Solutions to

MATH 221 EXAM 3 NAME:____________________

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Please Show Your Work To Ensure Full Credit.

1. (42 pts) Evaluate each of the following:

a.(8 pts) $\mathcal{L}\left[e^{-6t}\cos(2t)\right](s) = L\left[\cos(2t)\right](s+6) = \dfrac{s+6}{(s+6)^2+4}$

b. (10 pts) $\mathcal{L}\left[t\sin(3t)\right](s) = (-1)^1\left(\dfrac{d}{ds}\right) L\left[\sin 3t\right](s)$

$$= -\left(\frac{d}{ds}\right)\frac{3}{s^2+9} = \frac{6s}{(s^2+9)^2}$$

c. (12 pts) $\mathcal{L}^{-1}\left[\dfrac{s-1}{s^2+6s+45}\right] = L^{-1}\left[\dfrac{s-1}{(s^2+6s+9)+36}\right]$

$$= L^{-1}\left[\frac{s-1}{(s+3)^2+36}\right] = L^{-1}\left[\frac{s+3-4}{(s+3)^2+36}\right]$$

$$= L^{-1}\left[\frac{s+3}{(s+3)^2+36}\right] - \frac{4}{6}L^{-1}\left[\frac{6}{(s+3)^2+36}\right]$$

$$= e^{-3t}\cos 6t - \frac{2}{3}e^{-3t}\sin 6t.$$

d. (12 pts) $\mathcal{L}^{-1}\left[e^{-\pi s}\frac{1}{(s-3)s}\right]$. First note: $\frac{1}{(s-3)s} = \frac{1}{3}\left(\frac{1}{s-3} - \frac{1}{s}\right)$

So, $L^{-1}\left[e^{-\pi s}\frac{1}{(s-3)s}\right] = L^{-1}\left[\frac{e^{-\pi s}}{3}\left(\frac{1}{s-3} - \frac{1}{s}\right)\right]$

$= \frac{1}{3} L^{-1}\left[e^{-\pi s} L\left[e^{3t} - 1\right](s)\right]$

$= \frac{1}{3} U_{\pi}(t)\left[e^{3(t-\pi)} - 1\right].$

2. (12 pts) Use the Laplace transform to solve the integral equation: (Put: $Y(s) = L[y(t)](s)$)

$$y(t) + 6\int_0^t y(\tau)\sinh 3(t-\tau)\, d\tau = \cosh 3t.$$

$\Leftrightarrow L[y(t)](s) + 6\, L[y(t) * \sinh 3t](s) = L[\cosh 3t]s.$

$\Leftrightarrow Y(s) + 6\, Y(s) \cdot \frac{3}{s^2-9} = \frac{s}{s^2-9}$

$\Leftrightarrow Y(s)\left[1 + \frac{18}{s^2-9}\right] = \frac{s}{s^2-9}$

$\Leftrightarrow Y(s)\left[\frac{s^2+9}{s^2-9}\right] = \frac{s}{s^2-9} \Leftrightarrow Y(s) = \frac{s}{s^2+9}$

$\Leftrightarrow y(t) = L^{-1}\left[\frac{s}{s^2+9}\right] = \cos 3t.$

3. (14 pts) Consider the system of ODE's:

$$\begin{cases} \frac{dx}{dt} = x + 3y \\ \frac{dy}{dt} = 3x + y \end{cases} \tag{0.1}$$

Find **only one** solution to the system that corresponds to the positive eigenvalue of the matrix A associated with the system above.

① E-values for $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$: Solve the eqn.: $\det(A - \lambda I) = 0$

$\Leftrightarrow \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = 0 \Leftrightarrow 1 - \lambda = \pm 3$

$\Leftrightarrow \lambda = 4, -2$

are the e-values.

② "E-vector for $\lambda = 4$": Find $u = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ s.t.

$(A - 4I)u = 0 \Leftrightarrow \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} -3\alpha + 3\beta \\ 3\alpha - 3\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \alpha = \beta$. Take $\beta = 1 \Rightarrow \alpha = 1$.

So, $\left(\lambda = 4, u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ is an e-pair, which gives one solution $Y_1(t) = e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4. (12 pts) Find $Y(s) = \mathcal{L}[y(t)](s)$ where $y(t)$ is the solution of the IVP:

$$y'' + 2y' + y = e^t U_3(t); \qquad y(0) = -2, \quad y'(0) = 3.$$

You Need Not Find y(t).

$L[y'' + 2y' + y](s) = L[e^t U_3(t)](s) \Leftrightarrow$

$s^2 Y(s) + 2s - 3 + 2\{s Y(s) + 2\} + Y(s) = e^{-3s} L[e^{t+3}](s)$

$= e^3 e^{-3s} \frac{1}{s-1}$.

$\Leftrightarrow Y(s)[s^2 + 2s + 1] = -1 - 2s + e^3 e^{-3s} \frac{1}{s-1}$.

$\Leftrightarrow Y(s) = \frac{-1}{(s+1)^2} - \frac{2s}{(s+1)^2} + e^3 e^{-3s} \frac{1}{(s-1)(s+1)^2}$.

so equil. solutions are $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \alpha \\ 2 \end{pmatrix}$; $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$; $\alpha \in \mathbb{R}$.

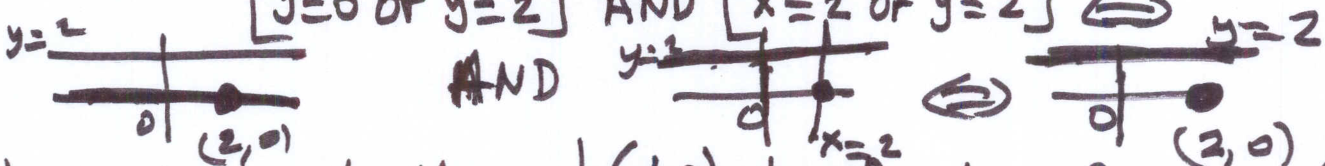
5. (20 pts) Consider the system

$$\begin{cases} \dfrac{dx}{dt} = -y(y-2) \\[2ex] \dfrac{dy}{dt} = (x-2)(y-2) \end{cases} \tag{0.2}$$

(a) (6 pts) Find all equilibrium solutions of the system (0.2) above.

Equil. pts: $-y(y-2)=0$ & $(x-2)(y-2)=0 \iff$

$[y=0 \text{ or } y=2]$ AND $[x=2 \text{ or } y=2] \iff$



Hence, the equil. pts are $\{(\alpha, 2) : \alpha \in \mathbb{R}\} \cup \{(2,0)\}$

(b) (2 pts) What is the solution $(x(t), y(t))$ that starts at $(x(0), y(0)) = (4, 2)$?

Since $(4,2)$ is an equil. pt. then $(x(t), y(t)) = (4,2)$, for all t.

(b) (12 pts) Determine and describe all the trajectories of the system (0.2). Sketch the graph of only TWO trajectories (with orientation): one satisfies $(x(0), y(0)) = (2, 3)$, the second satisfies $(x(0), y(0)) = (2, -3)$; and briefly describe the behavior of each corresponding solution $(x(t), y(t))$ as t gets large.

The phase eqn. is $\dfrac{dy}{dx} = \dfrac{(x-2)(y-2)}{-y(y-2)} = \dfrac{x-2}{-y}$ (separable eqn.)

$\iff \int -y\,dy = \int (x-2)\,dx \iff -\frac{1}{2}y^2 = \frac{1}{2}(x-2)^2 - \frac{1}{2}C$

$\iff (x-2)^2 + y^2 = C$ (the integral curves are circles).
centered at (2,0).

From picture, both trajectories approach the pt B as $t \to +\infty$. To find B: Note the drawn circle is

$(x-2)^2 + y^2 = 9$

so, if $y=2 \Rightarrow (x-2)^2 + 4 = 9 \Rightarrow x-2 = \pm\sqrt{5}$

$\Rightarrow x = \cancel{2+}\,\pm\sqrt{5}$, i.e. $x = 2-\sqrt{5}$ and $B = (2-\sqrt{5}, 2)$.
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