FIBONACCI'S RABBITS

Math 107 Project, Due: April 18, 2002.
Spring 2002

The original problem that Fibonacci investigated in the year 1202 was about how fast rabbits could breed in ideal circumstances. Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was... How many pairs will there be in one year?

Well, the problem is not that difficult, at the end of the first month, they mate, but there is still one only 1 pair. At the end of the second month the female produces a new pair, so now there are 2 pairs of rabbits in the field. At the end of the third month, the original female produces a second pair, making 3 pairs in all in the field. At the end of the fourth month, the original female has produced yet another new pair, the female born two months ago produces her first pair also, making 5 pairs. So, it is easy to see that the number of pairs of rabbits in the field at the start of each month is $1, 1, 2, 3, 5, 8, 13, 21, 34, ...$ This sequence of positive integers is known as the Fibonacci sequence and it can be given by $\{F_n : n = 1, 2, 3, ...\}$; where $F_1 = 1,\ F_2 = 1$, and

$$F_{n+1} = F_n + F_{n-1} \text{ for } n = 2, 3, 4, ... \tag{0.1}$$

In this project, you are asked to do the following:

(1) **The Golden Ratio:** Let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. A quick calculation shows that a and b are the roots of the equation $x^2 - x - 1 = 0$. Prove that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = a$. (*the number a was known (but not understood) to the ancient Greeks as the Golden Ratio. Check the web site* **http://www.geocities.com/cyd_conner/page1.html** *for more information about the name Golden Ratio and other fascinating facts about the Fibonacci sequence and how it relates to arts, music, and nature.*

Hint: Suppose $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = R$. Use the definition of the F_n's in equation (0.1) and argue that $\lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \frac{1}{R}$. Show that R has to be a.

(2) **The Fibonacci sequence in closed form:** Prove that F_n can be expressed by: $F_n = \dfrac{a^n - b^n}{a - b}$ for $n = 1, 2, 3, ...$
Hint: Set $G_n = \frac{a^n - b^n}{a-b}$ and show that G_n satisfies equation (0.1). That is, G_n satisfies the recurrence formula $G_{n+1} = G_n + G_{n-1}$ for every $n = 2, 3, 4, ...$ Here, do not forget that a and b satisfy the equation $x^2 - x - 1 = 0$.

(3) **Interval of convergence of a power series:** Consider the power series:

$$\sum_{n=1}^{\infty} F_n x^n. \tag{0.2}$$

Use the ratio test to determine the open interval on which the power series (0.2) converges.
Hint: You have already computed something useful in part (1).

(4) **Taylor and Fibonacci:** Show that the Taylor series of the function $f(x) = \dfrac{x}{1 - x - x^2}$ about $x = 0$ is given by:

$$\frac{x}{1 - x - x^2} = \sum_{n=1}^{\infty} F_n x^n, \tag{0.3}$$

where F_n is the Fibonacci sequence. *The function* $f(x) = \frac{x}{1-x-x^2}$ *is called the generating function of the Fibonacci sequence.*
Hint: It is not easy to show that $\dfrac{f^{(n)}(0)}{n!} = F_n$ for all n, and thus obtaining (0.3) has to be done in a clever way. Here is one way to do it: Call $H(x)$ the sum of the series (0.2) on the interval of convergence you found in part (3), i.e., set $H(x) = \sum_{n=1}^{\infty} F_n x^n$. By keeping in mind (0.1), compute $(1 + x)H(x)$ and then find the value of $x(1 + x)H(x)$.

(5) Write a short essay that does not exceed one page on any useful/interesting facts about the Fibonacci sequence. Here, you are encouraged to search the web for such information.