FINAL EXAM

Math 221-Sec 005, Spring Semester 2005

Name (Print):______________________________

Student ID Numer:______________________________

Professor Mohammad Rammaha

INSTRUCTIONS:

- *Be sure to write your name on each page. There are 7 pages of questions and this cover sheet.*
- *SHOW ALL YOUR WORK. Partial credit will be given only if your work is relevant and correct.*
- *Please make your work as clear and easy to follow as possible.*
- *Do not spend too long on any one problem—note the point value of the problem when deciding how much time to spend! You do not need to work the problems in the order in which they appear.*
- *This examination is closed book.*

Question	Points	Score
1	16	
2	10	
3	16	
4	18	
5	22	
6	10	
7	10	
8	8	
9	20	
10	12	
11	16	
12	14	
13	28	
Total	200	

1. [16 Points] Solve the IVP: $(x+1)y' - 3y = 6x, \quad y(0) = 1, \quad x > -1.$

2. [10 Points] Find the largest interval on which the following IVP has a unique solution. To receive any credit, you must show technical details that support your answer. You need not solve the IVP:

$$(x-3)y' + \left(\sqrt{x+1}\ \sin x\right) y = e^x, \quad y(0) = 3.$$

3. [16 Points] Find the **exact** form of the particular solution y_p of the following differential equation if the undetermined coefficient method is used; **But do not evaluate the coefficients**:

$$D(D^2 - 1)y = 3x + 4e^{-x}$$

4. [18 Points] Suppose a tank originally contains 50 gallons of water with y_0 lb of sugar in solution. Water containing 2 lb of sugar per gallon is pumped into the tank at a rate of 3 gallon per minute and the mixture is drained out at the same rate.

a) [8 Points] Set up the initial value problem (**but do not solve**) whose solution gives $y(t)$, the amount of sugar in the tank at any time t.

b) [10 Points] **Without** solving the IVP in part a), **but with justification**, sketch reasonable graphs of two solutions $y_1(t)$ and $y_2(t)$ where $y_1(0) = 0$ and $y_2(0) = 100$, and determine the behavior of each solution as $t \to \infty$.

5. [22 Points] Consider the system of ODE's:

$$\begin{cases} \dfrac{dx}{dt} = 4y(1-x) \\[2ex] \dfrac{dy}{dt} = x(x-1) \end{cases} \tag{1}$$

a) [5 Points] Find **all** critical (equilibrium) points of the system (1) above.

b) [5 Points] Find the solution of the system (1) that satisfies the initial condition $x(0) = 1$, $y(0) = 4$. Make sure to explain your answer.

c) [12 Points] **Determine** and **describe** all trajectories for system (1). Sketch the graphs of only two trajectories (with orientation) in the phase plane. One satisfies $(x(0), y(0)) = (2, 0)$, and the second satisfies $(x(0), y(0)) = (0, 1)$. Describe the behavior of the corresponding solutions $x(t)$ and $y(t)$ as t gets large.

6. [10 Points] **Describe and carefully sketch** the region in the xy−plane where the hypotheses of Picard's existence and uniqueness theorem are satisfied (so that there is a unique solution through each given initial point in this region):

$$y' = \sqrt{y}\ \ln(1 - x^2 - y^2), \quad y(x_0) = y_0.$$

7. [10 Points] Find an explicit formula for the solution of the IVP:

$$\frac{dy}{dx} = 2x(9 + y^2), \quad y(0) = 3.$$

8. [8 Points] Determine whether the following set of functions are linearly dependent or independent on $(-\infty, \infty)$: (**Make sure to provide a proof**)

$$f_1(x) = \sin x, \quad f_2 = x^2 - 3\sin x, \quad f_3(x) = 4x^2 + \sin x.$$

9. [20 Points] Find the general solution of the following equations:

a) [8 Points] $y'' - 2y' + 2y = 0$.

b) [12 Points] $D(D+1)^2(D^2+9)y = 0$.

10. [12 Points] Given that $y_1(x) = x+1$ is a solution of the equation: $(x+1)^2y'' - (x+1)y' + y = 0$, $x > -1$. Find a second solution $y_2(x)$ of the equation that is linearly independent from $y_1(x)$ on $(-1, \infty)$, and write down the general solution to the equation.

11. [16 Points] Given that $y_1(x) = x^2$ and $y_2(x) = x^3$ are two linearly independent solutions of the ODE: $x^2y'' - 4xy' + 6y = 0$, for $x > 0$. Use the method of variation of parameters to find the general solution of the non-homogeneous equation

$$x^2y'' - 4xy' + 6y = x^2, \quad x > 0.$$

12. [14 Points] Find $Y(s) = L[y(t)](s)$, where $y(t)$ is the solution of the IVP:
$y'' + 6y' + 9y = e^{-t}U_3(t),\ \ y(0) = -2,\ y'(0) = 3$. **Do not solve for** $y(t)$**!!**

13. [28 Points] Evaluate each of the following:

a) [8 Points] $L\left[t\,e^t \sin 3t\right](s)$

b) [8 Points] $L\left[\int_0^t (\cosh 4\tau)\,(t-\tau)^{19} d\tau\right](s)$

c) [12 Points] $L^{-1}\left[e^{-7s}\frac{s-1}{s^2+4s+13}\right]$