AN EQUIVARIANT SPHERE THEOREM

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1. *Introduction*

Let M be a connected 3-manifold acted on by a group G. Suppose M has a triangulation invariant under G. In this paper it is shown that if there exists an embedded 2-sphere S which does not bound a 3-ball, then there exists such an S for which $gS = S$ or $gS \cap S = \varnothing$ for every $g \in G$. This result was proved by Meeks, Simon and Yau [3] using analytic techniques. The proof given here is self-contained and elementary.

The proof involves looking at embedded 2-spheres which are in general position with respect to the given triangulation. Such a sphere is called minimal if it does not bound a 3-ball and the number of intersections with the 1-skeleton of the triangulation is the smallest possible. The key result proved in this paper is that given a finite set of minimal spheres satisfying a general position condition, there is a finite set of 'standard' disjoint minimal spheres whose union has the same intersection with the 1-skeleton as the original spheres. The set of disjoint spheres is unique up to a homeomorphism of M which fixes the 2-skeleton.

In §4 it is shown that if $G\backslash K$ is finite then there is a G-equivariant decomposition of M with irreducible factors. We are then able to deduce from the ordinary loop theorem an equivariant version of the projective plane theorem. In §5 the arguments of the previous sections are modified to provide a proof of the equivariant loop theorem [2].

I think that many of the topological results obtained using analytic minimal surface theory can also be derived using the techniques of this paper.

I am grateful to Andrew Bartholomew for pointing out an error in an earlier version of this paper. I thank both Peter Scott and the referee for their helpful comments.

2. *Minimal spheres*

Let M be a connected 3-manifold. Suppose M is triangulated, that is, there is a 3-dimensional simplicial complex K such that $M = |K|$.

We consider surfaces $S \subset M$ with the following properties:

S1. S can be triangulated (using some subdivision of K).

S2. S and K are in general position. Thus if K^1 is the 1-skeleton of K, $S \cap |K^1|$ is a finite set of points.

If S satisfies S1 and S2 let $\| S \| = \#(S \cap |K^1|)$.

PROPOSITION 2.1. *Let* $S \subset M$ *satisfy* S1 *and* S2 *and suppose* S *is a* 2-*sphere. If* $\| S \| = 0$, *then* S *bounds a* 3-*ball in* M.

Received 7 January 1985.

1980 *Mathematics Subject Classification* 57M35.

Bull. London Math. Soc. 17 (1985) 437–448

Proof. Let σ be a 2-simplex of K. Now $|\sigma| \cap S$ is a set of disjoint closed curves. The Proposition is proved by induction on $n(S) = \#\{c \mid c \text{ is a component of } |\sigma| \cap S$ for some 2-simplex σ of $K\}$.

If $n(S) = 0$ then S is contained in a 3-simplex and the result is clear. If not choose a closed curve ℓ in S such that $\ell = \partial\Delta$ where Δ is a disk contained in some 2-simplex of K and Δ contains no other such curve. Carry out surgery along Δ and obtain two 2-spheres S_1 and S_2 which we may assume satisfy S1 and S2. Now $n(S_1)+n(S_2) = n(S)-1$. Thus, by induction, $S_1 = \partial B_1$ and $S_2 = \partial B_2$ where B_1 and B_2 are 3-balls in M. Now $S_1 \cap S_2 = \emptyset$. Thus either $S_1 \subset B_2$ or $S_1 \subset M - B_2$. If $S_1 \subset B_2$ then we may assume $B_1 \subset B_2$ by the Schoenflies Theorem. But now there is a 3-ball B obtained by removing from B_2 a neighbourhood of $\Delta - \ell$ together with the interior of B_1 and $\partial B = S$ as required. A similar argument works if $S_2 \subset B_1$. If neither of these cases occurs, $B_1 \cap B_2 = \emptyset$ and we take B to be the union of B_1, B_2 and a neighbourhood of Δ.

DEFINITION. A 2-sphere $S \subset M$ satisfying S1 and S2 is called *minimal* if S does not bound a 3-ball in M and $\| S \|$ takes the smallest value consistent with this property.

Let $\Omega(M)$ denote the set of minimal 2-spheres in M. The constant $\| S \|$, $S \in \Omega(M)$ is denoted $k = k(M)$. Thus $k > 0$ by Proposition 2.1.

Two surfaces $S, S' \subset M$ satisfying S1 and S2 are said to be disk equivalent if there is a sequence $S = S_1, S_2, \ldots, S_n = S'$ of surfaces satisfying S1 and S2 and such that for each j, $1 < j \leqslant n$, there exist disks $\Delta_j \subset S_j$, $\Delta'_j \subset S_{j-1}$ such that $\Delta_j \cup \Delta'_j$ bounds a 3-ball $B_j \subset M$, $S_j - \Delta_j = S_{j-1} - \Delta'_j$, $B_j \cap S_j = \Delta_j$ and $B_j \cap S_{j-1} = \Delta'_j$.

It is easy to see that disk equivalent surfaces are isotopic.

PROPOSITION 2.2. *Let $S \in \Omega(M)$. There exists $S' \in \Omega(M)$ such that S' is disk equivalent to S and for each 2-simplex σ, $|\sigma| \cap S' \subset |\sigma| \cap S$, and $|\sigma| \cap S'$ contains no simple closed curves.*

Proof. Suppose that $|\sigma| \cap S$ contains a simple closed curve ℓ. Choose ℓ so that it is innermost. As in Proposition 2.1 do surgery along the disk in σ bounded by ℓ and obtain surfaces S_1 and S_2. Clearly $\| S \| = \| S_1 \| + \| S_2 \|$. If neither $\| S_1 \|$ nor $\| S_2 \|$ is zero then by the minimality of $\| S \|$ both S_1 and S_2 must bound balls and a repeat of the argument of Proposition 2.1 shows that S bounds a ball. Hence we may assume $\| S_2 \| = 0$. But now by Proposition 2.1, $S_2 = \partial B$, where B is a 3-ball. Also $S_1 \cap B = \emptyset$ otherwise S_1 would bound a ball again giving a contradiction. It follows that S and S_1 are disk equivalent. Repeating this argument we eventually find S' such that $|\sigma| \cap S'$ contains no simple closed curves for each 2-simplex σ.

PROPOSITION 2.3. *Let $S \in \Omega(M)$. Let σ be a 2-simplex of K. Let c be a component of $|\sigma| \cap S$ such that c is not a simple closed curve. Then c is a line joining points on distinct faces of σ.*

Proof. By Proposition 2.2 we can assume that $S \cap |\sigma|$ consists of lines joining points on the boundary of $|\sigma|$. Suppose there is a line ℓ joining two points a, b on the same edge e. By replacing ℓ by another line in $S \cap |\sigma|$ if necessary it may be assumed that the disk Δ in $|\sigma|$ bounded by ℓ and the interval ab in e satisfies $\Delta \cap S = \ell$.

Let R be a regular neighbourhood of Δ in an appropriate subdivision of M. Now ∂R is a 2-sphere and $\partial R \cap S$ is a simple closed curve ℓ'. Also ℓ' bounds disks Δ_1, Δ_2

in ∂R such that $\| \Delta_1 \| = 0$ and $\| \Delta_2 \| = 2$. In S, ℓ' spans a disk Δ_3 such that $\| \Delta_3 \| = 2$. Now S is disk equivalent to S_1 in which Δ_3 is replaced by Δ_1. But S_1 bounds a ball if and only if S bounds a ball. Since $\| S_1 \| = k-2$ we have a contradiction.

PROPOSITION 2.4. *Let $S \in \Omega(M)$. There exists $S' \in \Omega(M)$ such that $S \cap |\rho| = S' \cap |\rho|$ for every 1-simplex ρ of σ and for each 2-simplex σ of K, $S' \cap |\sigma|$ is a set of disjoint straight lines, that is, the convex hulls in $|\sigma|$ of their end points.*

Proof. By Proposition 2.2 and the subsequent remark it can be assumed that $S \cap |\sigma|$ contains no simple closed curves. By Proposition 2.3 the components of $S \cap |\sigma|$ join points on distinct edges. Thus $|\sigma|$ is as in Fig. 1. It is easy to see that there is a homeomorphism $\mu_\sigma : |\sigma| \to |\sigma|$ which fixes $\partial |\sigma|$ and which maps the lines of $S \cap |\sigma|$ into straight lines. If γ is a 3-simplex, there is a homeomorphism

$$\mu_\gamma : |\gamma| \to |\gamma|$$

which restricts to μ_σ for each face σ of γ. Clearly there is a homeomorphism $\mu : M \to M$ which restricts to μ_γ on each 3-simplex γ. Put $S' = \mu S$.

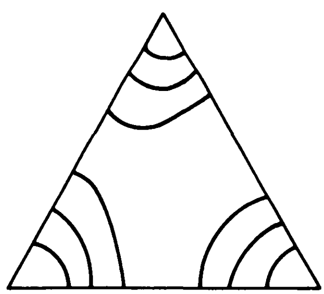


FIG. 1.

PROPOSITION 2.5. *Let $S \in \Omega(M)$. If $S \cap |\sigma|$ contains no simple closed curves for each 2-simplex σ then for each 3-simplex γ, $S \cap |\gamma|$ is a union of disjoint disks. Each disk intersects $|\gamma^1|$ in either 3 or 4 points.*

Proof. Suppose $S \cap |\gamma|$ has a component c which is not a disk. There is a simple closed curve ℓ in $c - \partial |\gamma|$ which does not bound a disk in c. Now ℓ bounds a disk Δ in the interior of $|\gamma|$. Assume that Δ and S are in general position. Look at an innermost circle in $\Delta \cap S$. If this bounds a disk in $|\gamma| \cap S$ then we can alter Δ so as to remove this intersection. Eventually we obtain a disk Δ' such that $\ell' = \partial \Delta' = \Delta' \cap S$ and ℓ' does not bound a disk in $|\gamma| \cap S$. Now do surgery along Δ' so that we obtain new 2-spheres S_1 and S_2 from S. A repeat of the argument of Proposition 2.2 shows that we may assume $\| S_2 \| = 0$. But if $\| S_2 \| = 0$ it means that S_2 can only intersect a 2-simplex in a union of simple closed curves. By our hypothesis on S we see that $S_2 \cap |\sigma| = \varnothing$ for every 2-simplex σ. Thus $S_2 \subset |\gamma|$ which contradicts the fact that ℓ' does not bound a disk in $|\gamma| \cap S$. Hence $S \cap |\gamma|$ is a union of disjoint disks.

Suppose there is a component D of $S \cap |\gamma|$ which intersects an edge of γ in more than one point. Then there is a disk Δ such that $\Delta \subset |\gamma|$, $\Delta \cap \partial |\gamma|$ is an interval ab in an edge of γ and $\Delta \cap S = \partial \Delta \cap S$ is a line ℓ joining a to b. Now repeating the argument of Proposition 2.3 shows that S is not minimal.

If D intersects each edge of $|\gamma|$ in at most one point it follows that, for each 2-face σ of γ, $D \cap |\sigma|$ is either empty or consists of a single line joining distinct edges. The Proposition follows easily.

DEFINITION. A surface S satisfying S1 and S2 is said to be *standard* if for every 2-simplex σ of K, $S \cap |\sigma|$ is a union of disjoint straight lines and for every 3-simplex γ of K, $S \cap |\gamma|$ is a union of disjoint disks.

PROPOSITION 2.6. *If S is a minimal sphere, then there is a standard minimal sphere $\tilde{S}$ such that $S \cap |K^1| = \tilde{S} \cap |K^1|$.*

Proof. This is immediate from Propositions 2.4 and 2.5.

The set of standard minimal spheres in M is denoted $\tilde{\Omega}(M)$.

Let S_1, S_2 be standard surfaces. We say that S_1 and S_2 are well placed if they are in general position, $|K^1| \cap S_1 \cap S_2 = \varnothing$, and for each 3-simplex $|\rho|$ there is no component of $S_1 \cap S_2$ which lies entirely in the interior of $|\gamma|$. Given standard S_1, S_2 such that $S_1 \cap S_2 \cap |K^1| = \varnothing$, we can find S_2' such that S_2' is a standard surface disk equivalent to S_2 and such that S_1 and S_2' are well placed.

Let $S \subset M$ satisfy S1 and S2. Let ℓ and ℓ' be simple closed curves in S, $\ell \cap |K^1| = \ell' \cap |K^1| = \varnothing$. We say that ℓ and ℓ' are parallel in S if there is an annulus $a \subset S$ such that $\ell \cup \ell' = \partial a$ and $\| a \| = 0$.

PROPOSITION 2.7. *If S_1, S_2 are well placed standard surfaces then no two components ℓ, ℓ' of $S_1 \cap S_2$ are parallel in both S_1 and S_2.*

Proof. Suppose ℓ and ℓ' are parallel in S_1 and S_2. Now $\ell \cap |K^2| \neq \varnothing$ since S_1 and S_2 are well placed. But if $\ell \cap |\sigma| \neq \varnothing$ for some 2-simplex σ then ℓ' must intersect the same component of $S_1 \cap |\sigma|$ as ℓ. Moreover these two points of intersection must lie in the same component of $S_2 \cap |\sigma|$ since ℓ and ℓ' are parallel in S_2. Hence there must be a component of $S_2 \cap |\sigma|$ which meets a component of $S_1 \cap |\sigma|$ twice. But as these components are straight lines this is impossible.

PROPOSITION 2.8. *Let $F \subset |K^1| - |K^0|$ and suppose that, for every 1-simplex ρ of K, $f(\rho) = \#(F \cap |\rho|)$ is finite. Suppose also that for each 2-simplex σ of K with faces ρ_1, ρ_2 and ρ_3 we have $f(\rho_1)+f(\rho_2)+f(\rho_3) = 2m$ where m is an integer and $f(\rho_i) \leqslant m$ for $i = 1, 2, 3$. There exists a standard surface S_F such that $S_F \cap |K^1| = F$. The surface S_F is unique up to a homeomorphism of M which leaves $|K^2|$ fixed.*

Proof. Let σ be a 2-simplex of K. If S_F exists then $S_F \cap |\sigma|$ must be as in Fig. 2; that is, it will consist of disjoint straight lines joining all the points of $F \cap |\sigma|$. Suppose there are β_1 lines joining ρ_1 and ρ_3, β_2 lines joining ρ_1 and ρ_3 and β_3 lines joining points in ρ_1 and ρ_2. Then

$$f(\rho_1) = \beta_2+\beta_3, \qquad f(\rho_2) = \beta_1+\beta_3, \qquad f(\rho_3) = \beta_1+\beta_2.$$

We know that $f(\rho_1)+f(\rho_2)+f(\rho_3) = 2m$ where m is an integer. Hence $\beta_1 = m-f(\rho_1)$, $\beta_2 = m-f(\rho_2)$, $\beta_3 = m-f(\rho_3)$. By hypothesis β_1, β_2 and β_3 are non-negative and we see that they are uniquely determined. Thus the pattern in Fig. 2 is uniquely determined. If γ is a 3-simplex of K and the points of F in $\partial|\gamma| \cap F$ are joined in the only way possible in $\partial|\gamma|$ so that all four faces are as in Fig. 2, then we obtain a collection of simple closed curves in $\partial|\gamma|$.

These curves must bound disjoint disks in $S_F \cap |\gamma|$. It follows easily that S_F exists and $S_F \cap |K^2|$ is uniquely determined.

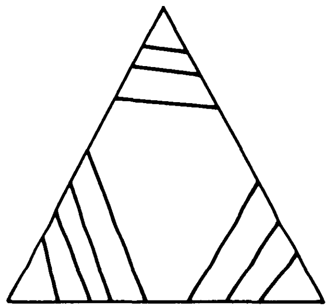


FIG. 2.

THEOREM 2.9. *Let $\Phi = \{S_1, S_2, \ldots, S_n\}$ be a finite subset of $\tilde{\Omega}(M)$. Suppose that if $i \neq j$ then $S_i \cap S_j \cap |K^1| = \varnothing$. Put $F = F(\Phi) = |K^1| \cap (\bigcup_{i=1}^n S_i)$. Then S_F is a union of n disjoint standard minimal spheres.*

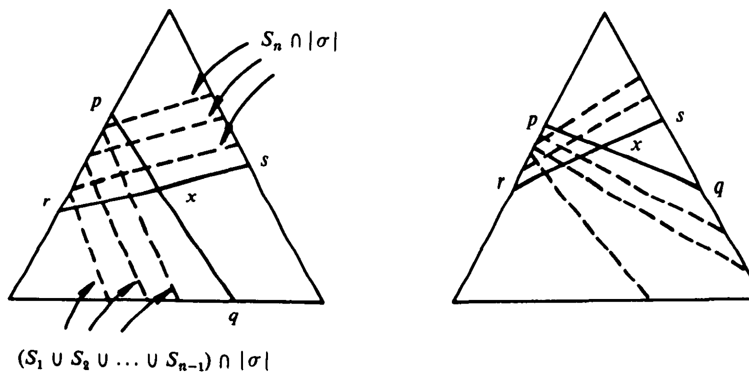
Proof. The proof is by induction on n. The result is clear if $n = 1$.

Assume the result is true for $n-1$. Let $F' = |K^1| \cap (\bigcup_{i=1}^{n-1} S_i)$. We know then that $S_{F'}$ is a union of $n-1$ disjoint minimal spheres. We can assume that $S_{F'} = \bigcup_{i=1}^{n-1} S_i$. Thus we may assume $S_i \cap S_j = \varnothing$ if $1 \leqslant i < j \leqslant n-1$. Also we may assume that S_i and S_n are well placed, $i = 1, 2, \ldots, n-1$. Let $I = S_n \cap (\bigcup_{i=1}^{n-1} S_i)$. Let $c(\Phi) = \#(I \cap |K^2|)$. If $c(\Phi) = 0$ then, since S_i and S_n are well placed, $S_i \cap S_n = \varnothing$ for $i = 1, \ldots, n-1$. Hence the Theorem is proved. Assume then that $c(\Phi) > 0$. We show that Φ can be altered (without changing F) so as to reduce $c(\Phi)$. Now I is a union of disjoint closed curves in S_n. Choose $\ell \subset I$ to be a closed curve such that $\ell = \partial\Delta$ where Δ is a smallest innermost disk in some S_i, $i = 1, 2, \ldots, n$; that is, $I \cap \Delta = \ell$, and $\|\Delta\|$ takes the smallest possible value. In fact we can assume $\Delta \subset S_n$. For if $\Delta \subset S_i$, $i = 1, 2, \ldots, n-1$, then ℓ bounds a disk Δ' in S_n and we can choose Δ' so that $\|\Delta'\| \leqslant \frac{1}{2}k$. If $\|\Delta\| < \|\Delta'\|$ then $\|\Delta \cup \Delta'\| < k$ and so $\Delta \cup \Delta' = \partial B$ where B is a 3-ball. Clearly S_n is disk equivalent to the 2-sphere S obtained from S_n by replacing Δ' by Δ. But $\|S\| < \|S_n\|$ and so we have a contradiction. Thus $\|\Delta\| = \|\Delta'\|$ and there is an innermost disk Δ'' such that $\Delta'' \subset \Delta'$ and $\|\Delta''\| = \|\Delta\|$. Hence we can assume $\Delta \subset S_n$. The argument above also shows that there is a disk $\Delta' \subset S_i$, for some $i = 1, 2, \ldots, n-1$, such that $\|\Delta'\| = \|\Delta\|$ and $\partial\Delta' = \ell$.

The first case we consider is when $\|\Delta\| < \frac{1}{2}k$. As above $\Delta \cup \Delta' = \partial B$ where B is a 3-ball. Assume that $\Delta' \cap S_n \neq \ell$. Then $\Delta' \cap S_n$ consists of a set of curves parallel to ℓ in S_i. This is because, by the choice of ℓ, there is no $\ell' \subset S_n \cap S_i$ which bounds a disk $\Delta_1 \subset S_i$ such that $\|\Delta_1\| < \|\Delta\|$. We may suppose that if C is a component of $B \cap S_n$, $C \neq \Delta$, and C is a disk then $\|C\| > \|\Delta\|$. For if there is a component of $B \cap S_n$ such that C is a disk and $\|C\| = \|\Delta\|$, then we can replace Δ by the component C for which ∂C is innermost in Δ'. Also $B \cap S_n$ has no component which is an annulus a such that $\|a\| = 0$. This is because no two components of $S_i \cap S_n$ are parallel in both S_i and S_n, by Proposition 2.7. Let ℓ' be the curve of $\Delta' \cap S_n$ which is innermost in Δ'. Now ℓ' bounds a disk D in $S_n \cap \overline{M-B}$ such that $\|D\| = \|\Delta\|$. For otherwise we obtain a contradiction by doing surgery on S_n along the disk D' in Δ' bounded by ℓ'. Now if $\Delta \subset D$ we have a contradiction by the parallel curve argument. If not, then by replacing D in S_n by D' and making a small alteration in the neighbourhood of D' we obtain a new surface S_n' which is disk equivalent to S_n but $\Delta' \cap S_n' = (\Delta' \cap S_n) - \{\ell'\}$. Since $B \cap S_n$ has no component which is an annulus

a such that $\| a \| = 0$, $B \cap S'_n$ has no component $C \neq \Delta$ which is a disk such that $\| C \| = \| \Delta \|$. If we repeat the argument we eventually obtain a disk D such that $\Delta \subset D$, giving a contradiction as required. Note that although S_n has been changed to a non-standard sphere the parallel curves giving the contradiction will occur in a part of the sphere which has not been changed. Thus $\Delta' \cap S_n = \ell$. The sphere S'_i obtained from S_i by replacing Δ' by Δ is disk equivalent to S_i. The sphere S'_n obtained from S_n by replacing Δ by Δ' is disk equivalent to S_n. Let $U_i = \tilde{S}'_i$ and $U_n = \tilde{S}'_n$. Put $U_j = S_j$ if $j \neq i$ and $j \neq n$. Now $(U_i \cup U_n) \cap |K^1| = (S_i \cup S_n) \cap |K^1|$. Put $\Phi' = \{U_1, \ldots, U_n\}$. I claim that $c(\Phi') < c(\Phi)$. Note that $F(\Phi') = F(\Phi)$. Also $U_1, U_2, \ldots, U_{n-1}$ are disjoint, since $S'_i \cap S_j = \varnothing$ if $j = 1, 2, \ldots, i-1, i+1, \ldots, n-1$, and standardizing S'_i does not affect this property.

Let σ be a 2-simplex of K which contains a point of ℓ. Let $p, q, r, s \in \partial |\sigma|$ be such that the line $pq \subset S_i$ and the line $rs \subset S_n$ and pq, rs intersect in a point x of ℓ. We can assume p and r lie in the same face. Thus the situation must be as in either Fig. 3(a) or Fig. 3(b)

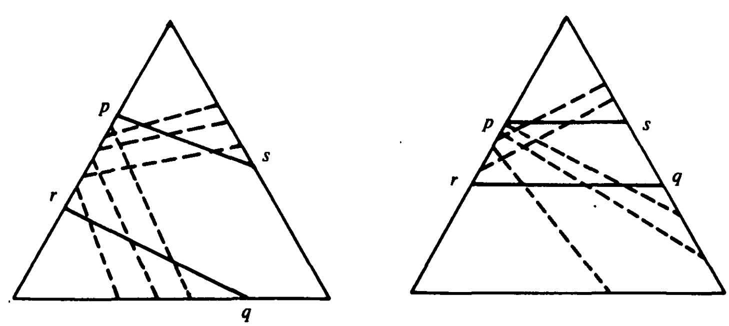


(a) (b)

FIG. 3.

Also either $xs \subset \Delta$ and $xq \subset \Delta'$ or $xp \subset \Delta'$ and $xr \subset \Delta$. For if, say $xs \subset \Delta$ and $xp \subset \Delta'$, $U_n \cap |\sigma|$ would have to have a line joining p and r contradicting Proposition 2.3. If $xs \subset \Delta$ then the only intersections of rs with $S_1, S_2, \ldots, S_{n-1}$ occur in the line segment rx and the only intersections of pq with S_n occur in the line segment px. A possible configuration is shown dotted in Fig. 3.

The changed configuration on replacing Φ by Φ' is shown in Fig. 4.



(a) (b)

FIG. 4.

It can be seen that the number of intersections is reduced by at least one (ℓ may intersect σ more than once).

It remains to treat the case when $\| \Delta \| = \frac{1}{2}k$. In this case $I = S_n \cap (\bigcup_{j=1}^{n-1} S_j)$ is a union of parallel curves in S_n. For each $j \in \{1, \ldots, n-1\}$, $S_n \cap S_j$ consists of at most one component, since we have remarked that curves cannot be parallel in both S_n and S_j. In this case also then we see that $\Delta' \cap S_n = \ell$. In this case we do not know that $\Delta \cup \Delta' = \partial B$ where B is a 3-ball. We show that after a possible relabelling that this can be assumed. If I has more than one component then there are two possible choices for ℓ, since there are two curves in S_n which are innermost. If $I = \ell$ then there are two possible choices for Δ. Thus there are always two possible choices for Δ. Also there are two possible choices for Δ', namely Δ' and $\overline{S_i - \Delta'}$. Suppose σ is a 2-simplex containing a point of ℓ. Let $rs \subset S_n$ be a component of $S_n \cap |\sigma|$ containing a point of ℓ. The situation will be as in Fig. 5. It is always possible to choose Δ and Δ' so

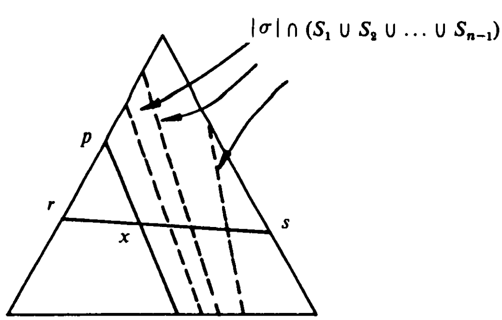


FIG. 5.

that $rx \subset \Delta$, $px \subset \Delta'$ where r and p are in the same face. It now follows from Proposition 2.3 that $\Delta \cup \Delta'$ is not a minimal sphere and so $\Delta \cup \Delta' = \partial B$ where B is a 3-ball. This case now reduces to the previous case. It follows then that there is a set $\hat{\Phi}$ of n disjoint minimal spheres such that $F(\hat{\Phi}) = F$. But by Proposition 2.8, S_F must be the union of these spheres.

3. *The main result*

Let M, K be as in §2. Let G be a group. Suppose there is an action of G on K (written on the left). We assume that $G \backslash K$ is a simplicial complex. Nothing is lost in assuming this, since if K does not have this property, K'', the second barycentric subdivision of K, with the induced G-action, does have this property. The action of G on K extends to an action on M. We assume that the stabilizer of each 3-simplex is trivial, since any element of G which fixes a 3-simplex must fix all of M. It is easy to see that the stabilizer of any simplex is finite.

THEOREM 3.1. *If $\Omega(M) \neq \varnothing$, then there exists $U \in \tilde{\Omega}(M)$ such that for every $g \in G$ either $gU = U$ or $gU \cap U = \varnothing$.*

Proof. Let $S \in \tilde{\Omega}(M)$. Let $\Sigma = \{gS \mid g \in G\}$. Since stabilizers are finite, for each 1-simplex μ of K there are only finitely many elements of Σ which intersect $|\mu|$. For each $T \in \Sigma$ choose $T' \in \tilde{\Omega}(M)$ so that there is a homeomorphism $\theta_T: T \to T'$ such that for each 1-simplex μ of K, θ induces an order-preserving bijection

$$\theta_\mu: T \cap |\mu| \to T' \cap |\mu|.$$

(Here we use the obvious order on $|\mu|$.) Choose the spheres T' so that if $T_1 \neq T_2$ then $T_1' \cap T_2' \cap |\mu| = \varnothing$ for every 1-simplex μ. We do not assume that the map $T \to T'$ is a G-map. Let $J = (\bigcup\{T' \mid T \in \Sigma\}) \cap |K^1|$. It is possible to change the choice of the spheres T' so that for any 1-simplex μ, $|\mu| \cap J$ is changed to another subset of $|\mu| - \partial|\mu|$ with the same number of points. Since $g\Sigma = \Sigma$, $\#(J \cap g|\mu|) = \#(J \cap |\mu|)$ for every $g \in G$. Thus we can choose the spheres T' so that $gJ = J$ for every $g \in G$.

If M is compact so that K is finite then it follows from Theorem 2.9 that S_J is a union of disjoint minimal spheres. Now $g(S_J \cap |K^2|) = S_J \cap |K^2|$ for every $g \in G$. Since $G\backslash K$ is a complex we can, in fact, choose S_J so that $gS_J = S_J$ for every $g \in G$. Thus the Theorem is proved. If M is not compact we need to do a bit more work.

Let γ_0 be a fixed 3-simplex of K. We can choose a finite subcomplex K_0 of K so that if $U \in \tilde{\Omega}(M)$ and $|\gamma_0| \cap U \neq \varnothing$ then $U \subset |K_0|$. Let Φ be the set of those T' which intersect $|K_0|$ and $J_0 = (\bigcup\{T' \mid T' \in \Phi\}) \cap |K^1|$. Now Theorem 2.9 applies to Φ and so S_{J_0} is a union of minimal spheres. But $|K_0| \cap J_0 = |K_0| \cap J$ and so $S_{J_0} \cap |K_0| = S_J \cap |K_0|$. Thus the components of S_J which intersect $|\gamma_0|$ are in $\tilde{\Omega}(M)$. The Theorem follows immediately, since γ_0 is arbitrary.

4. *Equivariant decompositions*

Again let M, K be as in §2. Let the group G act on K and M as in §3 so that $G\backslash K$ is a complex.

Let Σ be a G-set of disjoint 2-spheres in M. Let $N(\Sigma)$ be the 3-manifold (not usually connected) obtained from M by cutting along the 2-spheres of Σ and attaching a 3-ball to each 2-sphere in the boundary of the resulting manifold that arises from cutting along an element of Σ. Recall that a manifold N is called irreducible if each 2-sphere in N bounds a 3-ball.

THEOREM 4.1. *Suppose $G\backslash K = L$ is finite. There is a G-set Σ of disjoint 2-spheres in M such that $N(\Sigma)$ is irreducible.*

Proof. Let Σ be a G-set of disjoint 2-spheres in M and suppose $N(\Sigma)$ is not irreducible. There exists a 2-sphere $S' \subset N(\Sigma)$ such that S' does not bound a ball. In $N(\Sigma)$ there is a set Σ' of 2-spheres (all bounding 3-balls) corresponding to the elements of Σ. For each element of Σ there are two elements of Σ'. It can be assumed that S' does not intersect any $U \in \Sigma'$. For if $n(S)$ is the number of components of $S' \cap (\bigcup U \mid U \in \Sigma')$ then choose S' so that $n(S')$ is smallest. A familiar argument shows that $n(S') = 0$.

Since S' does not intersect any $U \in \Sigma'$, it corresponds in an obvious way to a 2-sphere $S \subset M$. Since S' does not bound a 3-ball in $N(\Sigma)$, S does not bound a 3-ball in M. Also S is disjoint from each 2-sphere in Σ. We restrict attention to those S' for which the corresponding S satisfies S1 and S2. Thus we can define $\|S'\|$ to be $\|S\|$. Among such spheres S' define S' to be minimal if $\|S'\|$ takes the smallest possible value.

The action of G on M induces an action of G on $N(\Sigma)$. The proof of Theorem 3.1 works equally well in this situation. Thus there exists a 2-sphere $S' \subset N(\Sigma)$ such that S' is minimal (and so S' does not bound a 3-ball in $N(\Sigma)$ and $n(S') = 0$) and $gS' = S'$ or $gS' \cap S' = \varnothing$ for every $g \in G$. Let S be the corresponding 2-sphere in M. Put $\Sigma' = \Sigma \cup \{gS \mid g \in G\}$. This is a G-set of disjoint 2-spheres. Also S is not parallel to any $U \in \Sigma$, that is, there is no submanifold A of M homeomorphic to $S^2 \times [0, 1]$

for which $\partial A = S \cup U$. The remainder of the proof of Theorem 4.1 consists of showing that if $\#(G\backslash\Sigma)$ exceeds a certain number then Σ contains parallel elements in different G-orbits. The argument used here is essentially that of Kneser (Hempel [**1**, p. 29]). However we have to be careful since $G\backslash M$ is not usually a 3-manifold.

Let τ be a 3-simplex of K and let X be the closure of a component of

$$|\tau| - \bigcup\{S \mid S \in \Sigma\}.$$

Then X is a 3-cell whose boundary is made up of standard disks in τ together with a connected submanifold in $\partial\tau$. We say that X is good if $X \cap \partial\tau$ is an annulus which contains no vertex of τ. For each 3-simplex τ, there are at most 6 bad cells X (see [**1**, p. 30]). If R is the closure of a component of $M - \bigcup\{S \mid S \in \Sigma\}$, then R is called good if for each 3-simplex τ of K, $R \cap |\tau|$ is good in the above sense. If R is good then it is either $S^2 \times I$ or the twisted I-bundle over P^2. Put $k(L) = \dim H^1(L; \mathbb{Z}_2) + 6t$ where t is the number of 3-simplexes of L. Suppose $\#(G\backslash\Sigma) > k(L)$. Let $\pi: M \to |L|$ be the projection map. If R is good then πR is good in $|L|$. Now $|L|$ has at most $6t$ bad components and so there are at most $6t$ orbits of bad components in M. Let R be a good component in M. The I-bundle structure on R maps to an I-bundle structure on πR. If πR is not twisted (that is, it is a trivial I-bundle), then R is not twisted. Suppose πR is twisted. If τ is a 3-simplex of L, $|\tau| \cap \pi R$ will consist of a number of components X such that $X \cap \partial|\tau|$ is an annulus containing no vertices of τ. For each edge e of τ for which $X \cap |e| \neq \emptyset$ take the mid point x_e of $X \cap |e|$. Let Δ_X be a standard disk in X determined by the points x_e. Let Z be the union of all the disks Δ_X as X ranges over all components of $\pi R \cap |\tau|$ and τ ranges over all simplexes of L. For any 2-simplex σ of L, $Z \cap |\sigma|$ is a union of disjoint lines joining points on the boundary. Let z be the 1-cochain

$$z: L^1 \to \mathbb{Z}_2, \quad z(e) = \#(|e| \cap Z) \pmod 2.$$

Then $\delta z = 0$. However, Z does not separate in πR, and hence in L, since πR is twisted. Thus z represents a non-trivial element of $H^1(L; \mathbb{Z}_2)$. The non-separating property in πR also ensures that if $R_1, R_2, \ldots, R_k$ are good components in M lying in different G-orbits and $\pi R_1, \pi R_2, \ldots, \pi R_k$ are all twisted, then the corresponding elements of $H^1(L, \mathbb{Z}_2)$ are linearly independent. Thus if $\#(G\backslash\Sigma) > k(L)$, there exists a good R in M such that πR is not twisted. Thus R is homeomorphic to $S^2 \times I$ and the two components of ∂R lie in different G-orbits. This completes the proof of Theorem 4.1.

Theorem 4.1 enables us to prove an equivariant version of the projective plane theorem for compact 3-manifolds (see [**1**, p. 54]).

If S is a 2-sphere or projective plane in M then there is a covering map $p: S^2 \to S$. Thus corresponding to S is an element $[S]$ of $\pi_2(M)$. This element is only determined up to action by elements of $\pi_1(M)$ and replacing $[S]$ by $-[S]$.

THEOREM 4.2. *Let M be a compact 3-manifold acted on by a group G. There is a G-set Σ of disjoint embedded 2-spheres and projective planes such that $\{[S] \mid S \in \Sigma\}$ generates $\pi_2(M)$ as a $\pi_1(M)$-module.*

Proof. Let $\tilde{M}$ be the universal cover of M. There is a group $\tilde{G}$ acting on $\tilde{M}$ for which there is an exact sequence

$$1 \longrightarrow \pi_1(M) \longrightarrow \tilde{G} \xrightarrow{\phi} G \longrightarrow 1.$$

Also if $\pi: \tilde{M} \to M$ is the covering map, then $\pi(gm) = \phi(g)\pi(m)$, $g \in \tilde{G}$, $m \in \tilde{M}$. Let $\tilde{M} = |\tilde{K}|$ where $\tilde{K}$ is a 3-dimensional complex covering K, and $|K| = M$. As in §3 we assume $G\backslash K$ is a complex and so $\tilde{G}\backslash\tilde{K} = G\backslash K$ is a finite complex. We can apply Theorem 4.1. Thus there exists a $\tilde{G}$-set $\tilde{\Sigma}$ of disjoint 2-spheres in $\tilde{M}$ such that $N(\tilde{\Sigma})$ is irreducible. Now if $\tilde{S} \in \tilde{\Sigma}$ then $S = \pi(\tilde{S})$ is either a 2-sphere or a projective plane. Let $\Sigma = \{\pi(\tilde{S}) \mid \tilde{S} \in \tilde{\Sigma}\}$. I claim that $\{[S] \mid S \in \Sigma\}$ generates $\pi_2(M)$ as a $\pi_1(M)$-module. This follows if we can show that $\{[\tilde{S}] \mid \tilde{S} \in \tilde{\Sigma}\}$ generates $\pi_2(\tilde{M})$. Now let $f: S^2 \to \tilde{M}$ be a map in general position. Assign an orientation to S^2.

Since $\tilde{M}$ is simply connected the homotopy class of f determines a unique element of $\pi_2(\tilde{M})$. Also $\pi_2(\tilde{M})$ is isomorphic to $H_2(\tilde{M})$ by the Hurewicz Isomorphism Theorem. By making small changes to $f(S^2)$ in the neighbourhood of singularities, we obtain an embedded oriented surface S_1 which represents the same element of $H_2(\tilde{M})$ as $f(S^2)$. If U is an embedded 2-sphere in $\tilde{M}$ then $[U]$ is in the subgroup Q of $\pi_2(\tilde{M})$ generated by $\{[\tilde{S}] \mid \tilde{S} \in \tilde{\Sigma}\}$. This is proved in the usual way by induction on the number of components of $U \cap (\bigcup\{\tilde{S} \mid \tilde{S} \in \tilde{\Sigma}\})$. If S_1 is a union of 2-spheres then $[f(S^2)] \in Q$. If not then S_1 can be reduced to such a union by surgery over a finite number of compressing disks the existence of which is guaranteed by the ordinary loop theorem. Thus $[f(S^2)] \in Q$ and $Q = \pi_2(\tilde{M})$ as required.

5. *The equivariant loop theorem*

Let M be a triangulated 2-manifold. Thus $M = |K|$ where K is a 2-dimensional simplicial complex. Suppose the group G acts on K so that $G\backslash K$ is also a complex. Extend this action to M. If we assume that any embedded simple closed curve in M separates M then exactly analogous theorems to Theorems 3.1 and 4.1 can be obtained. In fact in order to prove an analogous result to Theorem 3.1 it is only necessary to assume that there is a simple closed curve in M which does not bound a disk but which does separate M. This separation property is necessary in order to ensure that the curve does not meet a translate in a single point, when the arguments of Theorem 2.8 could not be applied. Thus we consider simple closed curves $C \subset M$ such that C is in general position with respect to K. Let $\Omega(M)$ be the set of those C which do not bound disks in M.

A 2-manifold N with $\partial N = \emptyset$ in which every simple closed curve bounds a disk is either $\mathbb{R}^2$ or S^2 These are the only 'irreducible' 2-manifolds without boundary.

THEOREM 5.1. *Let M be a 2-manifold with $\partial M = \emptyset$. Let $M = |K|$ where K is a simplicial complex. Suppose the group G acts on K and M so that $G\backslash K$ is a finite complex. Suppose every simple closed curve in M separates M, that is, suppose M is planar. There is a G-set Γ of disjoint simple closed curves such that each surface obtained by taking the closure of a component of $M - \bigcup\{C \mid C \in \Gamma\}$ and attaching disks to all the boundary curves is homeomorphic either to $\mathbb{R}^2$ or to S^2.*

Proof. As remarked above this is just a repeat of the argument of Theorem 4.1 (incorporating Theorem 3.1) adapted to 2-manifolds.

Note that the curves of Γ obtained are all in general position with respect to K. If σ is a 2-simplex of K and $C \in \Gamma$ then $C \cap |\sigma|$ will consist of straight lines joining distinct edges. (In fact a further argument shows that $C \cap |\sigma|$ has at most one component.)

THEOREM 5.2. *Let M be a 3-manifold. Let $M = |K|$ where K is a simplicial complex. Suppose the group G acts on K and M so that $G\backslash K$ is a finite complex. There exists a G-set Σ of disjoint disks properly embedded in M $(\partial M \cap \Delta = \partial\Delta$ if $\Delta \in \Sigma)$ such that the normal subgroup generated by $\Gamma = \{\partial\Delta \mid \Delta \in \Sigma\}$ is the kernel of the map $i_*: \pi_1(\partial M) \to \pi_1(M)$ induced by inclusion.*

Proof. Let $\tilde{M}$ be the universal cover of M. Let K and G be as in the proof of Theorem 4.2.

Let N be a component of $\partial\tilde{M}$. Let G_N be the stabilizer of N in $\tilde{G}$. The covering map π induces an injective map $G_N\backslash N \to \partial M$. Now any simple closed curve in N separates N. For if c is a simple closed curve in N then c bounds a disk Δ in $\tilde{M}$ by the ordinary Dehn's lemma. Cutting along Δ separates $\tilde{M}$, for otherwise $H_1(\tilde{M}) \neq 0$. Hence cutting along c separates N. Let Γ_N be the G_N-set of simple closed curves in N as given by Theorem 5.1. We can assume that if $g \in \tilde{G}$ then $\Gamma_{gN} = g\Gamma_N$. Thus the union of all the Γ_N as N ranges over all components of $\partial\tilde{M}$ is a $\tilde{G}$-set $\tilde{\Gamma}$. If $C \in \tilde{\Gamma}$ and $g \in \tilde{G}$ is such that $gC = C$ then we can assume that g does not transpose the two components of $N - C$. Here N is the component of $\partial\tilde{M}$ containing C. For if there is such a g which transposes these components, then we can replace each curve in the $\tilde{G}$-orbit of C by a pair of parallel curves lying each side of the original curve. This can be done so that the new set is a $\tilde{G}$-set with the required property.

If $C \in \tilde{\Gamma}$ then we say that a minimal disk spanning C is a disk Δ such that $\partial\Delta = C$, C and $\tilde{K}$ are in general position and $\|\Delta\|$ takes its smallest possible value.

Let $\Gamma_0 \subset \tilde{\Gamma}$ be a transversal for the G-action. For each $C \in \Gamma_0$ and $g \in G$ choose a $\Delta = \Delta(g, C)$ to be a minimal disk spanning gC. We make the choices so that $\Delta(g, C) = g\Delta(1, C)$. Now for each $\Delta = \Delta(g, C)$ choose Δ' so that there is a homeomorphism $\theta_\Delta: \Delta \to \Delta'$ which restricts to an order-preserving bijection $\Delta \cap |\mu| \to \Delta' \cap |\mu|$ for each 1-simplex μ. Also if $\Delta_1 \neq \Delta_2$ then

$$(\Delta_1' - \partial\Delta_1') \cap (\Delta_2' - \partial\Delta_2') \cap |\mu| = \varnothing.$$

As in the Proof of Theorem 4.1 it can be assumed that the set $J = \{|K^1| \cap \Delta' \mid \Delta \in \tilde{\Sigma}\}$ is a $\tilde{G}$-set. If $x \in J \cap \partial\tilde{M}$ so that $x \in C$ for some $C \in \tilde{\Gamma}$ then we give x a weighting $w(x) = |\tilde{G}_C|$. If $x \in J - \partial\tilde{M}$, put $w(x) = 1$. We can now adapt the proof of Proposition 2.8 to show that there is a unique $\tilde{G}$-subset S_J of $\tilde{M}$ such that for each 3-simplex γ of $\tilde{K}$, $|\gamma| \cap S_J$ consists of a union of standard disks, which intersect if at all in a subset of $\partial\tilde{M} \cap \partial|\gamma|$, and for each $x \in J$ there are $w(x)$ disks containing x. Repeating the arguments of Theorems 2.9 and 3.1 shows that S_J consists of a union of minimal disks which intersect if at all on their boundaries. If $C \in \tilde{\Gamma}$, there will be $|\tilde{G}_C|$ disks Δ such that $\partial\Delta = C$. By choosing a suitable subset of these disks we obtain a $\tilde{G}$-set $\tilde{\Sigma}$ of disjoint disks in $\tilde{M}$ such that $\tilde{\Gamma} = \{\partial\Delta \mid \Delta \in \tilde{\Sigma}\}$.

This choice is possible because if $gC = C$ then g fixes every disk Δ such that $\partial\Delta = C$. To see this suppose $\Delta_1, \Delta_2, \ldots, \Delta_r$ are all the disks of $\tilde{\Sigma}$ containing C. We can assume that the Δ_i are labelled so that there is a path joining points in ∂M and containing exactly one point from $\Delta_1, \Delta_2, \ldots, \Delta_k$ in that order. This order is independent of the path chosen provided it starts on the right side. If $gC = C$ then g fixes the components of $N - C$ and so g preserves the order of the disks. Thus if $gC = C$ then $g\Delta = \Delta$ for each $\Delta \in \tilde{\Sigma}$ such that $\partial\Delta = C$.

Cutting $\tilde{M}$ along the disks of $\tilde{\Sigma}$ produces manifolds with simply connected boundaries (either S^2 or $\mathbb{R}^2$). If a loop ℓ in ∂M represents an element in the kernel of the map $i_*: \pi_1(\partial M) \to \pi(M)$ then ℓ lifts to a loop in $\partial\tilde{M}$. Thus ℓ is freely homotopic to a product of the loops of the form ∂C, $C \in \tilde{\Gamma}$. Let $\Sigma = \{\pi\Delta \mid \Delta \in \tilde{\Sigma}\}$. Since $\tilde{\Sigma}$ is a $\tilde{G}$-set, it follows that Σ is a G-set of disjoint disks. Also the normal subgroup generated by $\{\partial\Delta \mid \Delta \in \Sigma\}$ is the kernel of i_*. This completes the proof of Theorem 5.2.

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