ESSENTIAL LAMINATIONS, EXCEPTIONAL SEIFERT-FIBERED SPACES, AND DEHN FILLING

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ABSTRACT

We show how essential laminations can be used to provide an improvement on (some of) the results of the 2π-Theorem; at most 20 Dehn fillings on a hyperbolic 3-manifold with boundary a torus T can yield a reducible manifold, finite π_1 manifold, or exceptional Seifert-fibered space. Recent work of Wu allows us to add toroidal manifolds to this list, as well.

Keywords: Essential lamination, Seifert-fibered space, Dehn filling

Introduction and outline

Much recent research in 3-manifold topology has been motivated by the conjecture that 'most' 3-manifolds are hyperbolic, i.e., their interiors admit a metric of constant curvature -1. One of the largest single pieces of evidence supporting this view is the so-called 2π-theorem of Gromov and Thurston (see [BH]), which essentially says that Dehn filling along a curve γ in the boundary T of a complete hyperbolic 3-manifold M of finite volume, which is represented by a geodesic of length greater than 2π, yields a manifold $M(\gamma)$ which admits a metric of (variable) negative curvature - see below for a more precise statement. Current efforts also lend increasing weight to the conjecture that 3-manifolds with a negative curvature metric are hyperbolic.

A (closed) 3-manifold N can fail to be hyperbolic in one of several ways: it can have finite fundamental group, it can contain an essential 2-sphere (i.e., N is reducible), it can contain an essential (embedded) torus, or it can be what is known as an exceptional Seifert-fibered space: N is foliated by circles, has infinite fundamental group, but contains no essential torus. In this paper, we focus mainly on this last possibility. Specifically, we show:

Theorem:. *Given a hyperbolic 3-manifold M with boundary a torus T, at most 20 Dehn fillings along T can yield manifolds which are reducible or have finite π_1 or are exceptional Seifert-fibered spaces.*

This result follows from three main facts. Bleiler and Hodgson [BH], following ideas of Adams [Ad1], have shown that the 2π-theorem implies:

Theorem [BH]. *: At most 24 Dehn fillings on a hyperbolic 3-manifold do not admit negatively-curved metrics.*

This is because 24 is the largest number of primitive points in a lattice, inside of a circle of radius 2π in the plane $\mathbf{R}^2$, each having distance from the origin at least 1, and whose basis vectors span a parallelogram of area $\geq \sqrt{3}$ (i.e., such that the quotient torus $\mathbf{R}^2/L$ has area $\geq \sqrt{3}$).

On the other hand, Gabai and Mosher (see [Mo] for details) have shown, using techniques from pseudo-Anosov flows, that every hyperbolic 3-manifold M with boundary a torus T contains a 'very full' essential lamination $\mathcal{L}$ disjoint from T. See [GO] for definitions and basic concepts concerning essential laminations. 'Very full' means that the complementary components of $\mathcal{L}$ are all solid tori, except for the component N containing T, whose metric completion is homeomorphic to $T \times I$ with some finite collection of parallel simple closed curves removed from one boundary component. These curves, projected to T and given parallel orientations, give a collection of curves (typically, a single curve) γ representing the element

$$(a,b) \in \mathbf{Z} \oplus \mathbf{Z} = H_1(T),$$

called the *degeneracy locus*. The number a/b is called the *degeneracy slope* of the lamination (w.r.t. our chosen basis for $H_1(T)$), and |gcd(a,b)| is its multiplicity.

We denote by $M(p/q)$ the result of p/q Dehn filling along T; the closed manifold obtained by gluing a solid torus to M along T so that the curve representing (p,q) in $H_1(T)$ is glued to the boundary of the meridian disk of the solid torus. It follows from standard arguments (see, e.g., [GK],[De1]) that this lamination $\mathcal{L}$ remains essential in the manifold $M(p/q)$ so long as $|aq-bp| > 1$, i.e., the curves representing (a,b) and (p,q) in T have intersection number greater than one.

Gabai [Ga] has termed *genuine* any essential lamination $\mathcal{L}$ in a 3-manifold M for which some component N of the manifold $M|\mathcal{L}$ obtained by splitting M open along $\mathcal{L}$ is not an I-bundle over a (usually non-compact) surface, with ∂N corresponding to the induced ∂I-bundle. In other words, a lamination $\mathcal{L}$ is not geuine if every component of $M|\mathcal{L}$ is an I-bundle. In [Br1], we noted that the classification of essential laminations in exceptional Seifert-fibered spaces [Br2],[Cl] implies that no essential lamination in an exceptional Seifert-fibered space is genuine, which gives the obvious corollary that any manifold containing a genuine essential lamination is not an exceptional Seifert-fibered space.

But it is easy to see that the very full lamination $\mathcal{L}$ in M is in fact a genuine essential lamination in $M(p/q)$, provided $|aq-bp| > 2$. This is because the component N of $M(p/q)|\mathcal{L}$ containing (the image of) T is a solid torus with a collection of essential curves γ on the boundary removed. γ has intersection number $|aq-bp| > 2$ with the meridian disk of the solid torus. This solid torus cannot, therefore, be given the structure of an I-bundle as described above; the meridian disk would have to intersect the curves γ exactly twice.

It follows, therefore, that the Dehn fillings which give exceptional Seifert-fibered spaces must lie on the five lines $ay - bx{=}k$, where $k{=}0$ (i.e., $x/y = a/b$), ± 1, or ± 2. Note that the lattice points lying on the lines $ay - bx = 1$ and $ay - bx = -1$ (i.e., $a(-y) - b(-x) = 1$), for example, give the same Dehn filling coefficients $x/y = (-x)/(-y)$. So the coefficients of the exceptional Seifert-fibered fillings all lie on the three lines $ay - bx{=}0,1,2$. In particular, since manifolds containing essential laminations are irreducible and have infinite fundamental group [GO], the coefficients of finite/reducible fillings all lie on the two lines $ay - bx{=}0,1$. Therefore, all finite/reducible/exceptional Seifert-fibered fillings lie on these three lines.

As we shall see, a straightforward calculation, following the lines of [BH], shows that, for a lattice in $\mathbf{R}^2$ as described above, at most 20 primitive lattice points within the disk of radius 2π can lie on three such lines. Since none of the three types of manifolds (finite π_1, reducible, or Seifert-fibered) can admit a metric of negative curvature - manifolds with such metrics have universal cover $\mathbf{R}^3$ (so are not finite/reducible) and contain no $\mathbf{Z}\oplus\mathbf{Z}$'s in their fundamental group (so are not exceptional Seifert-fibered spaces) - we immediately obtain the theorem.

1. The calculations

We first recall the setup of [BH]. We can assume that the universal cover of M is hyperbolic 3-space $\mathbf{H}^3$ with a collection of horoballs removed. We can expand the horoballs equivariantly until they first touch one another. One of these horoballs can be assumed to be centered at ∞, so its boundary horosphere $\mathbf{E}^2$ is a horizontal plane in the upper half-space model, which we can assume lies at height 1 above the x-y plane. The metric on $\mathbf{H}^3$, restricted to $\mathbf{E}^2$, is a Euclidean metric on $\mathbf{E}^2$. The fundamental group of the boundary torus T acts by Euclidean isometries on $\mathbf{E}^2$, and so acts on the points of tangency of $\mathbf{E}^2$ with the other horoballs. Since those horoballs have diameter one and are disjoint from one another, these points of tangency are at least distance one apart. These points form a lattice L in $\mathbf{E}^2$, which (after choosing an origin) represent the closed curves in T. Furthermore, by Adams [Ad1], a fundamental domain for this lattice has area at least $\sqrt{3}$. Any Dehn filling corresponding to a lattice point outside of the circle of radius 2π about the origin gives rise to a manifold admitting a metric of negative curvature.

Following [BH], by a Euclidean isometry, we can assume that L has basis consisting of $v_1{=}d(1,0)$ and $v_2{=}d(x_0,y_0)$, where $d \geq 1$, $0 \leq x_0 \leq 1/2$, $0 \leq y_0$, and $x_0^2 + y_0^2 \geq 1$. These are the so-called geometric coordinates on the cusp T; v_1 represents the shortest geodesic on T, and v_2 represents the shortest geodesic that is not a multiple of v_1. Furthermore, since the area of the parallelogram spanned by v_1 and v_2 (which is a fundamental domain for L) is $d^2y_0 \geq \sqrt{3}$, we have $y_0 \geq \sqrt{3}/d^2$.

With respect to this basis, our degeneracy locus is represented by

$$av_1{+}bv_2, \text{ for some } a,b{\in}\mathbf{Z}$$

For convenience, we will denote such lattice points as $(\mathbf{a},\mathbf{b}) = av_1{+}bv_2{\in}L$.

We wish to bound, for any fixed $(a,b) \neq (0,0)$, the number of points $(\mathbf{r},\mathbf{s})$ in our lattice L, representing simple closed curves (i.e., with $\gcd(r,s)$=1), of the form

$$(\mathbf{r},\mathbf{s}) = rv_1 + sv_2, \text{ where } as - br = 0,1, \text{ or } 2$$

and which lie inside of the circle of radius 2π. It is clear that for a fixed a/b, this number is maximized when $\gcd(a,b)$=1. If we set c=$\gcd(a,b)$ and a'=a/c, b'=b/c, then $as-br$=0,1,or 2 means $a's-b'r$=0,$1/c$, or $2/c$. Since $a's-b'r$ is an integer, this in turn means $a's-b'r$=0 only (if $c \geq 3$) or $a's-b'r$=0 or 1 (if c=2). These clearly yield a smaller sets of solutions than if we start with a' and b' in the first place (i.e., c=1). Therefore, in searching for an upper bound, we may assume that $\gcd(a,b)$=1.

Consider first the case $(\mathbf{a},\mathbf{b})=c(1,0)$ (i.e., b=0). Then $|as-br|=|cs|$=0,1, or 2 requires s=0,1, or 2. So we need to bound the number N of lattice points in

$A = A_0 \cup A_1 \cup A_2 = \{v_1\} \cup \{rv_1+v_2 : ||rv_1+v_2|| \leq 2\pi\} \cup \{rv_1+2v_2 :\ r$ is odd and $||rv_1+2v_2|| \leq 2\pi\}$.

In A_1 the lattice points are distance $||v_1|| \geq d$ apart, while in A_2 they are distance $\geq 2d$ apart, since only r odd will give $\gcd(rc,2)$=1.

These sets lie on the lines $y=y_0$, $y=2y_0$, and the segments of these lines lying inside the disk of radius 2π have length

$$2((2\pi)^2 - y_0^2)^{1/2} \leq 2((2\pi)^2 - (\sqrt{3}/d^2)^2)^{1/2} = 2(4\pi^2 - 3/d^4)^{1/2} = \ell_1 \text{ and}$$
$$2((2\pi)^2 - (2y_0)^2)^{1/2} \leq 2((2\pi)^2 - (2\sqrt{3}/d^2)^2)^{1/2} = 2(4\pi^2 - 12/d^4)^{1/2} = \ell_2 \ .$$

To find an upper bound on N, we therefore wish to maximize the quantities $N_1(d) = \ell_1/d$ and $N_2(d) = \ell_2/2d$, where d ranges over $[1,\infty)$. We can then fit in one more primitive lattice point than the integer parts of these quantities, on each segment. Both of these quantities, it turns out, are decreasing functions of d on $[1,\infty)$; their nearest critical points (which are local, hence global, maxima) are at $d^2=3/(2\pi)$ for N_1, and at $d^2=3/\pi$ for N_2. $N_1(1)=2((4\pi^2-3)^{1/2} \approx 12.079$, so there are at most 13 lattice points on the first line $as-br=1$ (i.e., A_1 contains at most 13 points), while $N_2(1)=2((4\pi^2-12)^{1/2}/2 \approx 5.242$, so A_2 contains at most 6 points. Consequently, A contains at most 20 = 1+13+6 points.

We obtain much the same picture as above, for any other choice of $(\mathbf{a},\mathbf{b})$, i.e., for $b \neq 0$. Points $(\mathbf{r},\mathbf{s})$ satisfying (*) $as-br=k$ are of the form $(r_n,s_n)=(r_0+na, s_0+nb)$, where (r_0,s_0) satisfies (*), and n is an integer. Consecutive points are therefore distance

$$||(\mathbf{r}_n,\mathbf{s}_n) - (\mathbf{r}_{n-1},\mathbf{s}_{n-1})|| = ||(\mathbf{a},\mathbf{b})|| = d||a(1,0)+b(x_0,y_0)||$$
$$= d((a+bx_0)^2 + (by_0)^2)^{1/2} \geq |b|dy_0 \geq |b|\sqrt{3}/d \geq \sqrt{3}/d$$

apart. Moreover, when k=2, only every other point $(\mathbf{r}_n,\mathbf{s}_n)$ will have $\gcd(r_n,s_n)$=1; this is perhaps most easily seen by considering cases. We may assume that one of r_0,s_0 is odd (if both are even, then one of r_0+a, s_0+b is odd, since a and b cannot both be even (they are relatively prime)). But then it is easy to see that both of r_0+a, s_0+b are even; $r_0b-s_0a=2$ implies that r_0b and s_0a are either both even or both odd. For example, a and b odd implies r_0 and s_0 are either both even or both odd, so both are odd (since one of them is), so the two sums are even. The other two cases are similar. One the other hand $as - br$=2 implies gcd(r, s) divides 2, so is either 2 (when both are even) or 1 (when one is odd). The above argument therefore shows that only every other pair of terms has gcd(r, s)=1.

In general, points of our lattice L are all at least distance d apart, and any line segment in the interior of the disk has length at most 4π, so we can obtain a fairly crude (though effective) estimate on the size of A; A_1 can have no more than $(4\pi/d)+1)$ points, and A_2 can have no more than $(4\pi/2d)+1)$ points (since only every second lattice point on the line has gcd(r, s)=1), so A has no more than

$$N_1'(d) = 1+(4\pi/d)+1)+(4\pi/2d)+1) = 3+(6\pi/d)$$

points. This is less than or equal to 20, provided $d \geq 6\pi/17 = \alpha$.

But in the present case (b$\neq$0) we also know that the lattice points on our line segments are at least distance $\sqrt{3}/d$ apart, so A has no more than

$$N_2'(d)=N_1'(\sqrt{3}/d)=3+(6\pi/(\sqrt{3}/d))=3+(6\pi/\sqrt{3})d$$

points. This is less than or equal to 20, provided $d \leq (17\sqrt{3})/(6\pi) = \beta$.

But $\alpha < 1.11 < 1.56 < \beta$, so d is either larger than α or smaller than β. So one of the two conditions must be satisfied, and so A never contains more than 20 primitive lattice points. This proves the theorem. ∎

Remark: Using the two bounds $3+(6\pi/d)$, $3+(6\pi/\sqrt{3})d$ for N together, in fact, implies that when b$\neq$0, $N \leq 17$. This is because one of the two bounds is always $\leq$17; the point where which is larger switches is at $d = 3^{1/4}$, and one of the bounds is always lower than the value of $N_1'=N_2'$ at this point, which is approximately 17.323. A more careful look the two main pieces $(4\pi/d)+1$, $(2\pi/d)+1$ of N_1' (and the analogous pieces of N_2') improves this upper bound to 16, since it is the integer parts of the two pieces that really contribute to N. For N_1' and N_2' the pieces also become equal at the same point $d = 3^{1/4}$, where they equal approximately 10.5484 and 5.7742, respectively. For one of N_1',N_2', each piece is less than each of these values, giving bounds of 10 and 5 on their contributions to N.

Recently Wu [Wu] has shown that if p/q Dehn filling along T produces a toroidal manifold (i.e., a manifold containing an embedded incompressible torus), then in our current notation $|aq-bp| \leq 2$, as well. This also follows from the results of [BR]. Taken together, the arguments here then show that at most 20 Dehn fillings on a hyperbolic manifold can produce a manifold which is either reducible, toroidal, has finite π_1, or is an exceptional Seifert-fibered space. According to the geometrization conjecture, this list contains all non-hyperbolic 3-manifolds. It is conjectured that the best upper bound on the number of non-hyperbolic Dehn fillings is actually 10; this hypothetical maximum is achieved by the exterior of the figure-8 knot. Recently Hodgson and Kerckhoff [HK] have shown that there is in fact a single universal bound on the number of non-hyperbolic Dehn fillings of a hyperbolic manifold; the arguments follow a line similar to the 2π-theorem, but involve a bound in the range of 4π to 8π for the length of the filling curves.

2. Concluding observations

Better bounds than what we have achieved here exist, when one makes additional assumptions about the manifold M. When the tunnel number of the manifold M (the minimum number of disjoint arcs one needs to drill out in order to turn M into a handlebody) is two or more, then Adams [Ad2] has shown that the fundamental domain of the lattice we have studied has area at least $3\sqrt{3}/2$. This better bound on the area leads, using the analysis of [BH], to a bound of 16 on the number of Dehn-filled manifolds which do not admit a negatively-curved metric. Unlike the previous bound of 24, the techniques we use here cannot obtain a correspondingly better bound; for some configurations of the lattice, these 16 points lie on our three parallel lines. In this case, if the degeneracy slope of the lamination is not the shortest geodesic, however, we can improve our bound of 16 to 13. This can be seen by replacing the function N_2' with $N_3'(d) = N_1'((3\sqrt{3})/(2d))$ and carrying out the refined analysis of the previous paragraph.

The existence of an essential lamination implies many strong properties for a 3-manifold. For example, as we have mentioned, manifolds with essential laminations (usually called *laminar* manifolds) have universal cover $\mathbf{R}^3$ [GO]. By combining this fact with the fact that manifolds with negative curvature metrics have universal cover $\mathbf{R}^3$, as well, the above analysis shows that all but at most 14 Dehn fillings (those lying on the lines $ay-bx$=0 or 1) of a hyperbolic 3-manifold M with one boundary component yield manifolds with universal cover $\mathbf{R}^3$. In the case of tunnel number at least two, all but at most 12 do. If the degeneracy slope is not the shortest geodesic, then all but 11 do, in general, and all but 9 do if M has tunnel number at least two.

Recently, essential laminations were used [BW] to completely classify the manifolds which are obtained by Dehn surgery on (non-torus) 2-bridge knots, according to whether they have finite π_1, a reducing sphere, an essential torus, an exceptional Seifert fibering, or a hyperbolic structure. They have also been used [DR] to show that non-trivial surgery on a (non-torus) alternating knot yields a manifold with universal cover $\mathbf{R}^3$, which proves (a strong version of) Property P for these knots.

In particular, essential laminations have demonstrated at least one advantage over other 'characteristic' objects in 3-manifolds. This is the property we have exploited here: there is a fairly easily recognized topological property of the lamination (genuine-ness) which implies that the ambient manifold M is not an exceptional Seifert-fibered space. This is probably the most useful topological criterion currently available for showing that a manifold does not admit an exceptional Seifert fibering. New constructions of essential laminations are therefore sure to yield similar results along the lines of this note.

The arguments here also raise several interesting questions. For example, if a manifold M were to admit essential laminations with distinct degeneracy slopes, then the finite/reducible/exceptional Seifert filling coefficients would have to lie on both sets of corresponding lines. At most 5 (primitive) lattice points can do so (just consider the slopes 1/0 and 0/1). It would be interesting to know when this occurs. Two such slopes cannot be found in general - they can't, for example, for the $(-2,3,7)$ pretzel knot exterior [BNR]. This does sometimes occur, however - for example, for the knot 8_{20} [De2].

We obtained better upper bounds on the number of 'bad' Dehn fillings when we assumed that $(\mathbf{a},\mathbf{b}) \neq (1,0)$, i.e., the degeneracy slope did not correspond to the shortest geodesic in T. It would probably be reasonable to expect that the shortest geodesic must occur as the degeneracy slope of an essential lamination, but this isn't true. According to SnapPea [We], the 18/1 curve on the boundary of the $(-2,3,7)$ pretzel knot exterior is the second shortest curve; the meridian is shortest. It would be interesting to determine when this sort of behavior does (and doesn't) occur.

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References

[Ad1] C. Adams *The noncompact hyperbolic 3-manifold of minimal volume*, Proc. AMS **100** (1987) 601–606

[Ad2] C. Adams, *Unknotting tunnels in hyperbolic 3-manifolds*, Math. Annalen **302** (1995) 177–195.

[BH] S. Bleiler and C. Hodgson, *Spherical space forms and Dehn filling*, Topology **35** (1996) 809–833.

[Br1] M. Brittenham, *Small Seifert-fibered spaces and Dehn surgery on 2-bridge knots*, to appear in Topology.

[Br2] M. Brittenham, *Essential laminations in Seifert-fibered spaces*, Topology **32** (1993) 61–85.

[BNR] M. Brittenham, R. Naimi, and R. Roberts, *Graph manifolds and taut foliations*, J. Diff. Geom. **45** (1997) 446–470.

[BR] M. Brittenham and R. Roberts, *When incompressible tori meet essential laminations*, submitted for publication.

[BW] M. Brittenham and Y.-Q. Wu, *The classification of Dehn surgeries on 2-bridge knots*, preprint.

[Cl] E. Claus, *Essential laminations in closed Seifert-fibered spaces*, Thesis, Univ. of Texas at Austin (1991).

[De1] C. Delman *Essential laminations and Dehn surgery on 2-bridge knots Topology and its Applications* **63**, (1995) 201–221.

[De2] C. Delman, *personal communication.*

[DR] C. Delman and R. Roberts, *Alternating knots have Property P*, preprint.

[Ga] D. Gabai, *Eight problems in the geometric theory of foliations and laminations on 3-manifolds*, in *Geometric Topology, vol 2*, ed. W. Kazez, Amer. Math. Soc., Providence, RI (1997) 1–33.

[GK] D. Gabai and W. Kazez, *Pseudo-Anosov maps and surgery on fibered 2-bridge knots*, Topology and its Appl. **37** (1990) 93–100.

[GO] D. Gabai and U. Oertel, *Essential Laminations in 3-Manifolds*, Annals of Math. **130** (1989) 41–73.

[GT] M. Gromov and W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987) 1–12.

[HK] C. Hodgson and S. Kerckhoff, *The shape of hyperbolic Dehn surgery space*, in preparation.

[Mo] L. Mosher, *Laminations and flows transverse to finite depth foliations*, preprint.

[We] J. Weeks, *SnapPea, a program for creating and studying hyperbolic 3-manifolds*, available by anonymous ftp from geom.umn.edu .

[Wu] Y.-Q. Wu, *Sutured manifold hierarchies, essential laminations, and Dehn surgery*, preprint.