# CANONICAL GENUS AND THE WHITEHEAD DOUBLES OF CERTAIN ALTERNATING KNOTS

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Abstract. We prove, for an alternating pretzel knot $K$, that the canonical genus of its Whitehead doubles $W(K)$ is equal to the crossing number $c(K)$ of $K$, verifying a conjecture of Tripp in the case of these knots.

## 0. Introduction

Every knot $K$ in the 3-sphere $S^3$ is the boundary of a compact orientable surface $\Sigma \subset S^3$, known as a Seifert surface for the knot $K$. The first proof of this was given by Seifert [14], who gave an algorithm which starts with a diagram of the knot $K$ and produces such a surface. The algorithm consists of orienting the knot diagram, breaking each crossing and reconnecting the resulting four ends according to the orientation, without re-introducing a crossing, producing disjoint Seifert circles in the projection plane, bounding (after offsetting nested circles) disjoint Seifert disks, and then reintroducing the crossings of $K$ by stitching the disks together with half-twisted bands.

The minimum of the genera of the canonical surfaces built by Seifert's algorithm, over all diagrams of the knot $K$, is known as the *canonical genus* or *diagrammatic genus* of $K$, denoted $g_c(K)$. The minimum genus over all Seifert surfaces, whether built by Seifert's algorithm or not, is known as the *genus* of $K$, and denoted $g(K)$.

Recently, Morton's inequality has been applied to the computation of the canonical genus of the Whitehead doubles $W(K)$ of certain knots. Tripp [16] posed the question whether the canonical genus of the Whitehead doubles of knots is equal to the crossing number of the original knot. This has been shown to be true in the case of torus knots, [16], and 2-bridge knots, [13]. This paper extends these results to show that the canonical genus of the Whitehead double of many other reduced prime alternating knots is equal to the crossing number of the original knots.

The basic tool is the following inductive step. Given a knot or link projection, $L$, we denote by $F(L)$ the flat double of $L$ which is the oriented boundary of an untwisted annulus with core circle $L$.

**Theorem.** *Let $L$ be a non-split link with a diagram $D'$ satisfying $c(D') = c(L)$ and the maximum $z-$degree of the HOMFLY polynomial of the flat Whitehead double, $F(D')$, being*

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$2c(D')-1$. *If $K$ is a link having diagram $D$ obtained by replacing a crossing in the diagram $D'$ with a full twist (so that $c(D) = c(D')+1$), then the maximum $z-$degree of the HOMFLY polynomial of the flat Whitehead double of $K$ is* $2c(D) - 1$.

This theorem serves as the inductive step for any initial link for which the hypothesis holds. A straightforward calculation, which can be completed with Mathematica, Knot Plot, or by brute force, shows that the hypothesis is satisfied for the alternating links having underlying basic polyhedra $1^*, 6^*, 8^*, 9^*, 10^*, 10^{**}$ (with a half-twist included to make it a knot instead of a link), $10^{***}$ (with three half-twists to make it a knot), $11^*$, and $11^{**}$ (with two half-twists added). This immediately implies the following theorem:

**Theorem 1.** *Let $K$ be an alternating knot with underlying polyhedra* $1^*, 6^*, 8^*, 9^*, 10^*, 10^{**}$*,* $10^{***}$*,* $11^*$*, and* $11^{**}$*. Then the canonical genus of the flat Whitehead double of the knot is equal to the crossing number of $K$.*

For underlying polyhedra $1^*$, this reproduces Gruber's result that alternating links that admit alternating "algebraic" diagrams satisfy Tripp's conjecture.

## 1. Previously Know Results

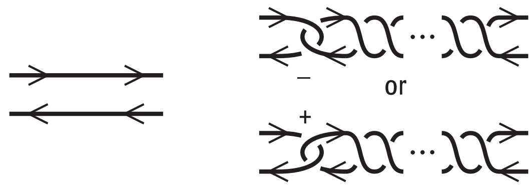
As mentioned above, the minimum of the genera of the canonical surfaces built by Seifert's algorithm, over all diagrams of the knot $K$, is known as the *canonical genus* or *diagrammatic genus* of $K$, denoted $g_c(K)$. The minimum genus over all Seifert surfaces, whether built by Seifert's algorithm or not, is known as the *genus* of $K$, and denoted $g(K)$. Like all such invariants, defined as the minimum over an infinite collection of configurations, the canonical genus and the genus can be extremely difficult to compute. However there are at least two situations where such a determination can be reasonably carried out.

The first is when $g_c(K) = g(K)$, i.e., when a canonical surface has minimum genus among all Seifert surfaces. A candidate for such a surface can, in principle (and usually in practice, as well), be verified to have minimum genus using sutured manifold theory [5]. This condition cannot always be met, however; the genus need not equal the canonical genus. For example, these genera are not equal for the Whitehead doubles of knots considered here and elsewhere, including in [13] and [16].

The second is when Morton's inequality, [11], is an equality. Morton's inequality states that the $z$-degree of the HOMFLY polynomial $P_K(v, z)$ of a knot $K$ is at most twice the canonical genus $2g_c(K)$ of $K$. A canonical surface $\Sigma$ whose genus is half of the $z$-degree, if it exists, must therefore have genus equal to the canonical genus. This condition also cannot always be met; the first examples where Morton's inequality was shown to be strict were found by Stoimenow in [15]. The authors also recently found several infinite families of examples, see [1].

Both methods succeed in computing the canonical genus of alternating knots as in [3], [6], and [12], while the second can compute the canonical genera of knots through 12 crossings, [15]. The first method provides an approach to computing the canonical genera of arborescent knots, [7]. In fact, the authors are aware of no example where both approaches

Figure 1. Whitehead doubles




are known to fail. The knots considered in [15] abd [1] can have their canonical genera computed by the first method, which is precisely why the second method is known to fail.

## 2. Definitions and Preliminaries

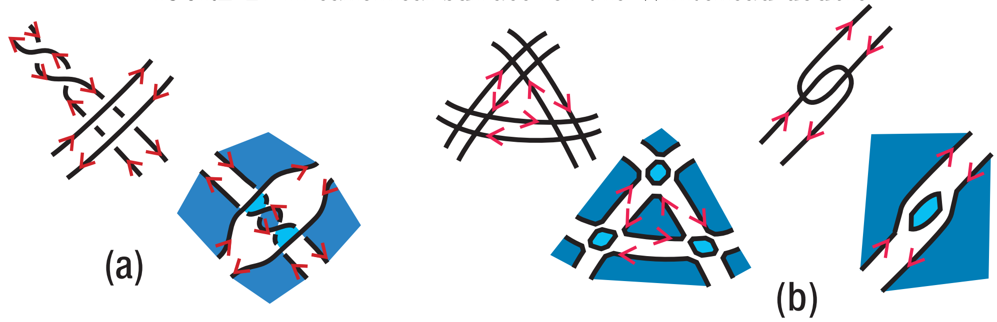
Recently there have been efforts to compute the canonical genera of Whitehead doubles $W(K)$ of knots $K$. Let $K$ denote a knot or link in the 3-sphere $S^3$, $N(K)$ a tubular neighborhood of $K$, $E(K) = S^3 \setminus \text{int}N(K)$ the exterior of $K$, and $\Sigma$ a Seifert surface for $K$, which we treat as embedded in $S^3$.

We take a diagrammatic approach to the construction of Whitehead doubles $W(K)$ of a knot $K$. Given a diagram $D$ of a knot $K$, we construct diagrams for the Whitehead doubles of $K$ as follows: Start with the diagram $D$ and choose an orientation for the underlying knot. Draw a second copy of $K$ pushed off of $K$ to the right and parallel to the original knot. Orient the new strand in the opposite direction. This gives an oriented link $F(K)$, which we will call the *flat double* of $K$. Note that we will always treat $F(K)$ as an *oriented* link, with the orientation just described.

The Whitehead double of $K$, $W(K)$, is obtained from $F(K)$ by replacing a pair of parallel arcs in the diagram by a twisted clasp, as in Figure 1. These twisted clasps consist of either a right-handed (+) or left-handed (-) clasp together with $\mid n \mid$ full twists to the parallel arcs. These are right-handed for $n > 0$ and left-handed for $n < 0$. If we need to specify a Whitehead double, rather than deal with them as a class, we will denote it by $W(K, n, +)$ or $W(K, n, -)$ . The interested reader may satisfy him/herself that this description of Whitehead doubles is equivalent to any other description he or she has encountered. Note that the notation $W(K, n, \pm)$ is very diagram-dependent. A different diagram will build the same collection of knots, but might label them differently depending upon the writhe of the diagram.

For a given non-trivial knot $K$, if we build its Whitehead doubles using a $c(K)$-minimizing diagram as above we can always hide the full twists of $W(K)$ inside of one of the small squares formed from a crossing of $K$ by isotopy as in Figure 2a. In this case, Seifert's algorithm will build a canonical surface for $W(K)$ of genus $c(K)$. To see this, note that the canonical surface is a checkerboard surface. This was our primary reason for building our Whitehead doubles with full twists in the parallel strands and then hiding them inside the doubling of a crossing of $K$. The local pictures are as in Figure 2b.

Figure 2. A canonical surface for the Whitehead double




Recently, Morton's inequality has been applied to the computation of the canonical genus of the Whitehead doubles $W(K)$ of knots $K$ for which the first approach can shed no light since $g(W(K)) = 1$ for all non-trivial $K$. Tripp, [16], computed the canonical genus of the doubles of the $(2, n)$-torus knots $T_{2,n}$, showing that $g_c(W(T_{2,n})) = n = c(T_{2,n})$ . For the proof, he demonstrates by induction on $n$ that the $z$-degree of the HOMFLY polynomial of the double is $2n = 2c(T_{2,n})$.

A general construction, for any knot $K$, provides a canonical surface with genus $c(K)$. We reproduce this construction in the next section. So we have the inequalities

$$2c(T_{2,n}) = 2n = \deg_z P_{W(T_{2,n})}(v, z) \leq 2g_c(W(T_{2,n})) \leq 2c(T_{2,n})$$

from which $g_c(W(T_{2,n})) = c(T_{2,n})$ follows.

Nakamura [13] has extended this argument to prove that for 2-bridge knots $K$, $g_c(W(K)) = c(K)$. The main part of the argument is again an inductive proof that $\deg_z P_{W(K)}(v, z) = 2c(K)$ .

In this paper we provide another extension of Tripp's result, to knots with underlying basic polyhedra $1^*, 6^*, 8^*, 9^*, 10^*, 10^{**}$ (with a half-twist added), $10^{***}$ (with three half-twists added), $11^*$, and $11^{**}$. Included in these are the alternating pretzel knots and alternating arborescent knots. We mention these specifically as the next families of knots for which the conjecture is proved.

Recall that alternating pretzel knots can be denoted $P(k_1, \dots k_n)$, $k_1, \dots, k_n \geq 1$. Note that $P(k_1, \dots k_n)$ is a knot if and only if either $n$ is odd and at most one $k_i$ is even, or $n$ is even and exactly one $k_i$ is even. We get the following immediate corollary.

**Corollary 2.** If $K$ is a pretzel knot $P(k_1, \dots k_n)$ with $k_1, \dots, k_n \geq 1$, then $g_c(W(K)) = k_1 + \dots + k_n = c(K)$ .

The main tool in the proof is a proposition which gives a method for building new knots $K$ satisfying $2c(K) = \deg_z P_{W(K)}(v, z)$ from old ones $K'$.

**Proposition 3.** If $K'$ is a knot satisfying $\deg_z P_{W(K')}(v, z) = 2c(K')$, and if for a $c(K')$-minimizing diagram for $K'$ we replace a crossing, thought of as a half-twist, with three half-twists (see Figure 4), producing a knot $K$, then $\deg_z P_{W(K)}(v, z) = 2c(K)$, and therefore $g_c(W(K)) = c(K)$ .

FIGURE 3. Pretzel knots

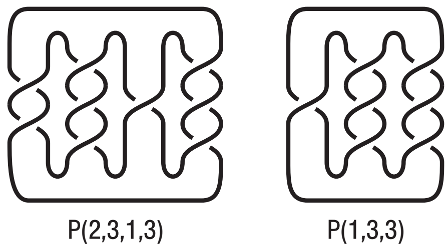
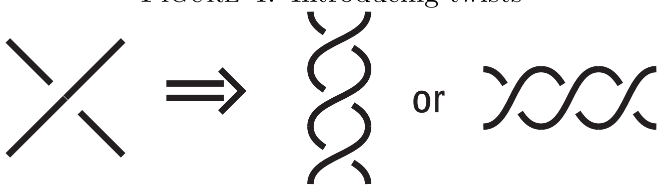



FIGURE 4. Introducing twists




Theorem 2 follows immediately by repeated application of Proposition 3 to the crossings of the $(2,n)$ torus knots $= P(1,\dots,1)$ (when all $k_i$ are odd) and the 2-bridge knots $[2,n-1] = P(2,1,\dots,1)$ (when one of the $k_i$ (without loss of generality we may assume that $k_1$) is even), where the initial hypothesis is satisfied by the results of Tripp and Nakamura. It should be clear that Theorem 1 is not the strongest result that can be proved in this way; the pretzel knots are just the simplest new class of knots encountered. A more general statement is:

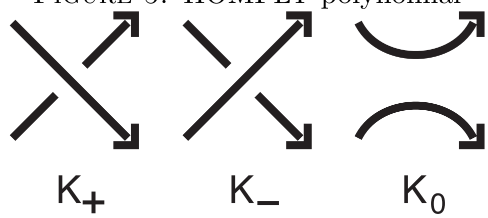
**Proposition 4.** Let $\mathcal{K}$ be the class of knots having diagrams which can be obtained from the standard diagram of the left- or right-handed trefoil knot $T_{2,3}$ by repeatedly replacing a crossing, thought of as a half-twist, by a full twist. Then for every $K \in \mathcal{K}$, $\deg_z P_{W(K)}(v,z) = 2c(K)$, and so $g_c(W(K)) = c(K)$ .

Using Tripp's and Nakamura's results, we could have stated this corollary using the a priori larger class obtained by starting with the reduced alternating diagrams of any 2-bridge knot. But we will show that all 2-bridge knots already lie in $\mathcal{K}$, and so this "larger" class would really just be $\mathcal{K}$. Stated differently, Proposition 4 implies the results of Tripp and Nakamura. We should note that the knots built from the left-handed trefoil are the mirror images of those built from the right-handed trefoil, and the $z$-degree of the HOMFLY polynomial is unchanged under taking mirror images. So it suffices to prove the result for only one of the two collections of knots.

## 3. Notations and preliminaries

Continuing our calculation of the genus, note that the canonical surfaces for all of the Whitehead doubles are homeomorphic, and so have the same genus; the surfaces differ only

Figure 5. HOMFLY polynomial




in the direction of twisting at the clasp and the number of full twists to the Seifert disk we "hid" the full twists of the doubling in. So we focus on the surface for the Whitehead double with no full twists, $W(K,0,+)$ . Its canonical Seifert surface $\Sigma$ has one Seifert disk for each complementary region of the diagram $D$ of $K$, one for each crossing of $K$, and one for the Whitehead doubling clasp. It has one half-twisted band for each crossing of $W(K,0,+)$, that is, $4c(K)+2$ twisted bands. So

$$\begin{aligned} 1-2g(\Sigma) &= \chi(\Sigma) \\ &= \mid \text{Seifert disks} \mid - \mid \text{bands} \mid \\ &= (c(K)+ \mid \text{regions of } D \mid +1) - (4c(K)+2) \\ &= \mid \text{regions} \mid -3c(K) - 1. \end{aligned}$$

But since $D \subseteq S^2$ is a 4-valent graph whose complementary regions are all disks, we have (using the usual notation) $v-e+f=2$, and $2e=4v$ (since all vertices have valence 4), so $e=2v$, and so $f = \mid\text{regions}\mid = 2+v = 2+c(K)$. So

$$1-2g(\Sigma) = 2+c(K)-3c(K)-1 = 1-2c(K).$$

So $g(\Sigma)=c(K)$, as desired.

This implies that $g_c(W(K)) \leq c(K)$ for all nontrivial knots $K$. Using Morton's inequality, we can establish the reverse inequality if we can show that $\deg_z P_{W(K)}(v,z) = 2c(K)$ . Our goal is to prove this equality for a class of knots that includes the alternating pretzel knots. Central to our proof is therefore a computation of the $z$-degree of the HOMFLY polynomial.

The HOMFLY polynomial [4] is a 2-variable Laurent polynomial defined for any oriented link, and may be thought of as the unique polynomial $P_K(v,z)$, defined on link diagrams and invariant under the Reidemeister moves, satisfying $P_{unknot}(v,z)=1$, and $v^{-1}P_{K_+} - vP_{K_-} = zP_{K_0}$, where $K_+, K_-, K_0$ are diagrams which all agree except at one crossing as in Figure 5. Here we are following Morton's convention in the naming of variables.

This skein relation gives an inductive method for computing the HOMFLY polynomial, since one of $K_+, K_-$ will be "closer" than the other to the unlink, in terms of unknotting number, while $K_0$ has fewer crossings. It also allows any one of the polynomials to be computed from the other two.

If we write $P_K(v,z)$ as a polynomial in $z$, having coefficients which are polynomials in $v$, then Morton showed in [11] that, for any oriented diagram $D$ of an oriented knot or link $K$, the $z$-degree $M(K)$ of $P_K(v,z)$ satisfies $M(K) \le c(D) - s(D) + 1$, where $c(D)$ is the number of crossings of the diagram and $s(D)$ is the number of Seifert circles of $D$. Since the Seifert surface $\Sigma$ built by Seifert's algorithm from this diagram has Euler characteristic

$$s(D) - c(D) = \chi(\Sigma) = (2 - |K|) - 2g(\Sigma),$$

where $|K|$ is the number of components of $K$, we have

$$M(K) \le c(D) - s(D) + 1 = 1 - \chi(\Sigma) = 2g(\Sigma) + (|K| - 1)$$

for every canonical Seifert surface for $K$. Consequently, for a knot $K$, $M(K)$ is bounded from above by twice the canonical genus of $K$, $2g_c(K)$. Our main interest in the HOMFLY polynomial will therefore be in a computation of this $z$-degree. In particular, since the degree of the sum of two polynomials cannot exceed the larger of their two degrees, and is equal to the larger if the degrees are unequal, we get the basic inequalities (letting $K_+$ denote the knot with diagram $D_+$, etc.) for the $z$-degree of $P_K$, $M(K)$:

$$\begin{aligned} M(K_+) &\le \max\{M(K_-), M(K_0) + 1\} \\ M(K_-) &\le \max\{M(K_+), M(K_0) + 1\} \\ M(K_0) &\le \max\{M(K_+), M(K_-)\} - 1 \end{aligned}$$

where equality holds if the two terms in the maximum are unequal, from the equations $P_{K_+} = v^2 P_{K_-} + vzP_{K_0}$, $P_{K_-} = v^{-2}P_{K_+} - v^{-1}zP_{K_0}$, and $P_{K_0} = v^{-1}z^{-1}P_{K_+} - vz^{-1}P_{K_-}$. For example, since if we change one of the crossings in the clasp of $W(K) = L_\pm$ we get the unknot $= L_\mp$, while if we break the crossing we get the link $F(K) = L_0$, we have $M(F(K)) \le \max\{M(W(K)), 0\} - 1$, with equality if $M(W(K)) > 0$. In particular, if we already know that $M(F(K)) > 0$, then $M(F(K)) = M(W(K)) - 1$.

## 4. Proof of the main result

In this section we prove the following main theorem.

**Theorem 5.** *If $L$ is a non-split link with a diagram $D'$ satisfying $c(D') = c(L)$ and $M(D(L)) = 2c(D') - 1$, and $K$ is a link having diagram $D$ obtained by replacing a crossing in the diagram $D'$ with a full twist (so that $c(D) = c(D') + 1$), then $M(F(K)) = 2c(D) - 1$.*

From this result, we will get Propositions 3 and 4 as well as Theorem 2 will follow.

*Proof.* Our main goal is to compute the $z$-degrees of the HOMFLY polynomials of Whitehead doubles $W(K, n, \pm)$ of knots $K$. First, we get rid of the clasp. As we just saw,

$$M(F(K)) \le \max\{M(W(K)), 0\} - 1 = M(W(K)) - 1.$$

So to prove Proposition 4 we wish to establish that if a knot $K$ has a diagram $D$ obtained from the standard diagram $D_0$ of the trefoil knot $T_{2,3}$ by repeatedly replacing crossings, treated as half-twists, by full twists, then for a doubled link with $n$ full twists $D(K, n)$, we have $M(D(K,n)) = 2c(K)-1$. We note that, by induction, this replacement process always builds alternating diagrams with no nugatory crossings. A nugatory crossing is a crossing in the diagram $D$ so that some pair of opposite complementary regions represent the same connected component of the complement of the projection of $K$. Such a phenomenon is preserved when passing between a diagram with a full twist and one with a half-twist; if the crossing is one of those in the full twist, then the half-twist will be nugatory, while if not, then the nugatory crossing disjoint from the half- or full twist will persist. So, in our arguments, we will always have $c(K) = c(D)$, i.e., $D$ is a diagram with minimal crossing number. So what we will show is that $M(F(K)) = 2c(D) - 1$.

Next we get rid of the $n$ full twists. Given a specific double $D(K, n)$ of knot $K$, changing one of the crossings among the full twists yields $D(K, \pm(|n| - 1))$ (after isotopy), while breaking the crossing yields the unknot $L$, with $M(L) = 0$. So our basic inequalities yield

$$M(D(K, \pm(|n| - 1))) \leq \max\{M(D(K, n)), 1\},$$

with equality so long as $M(D(K, \pm(|n| - 1))) \neq 1$. By induction, then, since for a non-trivial knot $K$ we have $c(K) \geq 3$, so long as we show that $M(D(K, 0)) = 2c(D) - 1$ we will have established that $M(D(K, n)) = 2c(D) - 1$ for all $n$. Since we will focus on the case $n = 0$, from this point we will use $F(K)$ to denote the flat double $F(K, 0)$.

In general, using the same argument as in the last section, if $D'$ is a diagram of a link $L$ with doubled link $D(L)$, since the projection of $L$ is a graph with $c(D')$ vertices and $c(D') + 2$ faces, Morton's inequality gives

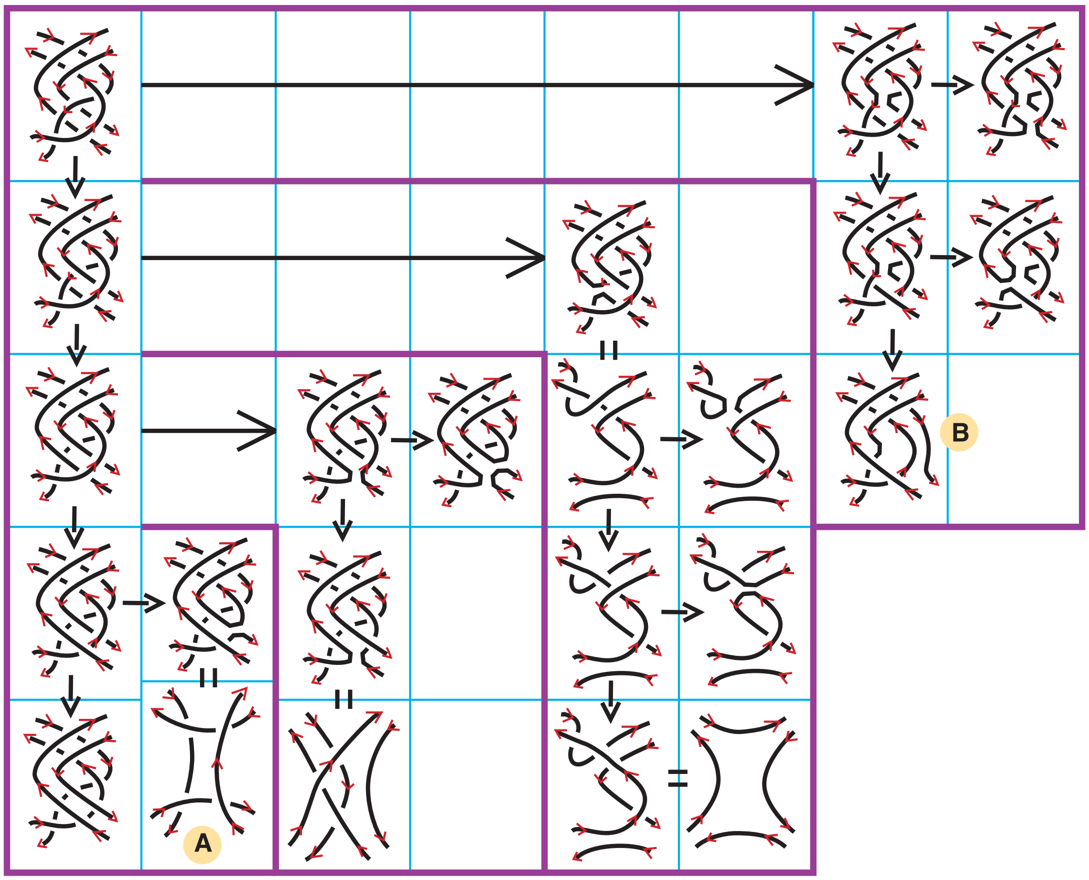
$$\begin{aligned} M(D(L)) &\leq c(F(D')) - s(F(D')) + 1 \\ &= (4c(D')) - (c(D') + (c(D') + 2)) + 1 \\ &= 2c(D') - 1. \end{aligned}$$

This basic estimate will be used several times in the proofs that follow.

Showing that $M(F(K)) = 2c(D) - 1$ has the advantage of making sense when the underlying $K$ is a link, and not just a knot; the Whitehead doubling process does not really apply to a link. It is therefore a better setting to use for an inductive proof; replacing a crossing in a knot diagram with a full twist produces a 2-component link, not a knot. We will therefore work in the setting of flat doubles $F(K)$ of knots or links $K$, following the orientation conventions established in section 1; the parallel doubled strands for every component of $K$ are oriented oppositely, so that the parallel strand is to the right.

From the discussion at the beginning of this section describing how information about $M(F(K))$ translates to information about $M(W(K))$, Theorem 5 implies Proposition 4 by induction on the number of full twists introduced, once we establish the base case. But a direct calculation (see also [16],[13]) establishes that $M(D(T_{2,3})) = 5 = 2 \cdot 3 - 1 = 2c(T_{2,3}) - 1$, giving the base case for the induction. Since at every stage the process of introducing full twists builds an alternating diagram $D$ for $K$ with no nugatory crossings, $c(K) = c(D)$ at every stage of the induction, and since the link projection is always connected, Menasco [10] implies that the underlying link is non-split. So Theorem 5 applies

FIGURE 6. Main skein tree diagram




at every step of the induction. Proposition ?? follows by applying Theorem 5 twice. And as remarked in the introduction, Theorem 2 follows from Proposition 3, using the two base cases the torus knots $T_{2,n} = K(1,\ldots,1)$, supplied by [16], and the twist knots $K(2,1,\ldots,1)$, supplied by [13]. We should note that the pretzel knots $K(n_1,n_2)$ are the torus knots $T_{2,n_1+n_2}$.

We now turn to the proof of Theorem 5. The proof is by induction on $c(K)$, and is essentially based on a large skein tree diagram calculation, with the diagram $F(D)$ for $F(K)$ at the root and $F(D')$ as one of the leaves. We will show that under the hypotheses of the proposition we have $M(F(K)) = M(F(L)) + 2$, which establishes the inductive step. Since we are only interested in $M(F(K)) = \text{maxdeg}_z(P_{F(K)})$, we will only track this quantity through the calculation, rather than the entire HOMFLY polynomial, using the inequalities established at the end of section 1.

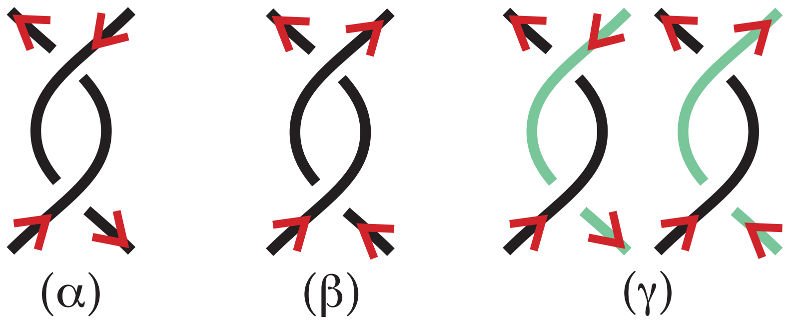
The main skein tree diagram that we employ is given in Figure 6. The upper left corner of the figure is a portion of our diagram $F(D)$ for $F(K)$ at the full twist of $K$ which exists by hypothesis, with an additional pair of canceling half-twists introduced on either side of a quartet of overcrossings. A horizontal step in the diagram involves breaking a crossing, according to the orientation of the strands; a downward step involves changing the same crossing. Except for the addition of a full-twist to the two parallel strands, the upper right corner of the figure is the corresponding portion of the diagram $F(D')$ of $F(L)$. The two right arrows across the top are the basis for our induction; they imply, essentially, that $P_{F(K)} = z^2v^{-2}P_{F(L)}+$ other terms. The proof involves showing that the other terms do not have high enough $z$-degree to interfere with the conclusion that $M(F(K)) = M(F(L))+2$. This demonstration will occupy the remainder of the section.

This in turn involves arguing up from the bottom of the skein tree, using the inequalities at the end of section 1 to bound the $z$-degrees of these other terms from above. The initial steps in this process are done using Morton's inequality, since we can compute the genus of the Seifert surface built from the diagram given to us at the leaves of the tree, or, more simply, the difference between this and the genus of the surface built from $F(D)$, by computing the changes in the number of crossings (which is evident) and the number of Seifert circles (which we can do since we know precisely where these circles come from for $F(D)$). In the generic case, this will be precisely how our estimates are made. But there is also the possibility that at some of the leaves our link will include a (possibly) twisted double of the unknot as a split component, and our estimates will then be off by the $z$-degree of its HOMFLY polynomial; they will be too low by 2 for each such component. The estimates need to be refined slightly in this case; for ease of exposition, we deal with this possibility at the end of the proof, and, in what follows now, act as if such a split component has not been produced.

To start, we make note that the local picture in the upper left corner of the skein diagram, if we were to undo the canceling pair of half-twists (since it is this simpler diagram that we are postulating minimizes the quantity $c(F(D)) - s(F(D)) + 1$), contains 8 crossings of $F(D)$ and contains parts of 7 Seifert disks - 2 from the crossings of $K$ and 5 from the complementary regions of $K$. Note that these 5 regions of $K$ are distinct; the one in the middle of the full twist is demonstrably so from the figure, and the other four come from the four regions surrounding a single crossing of $K'$, which are all distinct because this crossing is not nugatory (that is, opposite regions are distinct); otherwise, $c(D')$ would be greater than $c(L)$. As remarked earlier, replacing a crossing by a full twist cannot introduce a nugatory crossing. It is therefore this contribution, 8 crossings and 7 disks, that we will compare with the leaves of our skein tree to compute what genus surface each diagram will build, giving an upper bound, by Morton's inequality, on the $z$-degree of the HOMFLY polynomial of the underlying oriented link.

Our argument will really be three parallel arguments, based on the underlying orientations, and number of components, of the knot or link $K$ involved at the full twist where we are carrying out our skein tree argument. There are essentially three cases, given in Figure 7; ($\alpha$) and ($\beta$) are when the two strands at the full twist belong to the same component of $K$, with the opposite, respectively parallel, orientation, while ($\gamma$) is when the two strands lie in different components. We must at several points break our argument into three parallel ones, according to which of these three cases we find ourselves in, when, the diagrams

Figure 7. Three cases

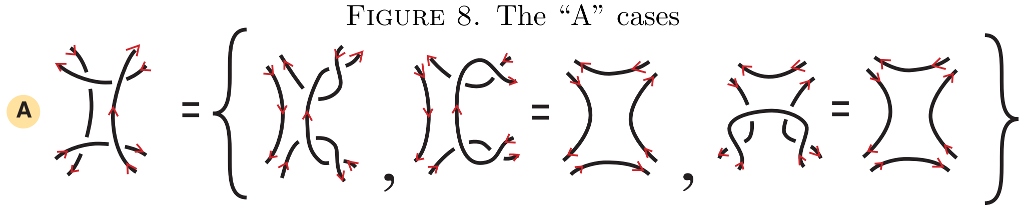



as given at the leaves of the tree will typically yield canonical surfaces whose genera are too high for our purposes. In each case we will wish to get rid of some of the crossings (thought of as half-twists) from the local diagrams at the leaves of the tree, in order to simplify the diagram and give us a lower genus Seifert surface. So we will push each half-twist through the rest of the link diagram (which consists entirely of pairs of parallel strands, having been untouched by the skein moves that we have carried out), along these parallel strands, which will return the crossing to another one of the four corners of the local diagram. This is precisely where the three cases arise; each behaves differently under this operation.

In case ($\alpha$), pushing a crossing from the upper left corner will return it to the upper right (and vice versa), while from the lower left it will return to the lower right. In case ($\beta$), pushing from the upper left will return to the lower right, and from the upper right will return to the lower left. In case ($\gamma$), pushing from the upper left returns to the lower left, and from the upper right returns to the lower right. The other point to note is that if we push from two corners in such a way that they return to the other two corners, then the net effect outside of the local diagram will be that all strands appear to be the same, but all of their orientations will be reversed. (If there are entire pairs of parallel components of $F(K)$ outside of the local diagram, we also interchange them by an isotopy, so that this is true for all components outside of the local diagram.) The point to this is that as a result, all of the Seifert circles outside of the local diagram, when running Seifert's algorithm, are the same after this pair of crossing pushes; they have simply had all of their orientations reversed. In particular, in our count of the number of Seifert circles at the leaves of the skein tree, we may act, after these pushes, as if the Seifert disks outside still come precisely from the crossings of $K$ and the complementary regions of $K$, as before in Figure 4 above.

In the case of the spot marked "A" in Figure 6, we have four extra crossings at the corners which if we leave in place will give a surface with genus which is too high for our purposes when we run Seifert's algorithm. Figure 8 shows the result of pushing a pair of the crossings (upper and lower left in the first two cases, upper right and left in the third). The results are identical, except that in the first case we have two extra pairs of full twists. We will show shortly that these full twists, like our argument at the beginning of this section passing from $D(K,n)$ to $D(K,0)$, do not affect the $z$-degree, but first let us analyze

Figure 8. The "A" cases




the $z$-degree in the second and third cases. These diagrams $D_A$ have 8 fewer vertices than the local diagram for $F(D)$, and 3 fewer Seifert disks. Therefore, for these diagrams, the resulting Seifert surface $\Sigma'$ has $c(D_A) = c(F(D)) - 8$ and $s(D_A) = s(F(D)) - 3$. By Morton's inequality, this yields

$$\begin{aligned} M(D_A) &\leq c(D_A) - s(D_A) + 1 \\ &= (c(F(D)) - s(F(D)) + 1) - 5 \\ &= (2c(K) - 1) - 5 \\ &= 2c(K) - 6 \end{aligned}$$

for the last two cases. This, we shall see, is a $z$-degree that is too low to interfere with our main calculation.

Removing the pair of full twists in the first case will give a diagram identical to the other two. The effect on the $z$-degree that these twists have can be determined from the skein tree in the left of Figure 9; each of the two times we break the crossing, the resulting short arc can be pushed out of the local diagram to return, since we are in case $(\alpha)$, to the other top or bottom corner. When it does, we see that the resulting link is another double $D(L)$ of some link $L$ (possibly with a full twist). The full twist can be removed by the method of the beginning of this section, without changing the $z$-degree, unless $L$ is the unknot, in which case we will have degree 2. Since our original knot or link $K$ had no nugatory crossings, however, the isotopy of the short arc has erased at least two crossings, in addition to the two crossings in our local diagram which have now disappeared (Figure 9, middle); the arc of $K$ between the two corners must meet other crossings, else it represents a monogon in the digram, and we have a nugatory crossing.

So $L$ has at least four fewer crossings than $K$, so $M(D(L)) \leq 2c(L) - 1 \leq 2c(K) - 9$. Consequently, as we move up the skein tree on the left of Figure 9, we have, since the link $K_{A2}$ is the same link found in the second and third cases,

$$\begin{aligned} M(K_{A1}) &\leq \max\{M(D_{A2}), M(D_{A3}) + 1\} \\ &\leq \max\{2c(K) - 6, 2c(K) - 8\} \\ &= 2c(K) - 6, \end{aligned}$$

so

FIGURE 9. Removing full twists

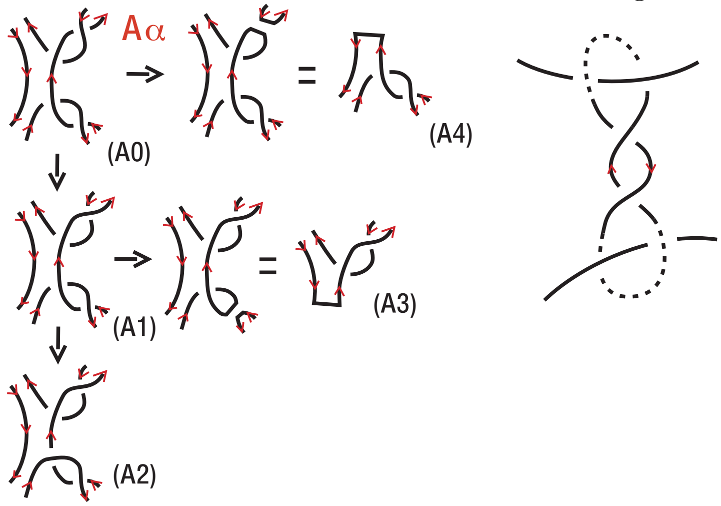
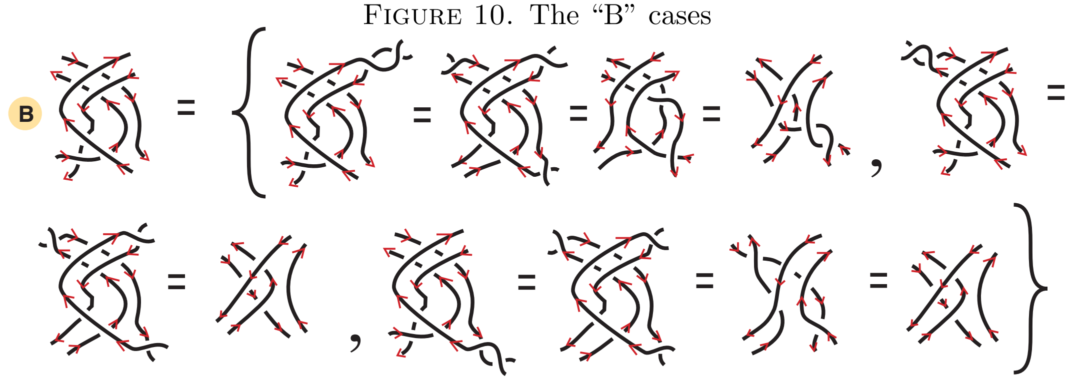



FIGURE 10. The "B" cases




$$M(K_{A0}) \le \max\{M(K_{A1}),$$
$$M(D_{A4}) + 1\} \le \max\{2c(K) - 6, 2c(K) - 8\} = 2c(K) - 6.$$

So in all three cases we have $M(D_A) \le 2c(K) - 6$.

In a similar vein, we analyze the $z$-degree for the link with diagram at spot "B" in our skein tree. The situation for the three cases is illustrated in Figure 10. In this situation we need to "invent" some half-twists to push, since the diagram does not obviously have a pair of half-twists to work with. When we do so, we obtain the diagrams in the figure. Again, the first case has an extra full twist that we must deal with, but first we look at the last two cases. Their local diagrams are identical and, compared with $F(D)$, $D_B$ has 5 fewer crossings and 2 fewer Seifert disks. So, by Morton's inequality,

$$\begin{aligned} M(D_B) &\le c(D_B) - s(D_B) + 1 \\ &= ((c(F(D)) - 5) - (s(F(D)) - 2) + 1) \\ &= (c(F(D)) - s(F(D)) + 1) - 3 \\ &= (2c(K) - 1) - 3 = 2c(K) - 4 \end{aligned}$$

for the last two cases.

In the first case we remove the full twist in the same manner that we did for spot "A"; this is illustrated on the right in Figure 9. Again we find that breaking a crossing of the full twist yields a double of a link with at least four fewer crossings (with a full twist), and so $M(K_{B2}) \le 2c(K) - 9$. The link given by the diagram $D_{B1}$ is not exactly the same as in the last two B-cases - one of the crossings is different - but it has the same projection to the plane and so has the same number of crossings and the same Seifert disks. So $M(K_{B1}) \le 2c(K) - 4$. Working our way up the skein tree then yields

$$\begin{aligned} M(K_{B0}) &\le \max\{M(K_{B1}), M(K_{B2}) + 1\} \\ &\le \max\{2c(K) - 4, 2c(K) - 8\} \qquad = 2c(K) - 4. \end{aligned}$$

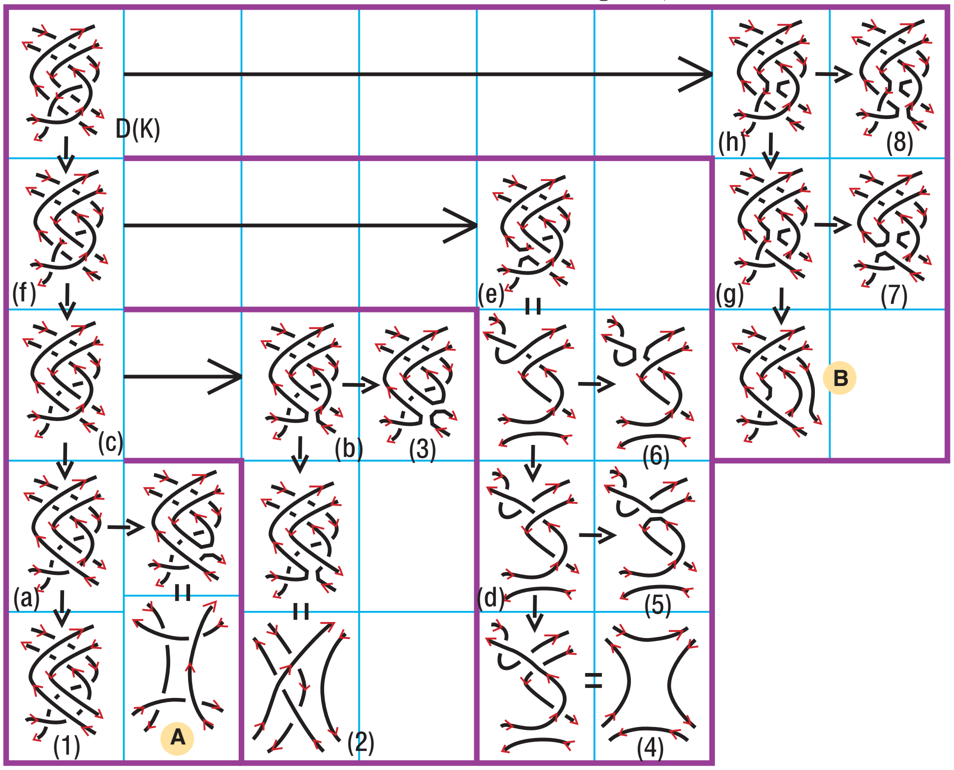
So in all three cases we have $M(D_B) \le 2c(K) - 4$.

To proceed with our degree calculation, we need similar estimates for all of the other leaves of our skein tree diagram. We reproduce our main skein tree diagram as Figure 11, with additional labels to mark the nodes of the tree. The link at the spot marked (1) is isotopic to the double of a link diagram $D'$ with two fewer crossings than $K$, so $M(K_1) \le 2c(D') - 1 = 2c(K) - 5$. At (2) we have the diagram encountered in the (first case of) spot "B", so $M(K_2) \le 2c(K) - 4$. At (4), we have the diagram encountered in case "A", so $M(K_4) \le 2c(K) - 6$. At (5), we have, after isotopy, the double of a link diagram with 2 fewer crossings than $K$, so $M(K_2) \le 2c(D_5) - 1 = 2c(K) - 5$.

At (3), (6) and (7), we have diagrams with short arcs in one corner that we can push out of the local diagram to return at one of the other three corners, after which, as in the arguments for spots "A" and "B", we have the double of a link with at least 2 fewer crossings; the crossings that were in the local diagram are now gone. (There are in fact still fewer crossings, but we won't need this fact.) In the ($\alpha$) case for (3), the ($\beta$) and ($\gamma$) cases for (6), and the ($\alpha$) and ($\gamma$) cases for (7), the double has a full twist. Still, as before, we find that $M(K_3) \le 2c(K) - 5$, $M(K_6) \le 2c(K) - 5$, and $M(K_7) \le 2c(K) - 5$.

Finally, at (8) we have a link, a double of our link $L$ with a full twist, to which to apply our inductive hypothesis. By hypothesis, $M(K_8) = M(D(L)) = 2c(D') = 2c(L) = 2c(K) - 2$. Having estimates for the $z$-degrees of the HOMFLY polynomials at all of the leaves of our skein tree diagram, we proceed to work our way up to the root of the tree, using the inequalities established at the end of section 1. Recalling that a vertical arrow represents changing a crossing (passing between $D_-$ and $D_+$) and a horizontal arrow

FIGURE 11. Main skein tree diagram, labeled




represents breaking the crossing (passing to $D_0$), and using the labeling scheme of Figure 11, we find:

$$M(K_a) \leq \max\{M(K_1), M(K_A)+1\} \leq \max\{2c(K)-5, 2c(K)-5\} = 2c(K)-5$$
$$M(K_b) \leq \max\{M(K_2), M(K_3)+1\} \leq \max\{2c(K)-4, 2c(K)-4\} = 2c(K)-4$$
$$M(K_c) \leq \max\{M(K_a), M(K_b)+1\} \leq \max\{2c(K)-5, 2c(K)-3\} = 2c(K)-3$$
$$M(K_d) \leq \max\{M(K_4), M(K_5)+1\} \leq \max\{2c(K)-6, 2c(K)-4\} = 2c(K)-4$$
$$M(K_e) \leq \max\{M(K_d), M(K_6)+1\} \leq \max\{2c(K)-4, 2c(K)-4\} = 2c(K)-4$$
$$M(K_f) \leq \max\{M(K_c), M(K_e)+1\} \leq \max\{2c(K)-3, 2c(K)-3\} = 2c(K)-3$$
$$M(K_g) \leq \max\{M(K_B), M(K_7)+1\} \leq \max\{2c(K)-4, 2c(K)-4\} = 2c(K)-4$$

So far we have used the inequalities; now we establish equalities.

$$M(K_h) \leq \max\{M(K_g), M(K_8)+1\} \leq \max\{2c(K)-4, 2c(K)-2\} = 2c(K)-2,$$

but since $M(K_8)+1 = 2c(K)-2$ by hypothesis, and $M(K_g) \leq 2c(K)-4$, the two quantities in the maximum are unequal, so $M(K_h) = 2c(K)-2$. Finally,

$$M(F(K)) \leq \max\{M(K_f), M(K_h)+1\} \leq \max\{2c(K)-3, 2c(K)-1\} = 2c(K)-1,$$

but since $M(K_h)+1 = 2c(K)-1$ and $M(K_f) \leq 2c(K)-3$, the quantities in the maximum are again unequal, so $M(F(K)) = 2c(K)-1$, as desired.

This completes the inductive step, and so the proposition is proved by induction, except for dealing with the possibility that the links at the leaves of the skein tree contain twisted doubles of unknots. This is really only an issue at the leaves where we needed to isotope a short arc through the exterior of the local diagram of the link before using Morton's inequality to estimate the $z$-degree, namely at the nodes labeled "A",(3),(6),"B", and (7). In the other cases we were applying either a local isotopy or our inductive hypothesis to make our estimates. It is a non-local isotopy, pushing a short arc through the diagram, which can expose unlinked components, which would render our estimates based on the number of crossings of the link inaccurate. However, in every case, the link we have after isotopy *is* a (possibly twisted) double of a link $L'$, obtained, essentially, by erasing one of the two arcs in the exterior of, but incident to, the original local diagram, as well as erasing the full twist in the local diagram, replacing the full twist with an unknotted arc joining the two remaining ends. Since by hypothesis the link $L$ is non-split, every unknotted component of $L'$ split off from the rest of $L'$, which contributes 2 to $M(D(L'))$ without contributing anything to $C(L')$ comes at the expense of $L'$ having at least two fewer crossings than $L$; the isotopy of the short arc must "release" these unknotted components from the main body of $L$. Put differently, every spanning disk $\Delta$ for the unknotted component must intersect $L$ in its interior, since otherwise the boundary of a neighborhood of $\Delta$ would provide a splitting sphere for $L$. So the disk bounding the unknotted, unlinked component, which exists after isotoping the arc, must intersect the arc, implying that the removal of the arc erases at least two crossings in the projection of $L$ (the number must be even). Consequently, the existence of an unknotted split component, rather then requiring us to add 2 to our $z$-degree, implies that it is actually *lower* by 2 than our local count might have led us to believe. So in addition to the two crossings which were lost in the local diagram, which dropped our estimate of the $z$-degree by 4, every unknotted split component also, on balance, implies a drop of at least 2 to this estimate. This implies, at all of the five leaves where this argument is relevant, that if an unknotted split component is encountered, we can still conclude that $M(D(L')) \leq 2c(K)-7$. A quick check above will show that this is as low as any of the estimates that were used in the arguments above, and so, if anything, will result in *lower* upper bounds than we have used. So it will not affect the final outcome.

With this last detail in place, Theorem 5 is proved.

□

## 5. Most non-alternating pretzels do not satisfy $\text{deg}_z P_{W(K)}(v,z) = 2c(K)$

We have shown that for the pretzel knots $K = P(n_1, \ldots, n_k)$ whose pretzel representation is alternating, that is, for which $n_1, \ldots, n_k \geq 1$, we have $M(W(K)) = 2g_c(K)$, so $g_c(K) = c(K) = n_1 + \cdots + n_k$, essentially by induction on the $n_i$. The inductive step, namely the statement of Theorem 5, can be turned around, however, to partially establish the opposite result for pretzel knots whose pretzel representation is not alternating, that is, the $n_i$ do not all have the same sign. In this instance we can show that $M(W(K)) < 2|n_1| + \cdots + 2|n_k|$.

This follows from the final computation in the proof of Theorem 5, since if at that point, we know the opposite fact that $M(K_8)$, namely the $z$-degree for the upper right corner, is *less* than $2c(D)-3$, then (since it must be odd) it is at most $2c(D)-5$, and so, borrowing the notation from the end of the proof, we have

$$M(K_h) \leq \max\{M(K_g), M(K_8)+1\} \leq \max\{2c(D)-4, 2c(D)-4\} = 2c(D)-4,$$

so

$$M(F(K)) \leq \max\{M(K_f), M(K_h)+1\} \leq \max\{2c(D)-3, 2c(D)-3\} = 2c(D)-3,$$

so

$$M(F(K)) \leq 2c(D)-3 < 2c(D)-1.$$

Put differently, if $M(F(K)) = 2c(D)-1$, then we *must* have $M(K_8) = 2c(D)-3$. But arguing our induction in reverse, if we start with a pretzel knot/link $P(n_1, \dots, n_k)$ having (without loss of generality) $n_1 > 0$, $n_2 < 0$, we can, by turning full twists into half-twists, return to a base case of $K' = P(1, -1, n_3, \dots, n_k)$ to which we can apply a type 2 Reidemeister move, $P(1, -1, n_3, \dots, n_k) = P(n_3, \dots, n_k)$. So, by the basic construction,

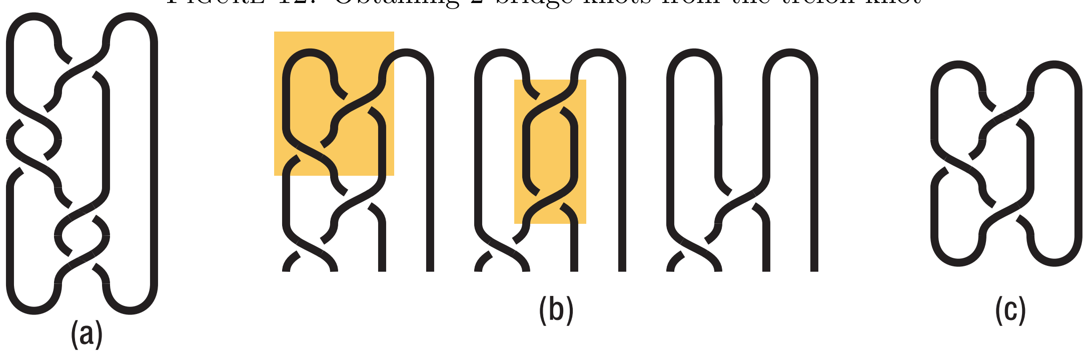
$$M(W(P(1,-1,n_3,\dots,n_k))) \leq 2|n_3|+\cdots+2|n_k| < 2|1|+2|-1|+2|n_3|+\cdots+2|n_k| = 2c(D'),$$

where $D'$ is the diagram for $K'$ before the Reidemeister move. Putting back all of the full twists to return to $K$, the above argument then shows that $M(F(K)) < 2c(D)-1$, so $M(W(K)) < 2|n_1|+\cdots+2|n_k|$. But when $1, -1 \notin \{n_1, \dots, n_k\}$, then the standard diagram of the link $K = P(n_1, \dots, n_k)$ is a reduced Montesinos diagram in the sense of [9], and so [[9], Theorem 10] $c(K) = |n_1| + \cdots + |n_k|$. So for non-alternating pretzel knots $K$ whose standard diagrams have at least two half-twists in each tangle, we have $M(W(K)) < 2c(K)$.

For example, $P(3, 3, -2)$ is the knot $8_{19}$, and $P(3, -3, 2)$ is the knot $8_{20}$, and in these cases we have $M(W(K)) < 2c(K)$. These two examples could of course be computed directly, and in fact they were, by the authors. It was, in fact, an attempt to show that $g_c(W(K)) < c(K)$ for these two examples that led the authors to the results presented here. Our work here does not establish this, of course; $M(W(K)) < 2c(K)$ implies that at *least* one of $M(W(K)) < 2g_c(W(K))$ or $g_c(W(K)) < c(K)$ (or both) hold, but to date we do not know which of the two might be true. The point to the line of reasoning of the previous paragraph, from our perspective, is that it “explains” the inequality $M(W(K)) < 2c(K)$; it shows that there is a skein tree diagram which demonstrates that the $z$-degree is “too low”, without resorting to the need for a fortuitous cancellation of terms to occur, to make the degree $2c(K)$ term vanish.

In a similar vein, the argument above also establishes that $M(W(K)) < 2c(K)$ holds for many alternating knots, namely, those that can be formed as a connected sum $K_1\#K_2$ for some $K_1 \in \mathcal{K}$, our class of knots from Proposition 4, and $K_2 \neq$unknot. Since these knots are alternating, we know that $c(K_1\#K_2) = c(K_1) + c(K_2)$. In this case if we focus on undoing the full twists in the $K_1$ summand, if $M(W(K_1\#K_2)) = 2c(K_1\#K_2) = 2c(K_1) + 2c(K_2)$ the argument above will allow us to work our way back to $M(W(T_{2,3}\#K_2)) = 2c(T_{2,3})+2c(K_2) = 2c(K_2)+6$ by undoing the full twists introduced to build $K_1$. However, there are still two more full twists in the $T_{2,3}$ summand to work with, and undoing them

FIGURE 12. Obtaining 2-bridge knots from the trefoil knot




yields a knot $K$ whose diagram is the projection of $K_2$ with a single, nugatory, crossing added. Therefore $c(K) = c(K_2)$, but having gotten here in two steps from $T_{2,3}\#K_2$ we *should* have $M(W(K)) = 2c(K_2)+2$, not $M(W(K)) = 2c(K_2)$. This contradiction implies that our original hypothesis, $M(W(K_1\#K_2)) = 2c(K_1\#K_2)$ is false. Since we must have $M(W(K_1\#K_2)) \leq 2c(K_1\#K_2)$, we must conclude that $M(W(K_1\#K_2)) < 2c(K_1\#K_2)$. Note that this does *not* tell us that $g_c(W(K_1\#K_2)) \neq c(K_1\#K_2)$. But it does establish that the method of proof used by Tripp, Nakamura, and ourselves to show that $g_c(W(K) = c(K)$, namely to show that $M(W(K)) = 2c(K)$, will not succeed for all alternating knots.

For our final result we establish the last assertion made in the introduction, namely that the class $\mathcal{K}$ built from the trefoil knots by replacing crossings by full twists already contains the 2-bridge knots, so that including them as initial objects would not enlarge the class. ($\mathcal{K}$ clearly already contains the $(2,n)$-torus knots.) To see this, we will work backwards from an alternating projection of a given 2-bridge knot $K$, successively replacing full twists with half-twists, and show that we can end up at the trefoil. Reversing this process then establishes the result.

The alternating projection for $K$ that we will use will be the alternating 4-plat projection of [[2], Proposition 12.13], denoted $\sigma_2^{a_1}\sigma_1^{-a_2}\cdots\sigma_2^{a_m}$ with $a_i > 0$ for $i = 1,\ldots,m$, and illustrated in Figure 12(a). Note that $m$ is necessarily odd. The idea is that by replacing full twists with half-twists within each block of twists $\sigma_\epsilon^{a_i}$, we can steadily lower each of the $a_i$, and eventually arrive at the knot represented by $\sigma_2^1\sigma_1^{-1}\cdots\sigma_2^1$. Then as shown in Figure 12(b), a further two replacements can, in effect, remove the first pair of $\sigma_2^1\sigma_1^{-1}$'s. Repeating this process will allow us to reach $\sigma_2^1\sigma_1^{-1}\sigma_2^1$, shown in Figure 12(c), which is the (left-handed) trefoil knot. Therefore, reversing this process, every 2-bridge knot can be obtained from the trefoil knot by a sequence of replacements of crossings by full twists, as desired. So all 2-bridge knots lie in our class $\mathcal{K}$, and so any knot obtained by replacements on a 2-bridge knot will also lie in $\mathcal{K}$.

## 6. Further considerations

Proving a result like ours for a class of knots naturally begs the question "For what other (classes of) knots does $g_c(W(K)) = c(K)$ hold?". The discussion of the previous section shows that the techniques we employ cannot establish this for all alternating knots, since the equality $\mathrm{maxdeg}_z(P_{W(K)}(v,z)) = 2c(K)$ need not hold for non-prime alternating knots. It does not, however, show that $g_c(W(K)) = c(K)$ does not hold. But based on the results that we have obtained here we feel that it is reasonable to make the

**Conjecture.** *If $K$ is a nontrivial prime alternating knot, and $W(K)$ is a Whitehead double of $K$, then $\mathrm{maxdeg}_z(P_{W(K)}(v,z)) = 2c(K)$, and therefore $g_c(W(K)) = c(K)$ .*

The argument given here implies that in trying to establish this conjecture, we may always inductively replace a pair of crossings forming a bigon by a single crossing. By Euler characteristic considerations, the projection of any knot, thought of as a graph in the 2-sphere, has a complementary region which is either bigon or a triangle. One may think of our results as therefore saying that a proof of the conjecture reduces to the cases where the projection has no bigons, i.e., has complementary regions that all have three or more vertices. This is because our process of "deflation", replacing a full twist with a half-twist, and the inverse process, keep us within the class of prime, reduced, alternating projections. That no nugatory crossings can be introduced has been established in Section 2 above. Menasco [10] has shown that a reduced alternating diagram of a non-prime alternating knot can be detected from the diagram, that is, there must be a circle in the plane of the projection that intersects the projection twice. But such a circle will exist after deflation if and only if there is such a circle before. If such a circle (before deflation) were to intersect the bigon region, deforming it to instead pass through one of the crossings identifies a nugatory crossing, a contradiction, so the same circle will work after deflation (showing $\Leftarrow$), while a circle after deflation stays away from the crossings of the projection, so since the opposite operation of inflation can be viewed as local to the crossings (by introducing a very "small" bigon), the same circle also works before deflation (showing $\Rightarrow$).

One may also build further infinite families of examples, along the lines of Proposition 4, by replacing the trefoil knot with any other link $K$ for which the equality $\mathrm{maxdeg}_z(P_{F(K)}(v,z)) = 2c(K) - 1$ has been established, for example by direct computation. One can, for example, verify directly that the Borromean rings $L$ satisfies $\mathrm{maxdeg}_z(P_{D(L)}(v,z) = 11 = 2c(L) - 1$, giving rise, using Theorem 5, to a different family of (alternating) knots $K$ satisfying $g_c(W(K)) = c(K)$ than the family given by Proposition 4.

Looking beyond the alternating knots, one might look for non-alternating knots $K$ to which the techniques of this paper apply, i.e., $\mathrm{maxdeg}_z(P_{W(K)}(v,z) = 2c(K)$. Our results would then imply that such knots form the basis for another infinite collection of examples. In a different direction, in the course of our proof we find that all of the Whitehead doubles of a knot $K$, when our techniques apply to it, have the same canonical genus. Is there a non-trivial knot $K$ having Whitehead doubles which have different canonical genera? (The trivial knot does.) Of course, their HOMFLY polynomials will have the same $z$-degree, so arguments like ours will be of no help in finding an example.

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