MATH 221 EXAM 1 NAME:________________

Summer I, 2007

Instructor: Mohammad Rammaha

Please Show Your Work Neatly And Clearly To Ensure Full Credit.

1. (14 points) Find an EXPLICIT FORMULA for the solution of the IVP: $\frac{dy}{dx} = 4xy^{\frac{3}{2}}$, $y(1) = 4$; and from the formula determine the domain of the solution.

$y^{-3/2}\,dy = 4x\,dx$ • Integrate to get:

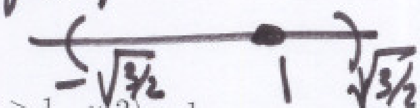
$\int y^{-3/2}\,dy = 4\int x\,dx \iff -2y^{-1/2} = 2x^2 + 2C$

$\iff \frac{1}{\sqrt{y}} = -(x^2 + C)\Big|_{x=1,\,y=4}$

we get: $\frac{1}{2} = -1 - C \Rightarrow C = -\frac{3}{2}$. So,

$\frac{1}{\sqrt{y}} = -\left(x^2 - \frac{3}{2}\right)$ which gives $y = \frac{1}{\left(x^2 - \frac{3}{2}\right)^2}$.

Domain of solution is $\left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$

Since $\longrightarrow$ 

2. (16 points) Solve the IVP: $(x-1)y' + 3y = \frac{x}{(x-1)^3}$, $x > 1$, $y(2) = 1$.

Normalize:

$$y' + \frac{3}{x-1}y = \frac{x}{(x-3)^4} \quad \text{———} \quad (*)$$

$$\text{I.F.} = e^{\int \frac{3}{x-1}dx} = e^{3\ln(x-1)} = (x-1)^3. \quad \text{Multiply by I.F.:}$$

$$(x-1)^3 y' + 3(x-1)^2 y = \frac{x}{x-1} \iff$$

$$\frac{d}{dx}\left[(x-1)^3 y\right] = \int \frac{x}{x-1}dx \iff (x-1)^3 y = \int \frac{x}{x-1}dx$$

$$= \int \frac{x-1+1}{x-1}dx = \int \left(1 + \frac{1}{x-1}\right)dx = x + \ln(x-1) + C.$$

So, $y(x) = \frac{x + \ln(x-1)}{(x-1)^3} + \frac{C}{(x-1)^3}$. But,

$1 = y(2) = 2 + C \Rightarrow C = -1 \Rightarrow y(x) = \frac{x + \ln(x-1) - 1}{(x-1)^3}$.

3. (12 points) Find the largest open interval on which the following IVP has a unique solution:
$(x-4)\ y' + (e^x \sin x)\ y = \ln(x-1), \quad y(2) = 777.$

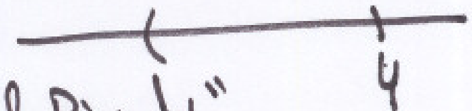
You must show technical details to receive credit.

Normalize to get: $y' + \frac{e^x \sin x}{x-4} y = \frac{\ln(x-1)}{x-4}$ — $(*)$

Here, $P(x) = \frac{e^x \sin x}{x-4}$ which is cts. on $\mathbb{R} \setminus \{4\}$

and $f(x) = \frac{\ln(x-1)}{x-4}$ which is cts. whenever $x>1$ & $x \neq 4$.

Since $2 \in (1,4)$ and both $P(x)$ & $f(x)$ are cts. on $(1,4)$ then the IVP above has a unique solution $y(x)$ defined on $(1,4)$; by "special Picard".



4. (14 points) A tank initially contains 100 gal of pure water. Brine containing 2 lb of salt per gallon enters the tank at the rate of 3 gal/min and the (perfectly mixed) solution is drained from the tank at the same rate.
(a) (10 points) Set up (BUT DON'T SOLVE) the initial value problem whose solution gives the amount $y(t)$ of salt in the tank at any time $t \geq 0$.

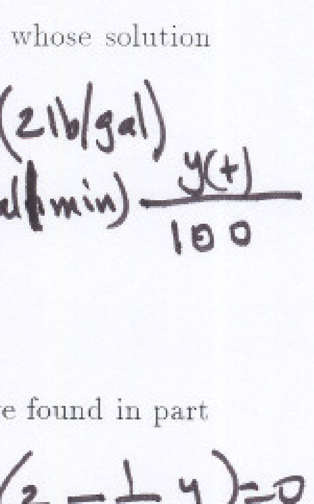
$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out} = (3\,\text{gal/min})(2\,\text{lb/gal}) - (3\,\text{gal/min})\frac{y(t)}{100}$$

or

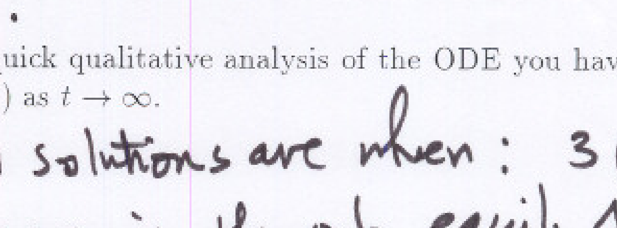
$$(*) \quad \begin{cases} \frac{dy}{dt} = 6 - \frac{3}{100}y = 3\left(2 - \frac{1}{100}y\right) \\ y(0) = 0. \end{cases}$$

(b) (4 points) By using a quick qualitative analysis of the ODE you have found in part (a), predict the value of $y(t)$ as $t \to \infty$.

The equilibrium solutions are when: $3\left(2 - \frac{1}{100}y\right) = 0$ i.e., $y(t) = 200$ is the only equil. sol. & we have the phase line diagram:



So, qualitatively, our solution looks like so:



Hence, $\lim_{t \to \infty} y(t) = 200$

5. (20 points) This problem deals with the ODE:

$$\frac{dy}{dx} = 1 + \left(\frac{y}{x}\right)^2. \tag{1}$$

(a) (10 points) Carefully explain why equation (1) is of the *homogeneous* kind and use an appropriate change of variables to reduce it into a separable equation, BUT DON'T SOLVE the resulting equation.

The eqn. has the form: $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ where $F(s) = 1 + s^2$.

Put $z = \frac{y}{x} \Leftrightarrow y = xz$. So, $\frac{dy}{dx} = x\frac{dz}{dx} + z$.

the eqn. becomes:

$$x\frac{dz}{dx} + z = 1 + z^2 \Leftrightarrow$$

$$x\frac{dz}{dx} = z^2 - z + 1 \Leftrightarrow$$

$$\frac{dz}{z^2 - z + 1} = \frac{dx}{x} \quad \text{(which is separable)}$$

(b) (10 points) Use the Euler method with step size $h = \frac{1}{3}$ to approximate the value of $y(2)$; where $y(x)$ is the solution of the IVP: $\frac{dy}{dx} = 1 + \left(\frac{y}{x}\right)^2$, $y(1) = 1$. Show the details of your work by using a proper table. Here, $f(x,y) = 1 + \left(\frac{y}{x}\right)^2$ and we have

$x_0 = 1 \quad x_1 = \frac{4}{3} \quad x_2 = \frac{5}{3} \quad x_3 = 2$

k	(x_k, y_k)	$m_k = 1 + \left(\frac{y_k}{x_k}\right)^2$	$y_{k+1} = y_k + h m_k$.
0	$(1, 1)$	2	$y_1 = 1 + 2\left(\frac{1}{3}\right) = \frac{5}{3} \cong y(4/3)$
1	$\left(\frac{4}{3}, \frac{5}{3}\right)$	$1 + \left(\frac{5/3}{4/3}\right)^2 = \frac{41}{16} \simeq 2.56$	$y_2 = \frac{5}{3} + \left(\frac{1}{3}\right)\left(\frac{41}{16}\right) = \frac{121}{48} \simeq 2.52$
2	$\left(\frac{5}{3}, \frac{121}{48}\right)$	$1 + \left(\frac{121/48}{5/3}\right)^2 \cong 3.29$	$y_3 \cong 2.52 + \frac{1}{3}(3.29) \cong 3.615$

So, $y(2) \cong 3.615$

6. (14 points) Consider the ODE: $y' = y^2(2-y)\sin y$.

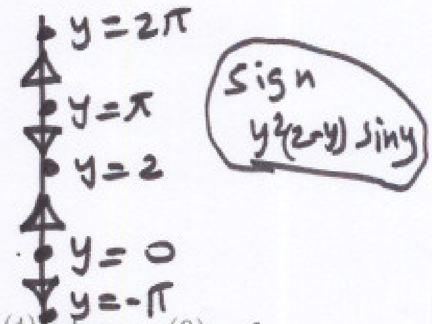
(a) (8 points) Find all equilibrium solutions and sketch the phase line diagram. Classify 3 nonnegative equilibrium solutions as sink (stable), source (unstable), or node (semi-stable). equil. solutions are when: $y^2(2-y)\sin y = 0$

$\Longleftrightarrow y = 0,\ y = 2,\ y = n\pi;\ n \in \mathbb{Z}.$

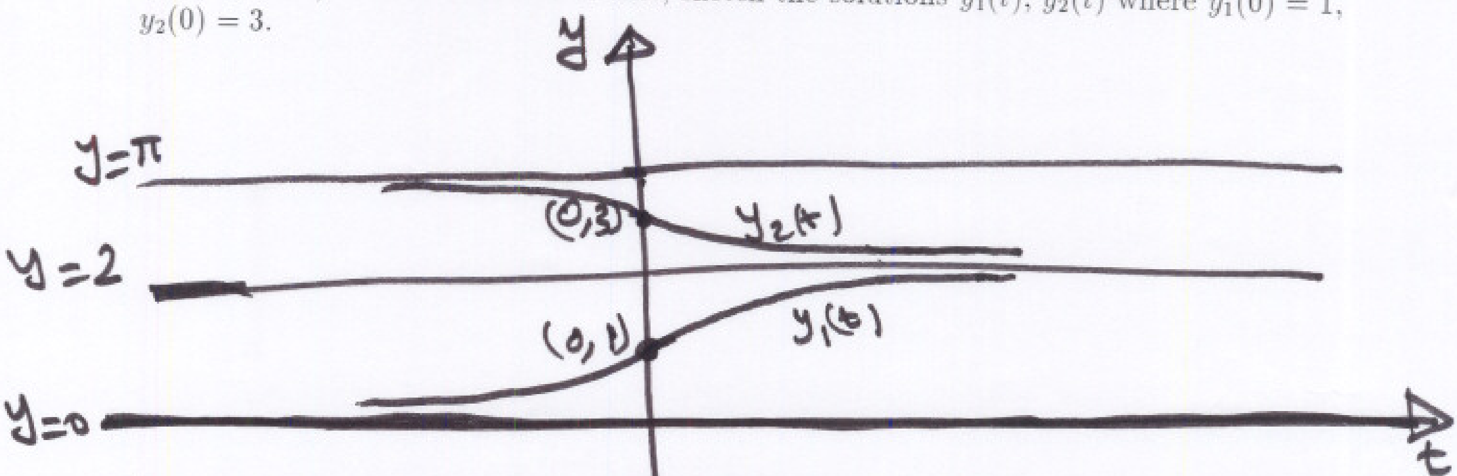
$y = 0$ is a source

$y = 2$ is a sink

$y = \pi$ is a source.



(b) (6 points) On the same set of axis, sketch the solutions $y_1(t)$, $y_2(t)$ where $y_1(0) = 1$, $y_2(0) = 3$.



7. (10 points) Describe and carefully sketch the region in the $xy-$plane where the hypotheses of Picard's existence and uniqueness theorem are satisfied (so that there is a unique solution through each given initial point (x_0, y_0) in this region).

$y' = y + \sqrt{x^2+y^2-1}, \quad y(x_0) = y_0.$ Here,

$f(x,y) = y + \sqrt{x^2+y^2-1}$ is cts. whenever $x^2+y^2 \geq 1$; and

$\frac{\partial f}{\partial y}(x,y) = 1 + \frac{y}{\sqrt{x^2+y^2-1}}$ is cts. (in $\mathbb{R}^2$) whenever $x^2+y^2 > 1$.

So, both f & $\frac{\partial f}{\partial y}$ are cts. whenever $x^2+y^2 > 1$ (which is ~~the~~ set of all pts. (x,y) exterior to the unit disc as shown excluding the circle $x^2+y^2 = 1$.