\mathcal{M}}(X, A) \cong \bigoplus_t H^{2nt-m}_{\mathcal{M}}(X \times \mathbb{P}(A), \mathbb{Z}(nt)) \to \bigoplus_t H^{2nt-m+2n-2}_{\mathcal{M}}(X, \mathbb{Z}(nt-n+1)),$$

which specializes to a surjection

$$CH(X, A) \twoheadrightarrow \bigoplus_q CH^{qn+1}(X; \mathbb{Z}_A).$$

We will not attempt to make this somewhat heuristic point of view rigorous, except for the following case: Assume for the rest of this section that $k = \mathbb{R}$, the field of real numbers, and $A = \mathbb{H}$, the quaternion division algebra.

Lemma 3.14. *If X is a smooth, real curve, then there is a natural isomorphism*

$$CH^2(X \times \mathbb{P}(\mathbb{H})) \cong CH^1(X, \mathbb{Z}_{\mathbb{H}}),$$

and thus

$$CH(X, \mathbb{H}) \cong \mathbb{Z} \oplus CH^1(X, \mathbb{Z}_{\mathbb{H}})$$

Remark 3.15. This lemma is part of more general fact, due to Kahn and Levine. Namely, the kernel of the surjection

$$M(\mathbb{P}(\mathbb{H})) \twoheadrightarrow \mathbb{Z}_{\mathbb{H}}$$

can be identified with $\mathbb{Z}(1)[2]$ so that we obtain a long exact sequence of the form

(3.16)
$$\cdots \to H^m_{\mathcal{M}}(X, \mathbb{Z}(q)) \xrightarrow{\pi^*} H^m_{\mathcal{M}}(X \times \mathbb{P}(\mathbb{H}), \mathbb{Z}(q)) \to H^{m-2}_{\mathcal{M}}(X, \mathbb{Z}_{\mathbb{H}}(q-1)) \to H^{m+1}_{\mathcal{M}}(X, \mathbb{Z}(q)) \to \cdots .$$

We can view this long exact sequence as a twisted form of the usual projective bundle formula as follows. If $\mathbb{H}$ splits over X so that $\mathcal{O}_X \otimes \mathbb{H} \cong \underline{\mathrm{End}}_{\mathcal{O}_X}(E)$ for a rank 2 locally free sheaf E on X, then $X \times \mathbb{P}(\mathbb{H}) \cong \mathbb{P}(E)$ and the sequence splits to give

$$H^m_{\mathcal{M}}(X \times \mathbb{P}(E), \mathbb{Z}(q)) \cong H^m_{\mathcal{M}}(X, \mathbb{Z}(q)) \oplus H^{m-2}_{\mathcal{M}}(X, \mathbb{Z}(q-1)).$$

In general, the sequence (3.16) does not split.

Specializing to Chow groups gives the right exact sequence

$$CH^q(X) \to CH^q(X \times \mathbb{P}(\mathbb{H})) \to CH^{q-1}(X, \mathbb{Z}_{\mathbb{H}}) \to 0. \tag{3.17}$$

If X is a curve and $q = 2$, we recover the isomorphism of the Lemma. For lack of a suitable reference, we will prove the lemma directly.

Proof of Lemma 3.14. We have a commutative diagram with exact rows

$$
\begin{array}{ccccccc}
\bigoplus_{x\in(X\times\mathbb{P}(\mathbb{H}))^{(1)}} \mathbb{R}(x)^\times & \longrightarrow & \bigoplus_{y\in(X\times\mathbb{P}(\mathbb{H}))^{(2)}} \mathbb{Z} & \longrightarrow & CH^2(X\times\mathbb{P}(\mathbb{H})) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K_1(\mathbb{R}(X)\otimes_{\mathbb{R}}\mathbb{H}) & \longrightarrow & \bigoplus_{z\in X^{(1)}} K_0(\mathbb{R}(z)\otimes_{\mathbb{R}}\mathbb{H}) & \longrightarrow & CH^1(X,\mathbb{Z}_{\mathbb{H}}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{Nrd}} & & \downarrow{\scriptstyle \mathrm{Nrd}} & & \downarrow & & \\
\mathbb{R}(X)^\times & \longrightarrow & \bigoplus_{z\in X^{(1)}} \mathbb{Z} & \longrightarrow & CH^1(X) & \longrightarrow & 0
\end{array}
$$

Here, the upper-left vertical map is the trivial map on the summands for which x does not map to the generic point of X, and on the other summands, it is the unique map such that the composite $\mathbb{R}(x)^\times \to \mathbb{R}(X)^\times$ is the norm map for the finite field extension $\mathbb{R}(X) \subset \mathbb{R}(x)$. (Note that $\mathbb{H}$ splits over $\mathbb{R}(x)$ for every point $x \in X \times \mathbb{P}(H)$ and thus $K_1(\mathbb{R}(x) \otimes_{\mathbb{R}} \mathbb{H}) \cong \mathbb{R}(x)^\times$. The upper-left vertical map is thus given by the transfer map for $K_1(- \otimes_{\mathbb{R}} \mathbb{H})$.) Similarly, the upper-middle vertical map is given on the summand indexed by y by the unique map such that the composite $\mathbb{Z} \to \mathbb{Z}$ is multiplication by the degree of $[\mathbb{R}(y) : \mathbb{R}]$ (which, of course, is either 1 or 2). The composite of the two right-hand vertical maps is π_* where $\pi : X \times \mathbb{P}(\mathbb{H}) \to X$ is the projection map.

Since the upper-middle vertical map is surjective, so is $CH^2(X \times \mathbb{P}(\mathbb{H})) \to CH^1(X)$.

The upper-left vertical map is also surjective (see [21]). The kernel of the upper-middle vertical map is generated by zero cycles on $X \times \mathbb{P}(\mathbb{H})$ of the form $z \times \gamma$, where z is a closed point on X and γ is a zero cycle on $\mathbb{P}(\mathbb{H})$ of degree 0. Now, the degree map determines an isomorphism

$$\deg : CH^1(\mathrm{Spec}(F) \otimes_{\mathbb{R}} \mathbb{P}(\mathbb{H})) \cong \Gamma(\mathrm{Spec}(F), \mathbb{Z}_{\mathbb{H}})$$

for any field extension $\mathbb{R} \subset F$. (To see this, note that $CH^1(\mathrm{Spec}(F) \otimes_F \mathbb{P}(\mathbb{H})) \cong K_0(F \otimes_{\mathbb{R}} \mathbb{H})$ under the decomposition $K_0(\mathrm{Spec}(F) \otimes_{\mathbb{R}} \mathbb{P}(\mathbb{H})) \cong K_0(F) \otimes K_0(F \otimes_{\mathbb{R}} \mathbb{H}.)$ Thus $z \times \gamma$ determines the trivial element of $CH^1(\mathrm{Spec}(k(z)) \times_{\mathbb{R}} \mathbb{P}(\mathbb{H}))$ and hence there is a rational function f on $\mathrm{Spec}(\mathbb{R}(z)) \times_{\mathbb{R}} \mathbb{P}(\mathbb{H}))$ such that $\mathrm{div}(f) = z \times \gamma$. Regarding f as an element of the summand indexed by $x = \mathrm{Spec}(\mathbb{R}(z)) \times_{\mathbb{R}} \mathbb{P}(\mathbb{H})$ in the upper-left group, we see that the kernel of the upper-left map surjects onto the kernel of the upper-middle map. A diagram chase shows that $CH^2(X \times \mathbb{P}(\mathbb{H})) \to CH^1(X)$ is injective. □

We now present a detailed example. Suppose E is a real elliptic curve such that $E(\mathbb{R})$ forms a disjoint union of two circles. To be specific, say E is defined by the equation $y^2 = x^3 - x$, although any such curve should give an example of the phenomenon we seek. Let F be the field of rational functions on E. Since F is the function field of a real curve, we have $\Sigma_2(F) = \Sigma_4(F) = \Sigma(F)$ [16, 1.9]. (In general, $\Sigma_l(F)$ is the set of field elements that are sums of at most l squares, $\Sigma(F) = \cup_l \Sigma_l(F)$, and $\Sigma_l(F)^\times = \Sigma_l(F) - \{0\}$.) The kernel of $\mathrm{div} : F^\times \to \bigoplus_{x \in X^{(1)}} \mathbb{Z}$ is $\mathbb{R}^\times$, from which we deduce that the kernel of the map $\Sigma(F)^\times \xrightarrow{\mathrm{div}_{\mathbb{R}}} \bigoplus_x \mathbb{Z}_{\mathbb{H}}(x)$ is $(\mathbb{R}^\times)^2$. (Note that -1 is not a sum of squares in the function field of a curve with real points.) This leads to the exact sequence

$$0 \to \ker\Big(CH^1(E, \mathbb{Z}_{\mathbb{H}}) \to CH^1(E)\Big) \to F^\times/(\{\pm 1\}\cdot\Sigma(F)^\times) \xrightarrow{\overline{\text{div}}} \bigoplus_{x\in E(\mathbb{R})} \mathbb{Z}/2 \to \operatorname{coker}\big(CH^1(E,\mathbb{Z}_{\mathbb{H}}) \to CH^1(E)\big) \to 0,$$

where $\overline{\text{div}}$ is the map sending a rational function to its valuation mod 2 on just the real points of E. (One can show directly that a sum of squares has even valuation at all real points, but this is implied by the above considerations.) In particular,

$$\ker\big(CH^1(E,\mathbb{Z}_{\mathbb{H}}) \to CH^1(E)\big)$$

is non-zero if and only if there is a rational function that is not a sum of squares and yet has even valuation at all real points of E. We claim the function $f = x+\alpha$, for any $-1 \le \alpha \le 0$, is such an example. Indeed, we have $\overline{div}(f) = 0$, since f has a double pole at infinity and a double zero at the only real zero (since $(0,0)$ and $(1,0)$ in $E(\mathbb{R})$ have vertical tangent lines — in the case $f = x+\alpha$ for $-1 < \alpha < 0$, there are no real zeros). We note in passing that the function $f = x+1$ also has this property, since $(-1,0)$ has a vertical tangent line, but it turns out to be a sum of squares (see below).

For $f = x+\alpha$, $-1 \le \alpha \le 0$, we show f does not belong to $\{\pm 1\}\cdot\Sigma(F)^\times$. Since $f \in A = \mathbb{R}[x,y]/(y^2 - x^3 + x)$, if f were a sum of squares in F, then we would have an equation of the form $fg^2 = h^2 + l^2$, for some $g, h, l \in A$ and $g \neq 0$. (Recall $\Sigma_2(F) = \Sigma(F)$, so that two terms on the right suffice — this is not a meaningful aspect of the argument, however.) Now evaluate this equation on the topological space $E(\mathbb{R})$. On one component the function $f = x+\alpha$ is positive and on the other it is negative. Since a square of an element of A is non-negative on all of $E(\mathbb{R})$, we see that g has to be identically zero on all of $E(\mathbb{R})$. But then g has infinitely many zeroes and thus $g = 0$, a contradiction. Similarly, $-f$ cannot be a sum of squares, and so $f \neq \{\pm 1\}\cdot\Sigma(F)^\times$.

We have shown $\ker\big(CH^1(E,\mathbb{Z}_{\mathbb{H}}) \to CH^1(E)\big) \neq 0$, with each $-1 \le \alpha \le 0$ providing a non-zero element arising from the function $x+\alpha$. For example, the function x determines the element $2[P] - 2[\infty] \in CH^1(E,\mathbb{Z}_{\mathbb{H}})$, where $P = (0,0)$, which is non-zero but maps to 0 in $CH^1(E)$, since $div(x) = 2[P] - 2[\infty]$.

In fact, the elements in $\ker\big(CH^1(E,\mathbb{Z}_{\mathbb{H}}) \to CH^1(E)\big)$ determined by $x+\alpha$ for $-1 \le \alpha \le 0$ coincide, as we now show. We first observe that although $x+1$ lies in the kernel of $\overline{div}$, it does not induce a non-zero element in $\ker\big(CH^1(E,\mathbb{Z}_{\mathbb{H}}) \to CH^1(E)\big)$, since it is a sum of squares. Indeed, we have $(x+1)(x+\beta)^2 = y^2 + (2\beta+1)x^2 + (\beta+1)^2x + \beta^2$ in A, and there are values of β so that $2\beta+1 > 0$ and $(\beta+1)^4 - 4(2\beta+1)\beta^2 = 0$, making $(2\beta+1)x^2 + (\beta+1)^2x + \beta^2$ a sum of squares. Since $x+1$ is a sum of squares in F, we have that $x(x-1) = y^2/(x+1)$ is also sum of squares and hence x and $x-1$ determine the same element in $\ker\big(CH^1(E,\mathbb{Z}_{\mathbb{H}}) \to CH^1(E)\big)$. Since $x+1$ belongs to $\Sigma(F)^\times$, so does $x+\alpha$ for each $\alpha \ge 1$. Another calculation now shows that $x+\alpha$ for $-1 \le \alpha \le 0$ all determine the same element of $\ker\big(CH^1(E,\mathbb{Z}_{\mathbb{H}}) \to CH^1(E)\big)$. More generally, if $f \in A$ is a function of just x and $\overline{div}(f) = 0$, then the element in $\ker\big(CH^1(E,\mathbb{Z}_{\mathbb{H}}) \to CH^1(E)\big)$ determined by f coincides with that determined by x. It seems likely, then, that $\ker\big(CH^1(E,\mathbb{Z}_{\mathbb{H}}) \to CH^1(E)\big) \cong \mathbb{Z}/2$, but we have not managed to show this.

This example serves several purposes. On the one hand, it shows $CH(-,\mathbb{H})$ is an interesting invariant of real varieties, since even for a smooth curve X, we have

$$CH(X,\mathbb{H}) \cong \mathbb{Z} \oplus CH^1(X,\mathbb{Z}_{\mathbb{H}})$$

and thus $CH(X,\mathbb{H})$ differs from the ordinary Chow group of X.

This example also shows that one cannot hope to define Chern classes in $CH^t(-,\mathbb{Z}_{\mathbb{H}})$ via a splitting principle. For note that we have a commutative diagram

$$\begin{array}{ccc} CH^q(X,\mathbb{Z}_{\mathbb{H}}) & \longrightarrow & CH^1(X) \\ \big\downarrow & & \big\downarrow \\ CH^q(X\times\mathbb{P}(\mathbb{H}),\mathbb{Z}_{\mathbb{H}}) & \xrightarrow{\cong} & CH^q(X\times\mathbb{P}(\mathbb{H})). \end{array}$$

Our example shows that the top horizontal arrow can fail to be injective, and hence the map

$$CH^q(X,\mathbb{Z}_{\mathbb{H}}) \to CH^q(X\times\mathbb{P}(\mathbb{H}))$$

can fail to be injective too.

Finally, since the map

$$CH(X,\mathbb{H}) \to CH(X\times\mathbb{P}(\mathbb{H}),\mathbb{H})$$

is injective for all X, we see that our twisted Chow groups and those of Kahn and Levine do not coincide, in general.

4. Higher twisted K-groups, motivic groups, and total Chern classes

In this section we extend the definition of the twisted total Chern class map to the higher twisted K-groups, taking values in the twisted motivic groups. In fact, we define the twisted total Segre class map (which differs from the twisted total Chern class map only by an inverse) at the level of spectra.

4.1. **Twisted K-theory via twisted forms of Grassmann varieties.** Define $\mathrm{Grass}_A^m(-)$ to be the functor from k-varieties to sets represented by $\mathrm{Grass}_A(A^m)$. That is, for a k-variety X, define

$$\mathrm{Grass}_A^m(X) = \{\text{quotient objects } \mathcal{O}_X \otimes A^m \twoheadrightarrow P \text{ of } A\text{-bundles}\}.$$

Let $\mathrm{Grass}^m \rightarrowtail \mathrm{Grass}^{m+1}$ be the evident natural transformation induced by the surjection $\mathcal{O}_X \otimes A^{m+1} \twoheadrightarrow \mathcal{O}_X \otimes A^m$ onto the first m components, and define

$$\mathrm{Grass}_A = \varinjlim_m \mathrm{Grass}_A^m,$$

so that an element of $\mathrm{Grass}_A(X)$ is a quotient object $\mathcal{O}_X \otimes A^\infty \twoheadrightarrow P$ with the surjection factoring through $\mathcal{O}_X \otimes A^m$ for $m \gg 0$. Let $\mathrm{Grass}_A(X\times\Delta^\bullet)$ be the simplicial set $d \mapsto \mathrm{Grass}_A(X\times\Delta^d)$.

A choice of k-linear injection $\mu : k^\infty \oplus k^\infty \rightarrowtail k^\infty$ determines a binary operation operation μ^* on Grass_A which sends a pair of quotient objects $(\mathcal{O}_X \otimes A^\infty \twoheadrightarrow P, \mathcal{O}_X \otimes A^\infty \twoheadrightarrow Q)$ to the quotient object given by the composition of

$$\mathcal{O}_X \otimes A^\infty \twoheadrightarrow \mathcal{O}_X \otimes A^\infty \oplus \mathcal{O}_X \otimes A^\infty \twoheadrightarrow P \oplus Q$$

where the first map is induced by the dual of μ. (The dual of k^∞ is, of course, a countably infinite *product*, not sum, of copies of k, but since the quotient objects in question factor through $\mathcal{O}_X \otimes A^m$, for $m \gg 0$, this distinction matters not.)

More generally, a k-linear injection $(k^\infty)^{\oplus j} \rightarrowtail k^\infty$ determines a j-ary operation on Grass_A.

The reason we use injections instead of surjections of the form $k^\infty \twoheadrightarrow (k^\infty)^{\oplus j}$, as was done previously in this paper, is to employ a version of the linear isometries operad. Namely, define $\mathcal{I}$ to be the simplicial operad given by $j \mapsto \mathcal{I}(j)$, where $\mathcal{I}(j)$ is the simplicial set $d \mapsto \mathcal{I}(j)(d)$ with $\mathcal{I}(j)(d)$ defined to be the collection of $k\Delta^d$-linear splittable injections from $((k\Delta^d)^\infty)^{\oplus j}$ to $(k\Delta^d)^\infty$. Then $\mathcal{I}$ satisfies the axioms of an E_∞-operad. We have pairings

$$\mathcal{I}(j)(d) \times \mathrm{Grass}_A(X \times \Delta^d)^{\times j} \to \mathrm{Grass}_A(X \times \Delta^d)$$

compatible with the face and boundary maps, which induce pairings of simplicial sets

$$\mathcal{I}(j) \times \mathrm{Grass}_A(X \times \Delta^\bullet)^{\times j} \to \mathrm{Grass}_A(X \times \Delta^\bullet), j \geq 0.$$

These pairings satisfy the axioms of an operad action.

Definition 4.1. Define $\mathcal{K}^{\mathrm{geom}}(X, A)$ to be the Ω-spectrum associated to the simplicial set $\mathrm{Grass}_A(X \times \Delta^\bullet)$ equipped with the above action of the simplicial E_∞-operad $\mathcal{I}$.

Theorem 4.2. *For X smooth, there is a chain of natural homotopy equivalences joining $\mathcal{K}^{geom}(X, A)$ and $\mathcal{K}(X, A)$.*

Proof. This would be a special case of the main theorem of [15] (using also [11, 6.8]), except for the fact that non-commutative rings were not considered in that paper. Nevertheless, all the relevant proofs of [15] carry over without any modification into the non-commutative context, with the possible exception that we need to show $\mathcal{K}(X \times -, A)$ is homotopy invariant for X smooth. As noticed before, homotopy invariance holds since $\mathcal{K}(X \times \mathbb{A}^m, A)$ is canonically a summand of $\mathcal{K}(X \times \mathbb{A}^m \times \mathbb{P}(A))$, by Quillen's Theorem [22], and $X \times \mathbb{P}(A)$ is smooth. □

4.2. **Twisted Multiplicative motivic cohomology space.** In this section, we define an Ω-spectrum which is the natural target of the twisted total Chern class map. Its homotopy groups are related the the groups of units of twisted motivic cohomology in the expected manner.

To motivate the following definition, it is useful to first consider how the spectrum level total Segre class map will be defined. As we have already seen, a point in $\mathrm{Grass}_A(X)$ is a quotient object $\mathcal{O}_X \otimes A^\infty \twoheadrightarrow P$ to which we associate the closed immersion

$$\mathbb{P}_{(X,A)}(P) \rightarrowtail X \times \mathbb{P}(A^\infty).$$

We then take the effective cycle associated to this closed immersion, giving an element of $Z_{j-1}(X, \mathbb{P}(A^\infty)$, where $\mathrm{rank}_{\mathcal{O}_X}(P) = nj$. The slight additional twist in this section is that we use the notion of relative degree. That is, the Chow variety of effective $r-1$-cycles on $\mathbb{P}(A^m)$ decomposes by degree. The degree of an integral subvariety $Z \subset \mathbb{P}(A^m)$ is defined to be the degree of $Z_{\overline{k}} = Z \times_k \mathrm{Spec}\, \overline{k}$ in $\mathbb{P}(A^m)_{\overline{k}} \cong \mathbb{P}^{nm-1}_{\overline{k}}$, where $\overline{k}$ is the separable closure of k. Degrees of effective cycles are, of course, defined by taking the sum of the degrees of the subvarieties which comprise them, and the degree of a non-effective cycle is the difference of the degrees of its positive and negative parts. In this way, $Z_{r-1}(X, \mathbb{P}(A^\infty))$ decomposes by relative degree (i.e., degree over the separable closure of the generic point of X) and, in particular, the cycle associated to $\mathbb{P}_{(X,A)}(P) \rightarrowtail X \times \mathbb{P}(A^\infty)$ lies in the relative degree 1 part. Define $Z_{r-1}(-, \mathbb{P}(A^\infty))_1$ to be the subfunctor of $Z_{r-1}(-, \mathbb{P}(A^\infty))$ consisting of cycles of relative degree 1. (This refers to all cycles, not just effective ones.)

The target of the twisted total Chern class, defined on $\mathrm{Grass}_A(X \times \Delta^\bullet)$, is thus

$$\coprod_{r \geq 0} Z_{rj-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))_1$$

where $j = j(X, A)$ (see Definition 2.10). Here, we interpret Z_{-1} to refer to the one point space consisting of the empty cycle, which has degree one by convention and which acts as the identity element under the operation of linear join of cycles. As with Grass_A, there is an evident action of the operad $\mathcal{I}$, arising from the operation of linear join of cycles:

$$\mathcal{I}(j) \times (\coprod_{r \geq 0} Z_{rj-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))_1)^{\times j} \to \coprod_{r \geq 0} Z_{rj-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))_1.$$

This action is well-defined since the join of cycles of relative dimension $rj - 1$ and $sj - 1$ is $(r + s)j - 1$ and since the join of cycles of relative degree d and e has relative degree de.

Definition 4.3. For a smooth variety X, $\mathcal{H}_{mult}(X, A)$ is the homotopy-theoretic group completion of the $\mathcal{I}$-space

$$\coprod_{r \geq 0} Z_{rj-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))_1$$

where $j = j(X, A)$.

Proposition 4.4. *Assume X is a smooth k-variety and k is a field admitting resolutions of singularities. There are natural homotopy fibration sequences*

$$Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))_1 \to \mathcal{H}_{mult}(X, A) \to \Gamma(X, \mathbb{Z}_A)$$

and

$$Z_{n-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))_1 \to \mathcal{H}_{\mathcal{M}}(X, A) \xrightarrow{deg} \mathbb{Z}$$

of spectra. Consequently, for $q > 0$, we have a natural isomorphism

$$\pi_q \mathcal{H}_{mult}(X, A) \cong \pi_q \mathcal{H}_{\mathcal{M}}(X, A),$$

and for $q = 0$, we have a natural isomorphism

$$\pi_0 \mathcal{H}_{mult}(X, A) \cong \Gamma(X, \mathbb{Z}_A) \times CH(X, A)_1^\times,$$

where $CH(X, A)_1^\times$ denotes fiber over 1 of the degree map $CH(X, A) \xrightarrow{deg} \mathbb{Z}$, which is a subgroup of $CH(X, A)^\times$.

Proof. This is a generalization of [12, 1.4], which gives the case $A = k$, and the proof given here is quite similar; we refer the reader there for more details.

The second homotopy fibration sequence exists simply because

$$Z_{rj-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty)) \xrightarrow{deg} \mathbb{Z}$$

is a surjection of simplicial abelian groups for any $r \geq 1$. To see this, note that by Theorem 2.8 and the fact that linear join by a cycle of degree 1 preserves degrees, it suffices to check the case $rj = n$. In this case, it is obvious, since $X \times \mathbb{P}(A) \subset X \times \mathbb{P}(A^\infty)$ is a cycle of degree 1. In particular, we have the isomorphism

$$\pi_0(Z_{j-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))_1) \cong CH(X, A)_1^\times,$$

Moreover, this gives a subgroup of $CH(X,A)^\times$ since $CH(X,A)^\times_1$ can be identified with $1+deg^{-1}(0)$ and $deg^{-1}(0)$ is a nilpotent ideal in the ring $CH(X,A)$, a fact which can be seen by the splitting principle (Theorem 2.21) for twisted Chow groups.

The total space of $\mathcal{H}_{mult}(X,A)$ is the homotopy theoretic group completion H^+ of the homotopy-commutative H-space

$$H := \coprod_{r\geq 0} Z_{rj-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))_1.$$

We give an explicit model for H^+ by first analyzing $\pi_0(H)$. We have homotopy equivalences

$$Z_{j-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))_1 \xrightarrow{\sim} Z_{nj-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))_1$$

for all $n\geq 1$, given by taking linear join with γ^{n-1} for some chosen element $\gamma\in Z_{j-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))_1$. This allows us to map H to $\mathbb{N}\times Z_{j-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))_1$ such that $\pi_0(H)$ is identified with the sub-monoid of

$$\pi_0(\mathbb{N}\times Z_{j-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))_1)\cong \mathbb{N}\times CH(X,A)^\times_1$$

consisting of those pairs (n,δ) where $\delta=\gamma$ if $n=0$. Since $CH(X,A)^\times_1$ is a group, we see that upon inverting any element

$$\epsilon\in\pi_0 Z_{rj-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))_1\subset\pi_0(H)$$

such that $r\geq 1$, the monoid $\pi_0(H)$ becomes a group. In particular, it suffices to invert the class of

$$\epsilon=[X\times\mathbb{P}(A)]\in\pi_0 Z_{n-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))_1.$$

It follows (see [12, 1.4] for more details) that H^+ is homotopy equivalent to the mapping telescope of the sequence of pointed spaces

$$H\xrightarrow{\alpha}H\xrightarrow{\alpha}\cdots$$

where α is defined by taking linear join with $X\times\mathbb{P}(A)$ and the basepoint for the l-th occurrence of H in this sequence is the linear join of $X\times\mathbb{P}(A)$ with itself l times. We identify $\Gamma(X,\mathbb{Z}_A)$ with $j\mathbb{Z}\subset\mathbb{Z}$ and let $H\twoheadrightarrow j\mathbb{N}$ be the map sending $Z_{nj-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty)_1$ to nj. There is a commutative ladder of pointed spaces

$$\begin{array}{ccccc} H & \xrightarrow{f_1} & H & \xrightarrow{f_2} & \cdots \\ \downarrow & & \downarrow & & \\ j\mathbb{N} & \xrightarrow{+n} & j\mathbb{N} & \xrightarrow{+n\cdots} & \end{array}$$

(where the basepoints of the l-th copy of $j\mathbb{N}$ is ln). Upon forming mapping telescopes, we get a surjection $H^+\twoheadrightarrow j\mathbb{Z}=\Gamma(X,\mathbb{Z}_A)$. The induced sequence of maps on fibers of this ladder,

$$Z_{n-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))_1\to Z_{2n-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))_1\to Z_{3n-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))_1\to\cdots,$$

consists entirely of homotopy equivalences by Theorem 2.8 (the homotopy equivalence of this theorem clearly restricts to one on degree one subspaces), and hence the mapping telescope of this sequence is homotopy equivalent to its first member, $Z_{n-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))_1$.

Finally, the splitting of the induced short exact sequence

$$0\to\pi_0 Z_{n-1}(X\times\Delta^\bullet,\mathbb{P}(A^\infty))\to\pi_0\mathcal{H}_{mult}(X,A)\to\Gamma(X,\mathbb{Z}_A)\to 0$$

is given canonically by the composition of

$$\Gamma(X, \mathbb{Z}_A) = j\mathbb{N} \to \pi_0 Z_{j-1}(X \times \Delta^\bullet) \to \pi_0(H^+),$$

where the first map sends $j \in j\mathbb{N}$ to the identity element of $CH(X, A)^\times \cong \pi_0 Z_{j-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))$. □

4.3. **The higher total twisted Chern class map.** We can now define the higher total twisted Segre class map.

Definition 4.5. For a smooth variety X, define

$$s_A : \mathcal{K}(X, A) \sim \mathcal{K}^{\mathrm{geom}}(X, A) \to \mathcal{H}_{mult}(X, A)$$

be the map of spectra induced by the map of $\mathcal{I}$-spaces

$$\mathrm{Grass}_A(X \times \Delta^\bullet) \to \coprod_n Z_{nj-1}(X \times \Delta^\bullet, \mathbb{P}(A^\infty))_1$$

that sends a quotient $\mathcal{O}_X \otimes A^\infty \twoheadrightarrow P$ to the degree one cycle $\mathbb{P}_{(X,A)}(P) \subset X \times \mathbb{P}(A^\infty)$. Let

$$s_A : K_q(X, A) \to \pi_q \mathcal{H}_{\mathcal{M}}(X, A), \quad \text{for } q > 0,$$

and

$$s_A : K_0(X, A) \to CH(X, A)_1^\times$$

denote the induced maps on homotopy groups, using the isomorphisms of Proposition 4.4. The twisted total Chern class map c_A on $K_q(X, A)$ is defined by $c_A(x) = s_A(x)^{-1}$.

For the map on π_0 groups, we have actually used the isomorphism

$$\pi_0 \mathcal{H}_{mult}(X, A) \cong \Gamma(X, \mathbb{Z}_A) \times CH(X, A)_1^\times$$

of Proposition 4.4 together with the projection map to define s_A. This is done to agree with the definition of s_A on $K_0(X, A)$ given before (i.e., so that the following proposition, whose proof is easy, holds). The map to the other factor

$$K_0(X, A) \to \Gamma(X, \mathbb{Z}_A)$$

sends the class $[P]$ of an object $P \in \mathcal{P}(X, A)$ to $\mathrm{rank}_{\mathcal{O}_X}(P)/n$, and it is a surjection.

Proposition 4.6. *The maps $s_A : K_0(X, A) \to CH(X, A)^\times$ given by Definitions 2.12 and 4.5 coincide.*

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