Floer homology, orderable groups, and taut foliations of hyperbolic 3-manifolds:

An experimental study

Nathan M. Dunfield
(University of Illinois and IAS)

These slides already posted at:

http://dunfield.info/slides/IAS.pdf

Y^3: closed oriented irreducible with $H_*(Y;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$.

Conj: For an irreducible $\mathbb{Q}$HS Y, TFAE

(a) $\widehat{HF}(Y)$ is non-minimal.

(b) $\pi_1(Y)$ is left-orderable.

(c) Y has a co-orient. taut foliation.

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Heegaard Floer: An $\mathbb{F}_2$-vector space $\widehat{HF}(Y)$ where

$$\dim \widehat{HF}(Y) \geq \left|H_1(Y;\mathbb{Z})\right|$$

When equal, Y is an *L-space*.

L-spaces: Spherical manifolds, e.g. $L(p,q)$.

Non-L-spaces: $1/n$-Dehn surgery on a knot in S^3 other than the unknot or the trefoil.

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Left-order: A total order on a group G where $g < h$ implies $f \cdot g < f \cdot h$ for al $f, g, h \in G$.

For countable G, equivalent to $G \hookrightarrow \mathrm{Homeo}^+(\mathbb{R})$.

Orderable: $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$, F_n.

Non-orderable: finite groups, $\mathrm{SL}_n\mathbb{Z}$ for $n \geq 2$.

Y^3 is called *orderable* if $\pi_1(Y)$ is left-orderable.

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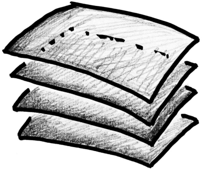

Taut foliation: A decomposition $\mathscr{F}$ of Y into 2-dim'l leaves where:

(a) Smoothness: $C^{1,0}$

(b) Co-orientable

(c) There exists a loop transverse to $\mathscr{F}$ meeting every leaf.

If Y has a taut foliation then $\widetilde{Y} \cong \mathbb{R}^3$ and so $\pi_1(Y)$ is infinite.

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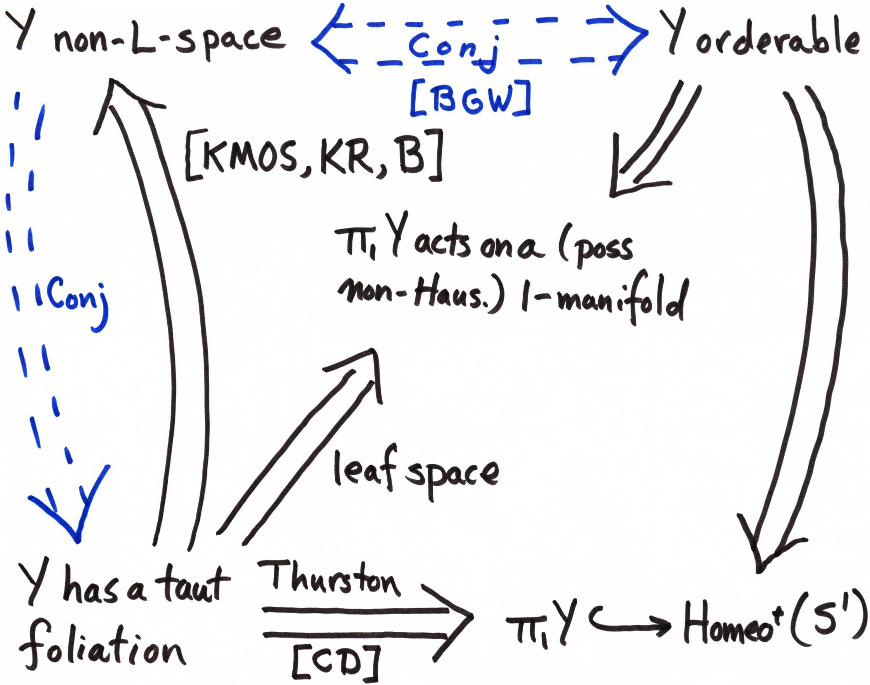

Y non-L-space
Conj
[BGW]
Y orderable
[KMOS, KR, B]
π₁ Y acts on a (poss non-Haus.) 1-manifold
Conj
leaf space
Y has a taut foliation
Thurston
[CD]
π₁ Y ↪ Homeo⁺(S¹)

Evidence for the conjecture:

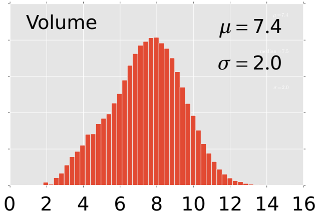
[Hanselman-Rasmussen^2-Watson, Boyer-Clay 2015] True for all graph manifolds.

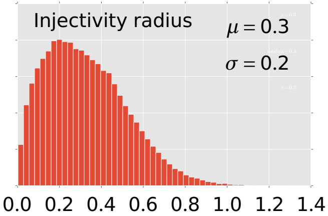
[Li-Roberts 2012, Culler-D. 2015] Suppose $K \subset S^3$ and $\Delta_K(t)$ has a simple root on the unit circle whose complement is lean. Then there exists $\epsilon > 0$ so that the conjecture holds for the r Dehn surgery on K whenever $r \in (-\epsilon, \epsilon)$.

[Gordon-Lidman, . . .]

A few rat'l homology 3-spheres:

265,503 hyperbolic $\mathbb{Q}$HSs which are 2-fold branched covers over non-alt links in S^3 with ≤ 15 crossings.





H-W census has 10,903 $\mathbb{Q}$HSs.

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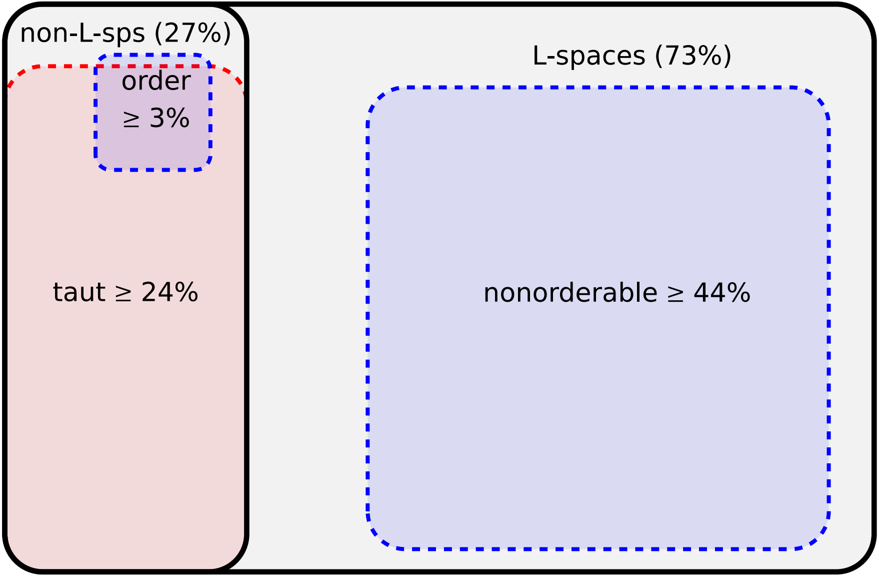
[Gordon-Lidman, ...]

Sample: 265,503 hyperbolic ℚHSs. Conjecture holds so far!



Finding 63,977 taut folations.

$\mathscr{T}$ a 1-vertex triangulation of Y.

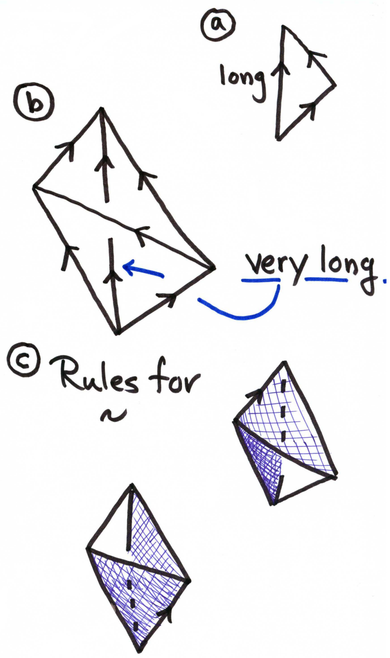
Def. A *laminar orientation* of $\mathscr{T}$ is:

(a) An orientation of the edges where every face is acyclic.

(b) Every edge is adjacent to a tet in which it is not very long.

(c) The relation on faces has one equiv class.

[D. 2015] *If Y has a tri with a laminar orient, then Y has a taut foliation.*



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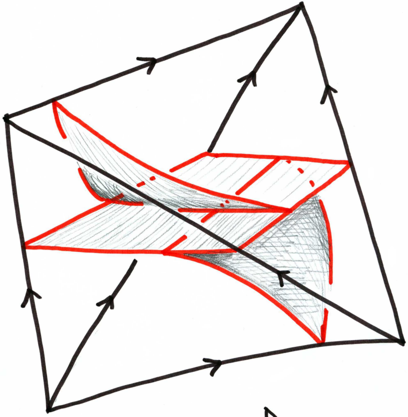
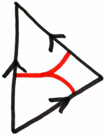
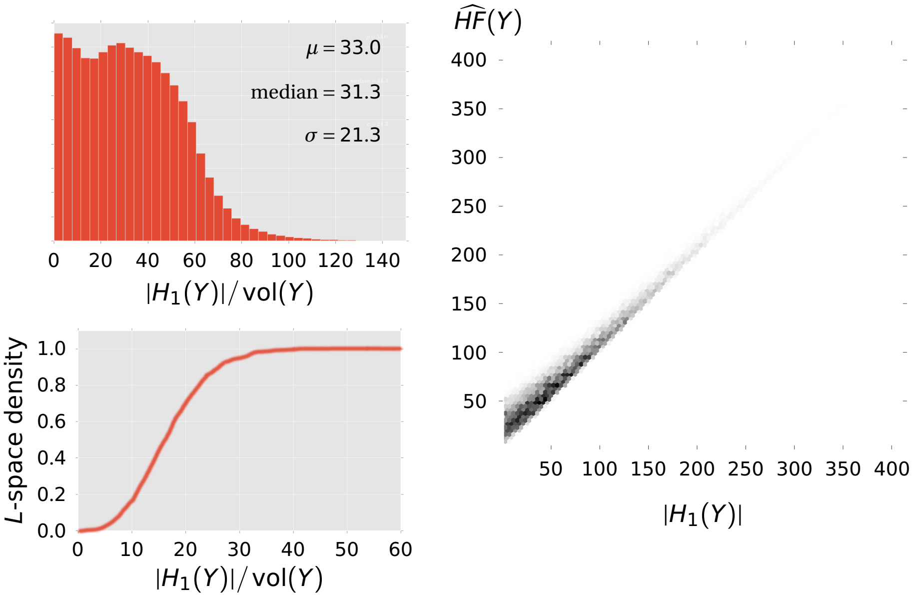




The pattern: Large $\left|H_1(Y)\right|$ *increases* the odds that Y is an L-space.



Computing $\widehat{HF}$: Used [Zhan] which implements the bordered Heegaard Floer homology of [LOT].

Nonordering $\pi_1(Y)$: Try to order the ball in the Cayley graph of radius 3-5 in a presentation with many generators. Solved word problem using matrix multiplication.

Ordering $\pi_1(Y)$: Find reps to $\widetilde{\mathrm{PSL}_2\mathbb{R}}$. Reps to $\mathrm{PSL}_2\mathbb{R}$ are plentiful (mean 8 per manifold) but the the Euler class in $H^2(Y;\mathbb{Z})$ must vanish to lift, so only get 7,382 orderable manifolds from 2.13 million $\mathrm{PSL}_2\mathbb{R}$ reps.