Math 990 Topics in Topology

Problem Set # 1

Two problems (of your choice) are due Thursday, February 7

1. Show that two elements u, v of the free group $G = F(x_1, \ldots, x_n)$ are conjugate ($v = wuw^{-1}$ for some $w \in G \Leftrightarrow v$, as a (reduced) word in the x_i, is a cyclic permutation of u; $u = x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$ and $v = x_{i_r}^{\epsilon_r} \cdots x_{i_k}^{\epsilon_k} x_{i_1}^{\epsilon_1} \cdots x_{i_{r-1}}^{\epsilon_{r-1}}$.

(N.B.: We can therefore test elements in the free group to decide if they are conjugate, i.e., we can solve the "conjugacy problem" in free groups.)

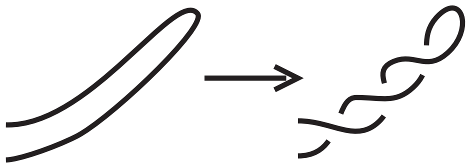
2. Show that using our more formal definition of a presentation that for a finite group G, the group with finite presentation $< \{g : g \in G\} \mid \{ghk^{-1} : g, h, k \in G \text{ and } gh =_G k\} >$ is isomorphic to G. (Build homomorphisms each way, show they are inverses.)

3. Show that the group $G =< a, b \mid a^3, b^2, (ab)^2 >$ is a non-trivial group. (Hint: find a non-trivial homomorphism to some small group.) Use this (or argue directly,) to show that the group $H =< a, b \mid a^3, b^4, ab^2a^2b^2 >$ is also non-trivial.

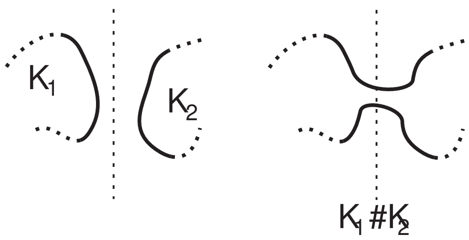
4. Show that the group with presentation $< x, y \mid x^2 = y^3, xyx = yxy >$ is the trivial group, i.e., show that the relators imply that $x = 1$ and $y = 1$. (One way: find may different ways to express the word x^2yxy in G ...)

5. Show that a group G is generated by finitely many x_i, all conjugate to one another $\Leftrightarrow$ G is finitely generated, and there is an x with $\{gxg^{-1} : g \in G\}$ a (not necessarily finite) generating set for G.

6. Show that in Johnson's construction of knots with a given quotient, in constructing the "tongue" we may allow it to twist around itself (see figure below) without interfering with the conclusion of the construction. (This allows us to build a wider class of knots with a given quotient.)



7. Show that the connected sum operation (figure below) results in a knot $K_1\#K_2$ whose group $G = \pi(\pi_1(K_1\#K_2) = \pi_1(\mathbb{R}^3 \setminus (K_1\#K_2))$ has quotients $\pi(K_1)$ and $\pi(K_2)$. (Hint: think in terms of "coloring" the connected sum so that its group maps onto each group (separately).)



8. Find several knots K whose knot groups $G = \pi_1(\mathbb{R}^3 \setminus K)$ have quotient the group of the trefoil knot (figure below). (Don't use the connected sum operation...)