Math107 Spring 2008
Calculus II University of Nebraska-Lincoln

Name: **Section #:**

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(print legibly)

Exam 2

Instructions

- Turn off all communication devices. If you do not do so, then you will not receive any credit for your exam.
- There are 7 pages in this exam with 6 problems. Before you begin, make sure that your exam has all 7 pages.
- The examination period is from 9:30am to 10:20am. If you wish to receive credit for your exam, then make sure that your exam is submitted for grading by 10:20am.
- You may NOT use a calculator during the exam.
- You may not use a text, notes, nor any other reference.
- To receive full credit for a problem, you must provide a correct answer and a sufficient amount of work so that it can be determined how you arrived at your answer.
- Clearly indicate what your solutions are and any work that you do not want to be included in the grading process.
- Write your solutions in an explicit form whenever possible.
- If you wish to speak with a proctor during the exam, then raise your hand and a proctor will come to you.
- Each problem will be graded out of 20 points.
- If it is determined that you have given or received any unauthorized aid during this exam, then you will receive no credit for your exam.

Problem	**Score**
1	
2	
3	
4	
5	
6	
Total	

1. For the following problems, let $a_n = \dfrac{2n+1}{n}$ for $n = 1, 2, 3, \ldots$.

   (a) Compute the first five terms in the sequence $\{a_n\}$.

   Solution:

$$\boxed{a_1 = \frac{3}{1}, a_2 = -\frac{5}{2}, a_3 = \frac{7}{3}, a_4 = -\frac{9}{4}, a_5 = \frac{11}{5}}$$

   (b) If the sequence $\{a_n\}$ converges, compute $\lim_{n\to\infty} a_n$. Otherwise, explain why the sequence diverges.

   Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{n} = \lim_{n\to\infty} \frac{n}{n}\frac{2+\frac{1}{n}}{1} = 2 + 0$$

$$\boxed{2.}$$

2. (a) Rewrite $\displaystyle\sum_{n=5}^{9} \frac{1}{\sqrt{n}}$ without Sigma notation. DO NOT SIMPLIFY YOUR ANSWER.

Solution:
We have that

$$\sum_{n=5}^{9} \frac{1}{\sqrt{n}} = \boxed{\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{9}}.}$$

(b) Is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ a p-series, geometric series, alternating series, or none of these. (Indicate all that apply.)

Solution:

This is $\boxed{\text{a } p\text{-series with } p = \frac{1}{2}.}$

(c) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converge? JUSTIFY YOUR ANSWER.

Solution:
This series $\boxed{\text{diverges}}$, since it is a p-series with $p = \frac{1}{2} < 1$.

This can also be verified using the integral test with $f(x) = x^{-\frac{1}{2}}$.

3. These problems refer to the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n}(x+1)^n$.

(a) What is the center of this power series?

Solution:

The center is at $\boxed{x=-1.}$

(b) What is the radius of convergence for this power series?

Solution:
We us the ratio test:

$$\lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}|x+1|^{n+1}}{\frac{n}{2^n}|x+1|^n} = \lim_{n\to\infty} \frac{n+1}{2n}|x+1| = \frac{1}{2}|x+1|;$$

By the ratio test, the series converges if $\frac{1}{2}|x+1|<1$ and it diverges if $\frac{1}{2}|x+1|>1$.

It follows that the series converges if $|x+1|<2$ and it diverges if $|x+1|>2$. The radius of convergence is $\boxed{2.}$

(c) What is the interval of convergence for this power series?

Solution:
From (b), we know that the series converges for x in $(-3,1)$ and diverges if $x<-3$ or $x>1$. It only remains to see what happens at $x=-3$ and at $x=1$.

At $x=-3$, the power series equals

$$\sum_{n=0}^{\infty} \frac{n}{2^n}(-3+1)^n = \sum_{n=0}^{\infty}(-1)^n n.$$

This series fails the n-th term test, so it diverges.

At $x=1$, the power series equals

$$\sum_{n=0}^{\infty} \frac{n}{2^n}(1+1)^n = \sum_{n=0}^{\infty} n.$$

This series also fails the n-th term test, so it diverges.

The interval of convergence is $\boxed{(-3,1).}$

4. Find the 3rd order Taylor polynomial $P_3(x)$ generated by $f(x) = \frac{5}{6} + \frac{1}{6}x^6$ at $x = 1$.

Solution:
We need to compute the derivatives up to order 3 for f and evaluate them at 1:

$$f(x) = \frac{5}{6} + \frac{1}{6}x^6 \qquad f(1) = 1$$
$$f'(x) = x^5 \qquad f'(1) = 1$$
$$f''(x) = 5x^4 \qquad f''(1) = 5$$
$$f'''(x) = 20x^3 \qquad f'''(1) = 20.$$

Assembling the Taylor polynomial yields

$$\boxed{P_3(x) = 1 + (x-1) + \frac{5}{2}(x-1)^2 + \frac{10}{3}(x-1)^3.}$$

5. For this problem, recall that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad \text{for all } x \text{ in } (-\infty, \infty).$$

Recall also that the radius of convergence for this power series is ∞.

(a) Provide a power series representation, with a center at $x = 0$, for e^{3x}. What is its radius of convergence?

Solution:
Substituting $3x$ for x in the power series representation for e^x, we find that

$$\boxed{e^{3x} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n.}$$

Since the radius of convergence for the power series for e^x is infinity, the radius of convergence for the power series for e^{3x} is also ∞.

(b) Provide a power series representation, with a center at $x = 0$, for $\int \frac{e^{3x} - 1}{x} \, dx$. What is its radius of convergence?

Solution:
The power series for $\frac{e^{3x}-1}{x}$ is

$$\frac{1}{x} \left\{ \left(1 + \sum_{n=1}^{\infty} \frac{3^n}{n!} x^n \right) - 1 \right\} = \sum_{n=1}^{\infty} \frac{3^n}{n!} x^{n-1}.$$

We can use the ratio test to find the radius of convergence for this series:

$$\lim_{n \to \infty} \frac{\frac{3^{n+1}}{(n+1)!} |x|^n}{\frac{3^n}{n!} |x|^{n-1}} = \lim_{n \to \infty} \frac{3}{n+1} |x| = 0.$$

This the series converges for all x and the radius of convergence is ∞.

Integrating term-by-term, we have

$$\int \frac{e^x - 1}{x} \, dx = \int \sum_{n=1}^{\infty} \frac{3^n}{n!} x^{n-1} \, dx = \sum_{n=1}^{\infty} \frac{3^n}{n!} \int x^{n-1} \, dx$$

$$= \boxed{C + \sum_{n=1}^{\infty} \frac{3^n}{(n!)n} x^n.}$$

From a theorem in the text, we know that the radius of convergence for this new series is also ∞. One could also use the ratio test to determine this.

6. (20 points) A 1 foot tall empty tank has the shape in the figure below. Compute the total amount of work done to fill the tank by pumping oil in through the bottom. Assume that the oil is pumped into the tank from ground level at $y = 0$, and assume that the weight density of the oil is 40 $\frac{\text{pounds}}{\text{feet}^3}$.